ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 7, pp. 2233–2258

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 7, pp. 2233–2258
Online ISSN: 1735-8515

DUALITY FOR THE CLASS OF A MULTIOBJECTIVE PROBLEM WITH SUPPORT FUNCTIONS UNDER K-G_f-INVEXITY ASSUMPTIONS

I.P. DEBNATH* AND S.K. GUPTA

(Communicated by Maziar Salahi)

Abstract. In this article, we formulate two dual models Wolfe and Mond-Weir related to symmetric nondifferentiable multiobjective programming problems. Furthermore, weak, strong and converse duality results are established under K-G_f-invexity assumptions. Nontrivial examples have also been depicted to illustrate the theorems obtained in the paper. Results established in this paper unify and extend some previously known results appeared in the literature.
Keywords: Multiobjective programming, K-G_f-invexity, support function, efficient solutions, duality.
MSC(2010): Primary: 90C26; Secondary: 49N15.

1. Introduction

Duality theory is an important part of the optimization theory. Special, dual problems of optimization, are applied to many types of optimization problems. They are used for the proof of optimality of solutions, for designing and a theoretical justification of optimization algorithms, for physical or economic interpretation of received solutions. Quite often dual problems introduce new meaning to modeled problems. For example, economic resources optimal allocation dual problems are usually models of rational pricing.

It is a possible situation where the dual problem to a dual optimization problem coincides with an initial optimization problem. This case is named symmetric duality [21]. It is well known that the symmetric duality is applied for linear programming problems. In general, this does not happen for nonlinear programming problems. Symmetric duality was first introduced by Dorn [7] and called the same to be symmetric if the dual of the dual can be recast as

Article electronically published on December 30, 2017.
Received: 5 July 2016, Accepted: 31 January 2017.
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the primal problem. Outstandingly, many researchers working in this direction, has developed the concept of symmetric duality.

Interestingly, multiobjective optimization has a vast number of applications, for example in goal programming, risk programming, etc. Miettinen [15] and Pardalos et al. [16] gave the conditions for optimality in the case of multi-objective programming problems. Further, using the concept of higher-order cone-preinvex and cone-pseudoinvex functions, Gupta and Jayswal [9] studied the duality relations for a higher-order symmetric Mond-Weir type multiobjective problem over cones, which therefore extends some of the results in [8, 14]. Introducing the concept of higher order strictly and strongly K-pseudoinvexity, recently, Suneja and Louhan [19] discussed recent developments in nondifferentiable multiobjective optimization under higher order K-invexity. Agarwal et al. [1] have given some corrective measures in the work of Chen [5]. Gupta et al. [10] constructed a pair of higher-order Wolfe type symmetric dual programs for nondifferentiable multiobjective programming problems over cones under (F, α, ρ, d)-convexity assumptions. Motivated by various concepts of generalized convexity, Jayswal and Kummari [11] studied higher order duality for multiobjective programming problem under (ϕ, ρ)-invexity assumptions. Recently, Jayswal and Kummari [12] established necessary and sufficient optimality conditions for a nondifferentiable minimax semi-infinite programming problems in complex spaces under invexity assumptions. Considering an improved definition of generalized type I univex function, Soleimani-damaneh [17] addressed the optimality and duality of multiobjective optimization problems.

Very recently, Dehui and Xiaoling [6] have established necessary and sufficient optimality conditions for a multiobjective programming problem with support functions and hence derived the duality theorems for general Mond-Weir type dual problem under (G, C, ρ)-convexity assumptions. Jiao [13] introduced new concepts of nonsmooth K-α-d_I-invex and generalized type I univex functions over cones using Clarke's generalized directional derivative and d_I-invexity for a nonsmooth vector optimization problem with cone constraints. Further, the author has also established sufficient optimality conditions and Mond-Weir type duality results under K-α-d_I-invexity and type I cone-univexity assumptions. In recent past, several definitions such as, nonsmooth univex, nonsmooth quasiunivex and nonsmooth pseudoinvex functions have been introduced by Xianjun [20]. Introducing these new concepts, sufficient optimality conditions for a nonsmooth multiobjective problem have been derived and then weak and strong duality results are established for a Mond-Weir type multiojective dual programs. Recently, Antczak [3] has established the saddle point criteria and Wolfe duality theorems for a class of nondifferentiable vector optimization problems.

In this article, we consider a concept of K-G_f-invexity and formulate Wolfe and Mond-Weir type symmetric dual models related to nondifferentiable multiobjective programming problems. Various nontrivial examples which shows the existence of K-G_f-invex and K-G_f-incave functions have been illustrated. Considering the Wolfe and Mond-Weir type symmetric primal-dual models, appropriate duality results have been established. Further, several examples verifying the weak duality results for both the Wolfe and Mond-Weir type primal-dual pairs have also been discussed in the paper.

2. Notations and preliminaries

Throughout this article, let R^n denotes n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Consider the following multiobjective programming problem:

$$\begin{aligned} (\mathbf{P}) \quad & K - \text{minimize } f(x) \\ & \text{subject to } x \in X^0 = \{x \in S : -g(x) \in C\}, \end{aligned}$$

where $S \subset R^n$ be open, $f : S \to R^k$, $g : S \to R^m$, K and C are closed convex pointed cones with nonempty interiors in R^k and R^m, respectively.

In this section, we provide some definitions that will follow-up throughout the manuscript.

Definition 2.1 ([10])**.** The positive polar cone C^* of C is defined as

$$C^* = \{z \in R^m : x^T z \geqq 0, \text{ for all } x \in C\}.$$

Definition 2.2 ([10])**.** A point $\bar{x} \in X^0$ is said to be an efficient solution of a multiobjective programming problem (P) if there exists no other $x \in X^0$ such that

$$f(\bar{x}) - f(x) \in K \backslash \{0\}.$$

Let $C_1 \subseteq R^n$ and $C_2 \subseteq R^m$ be closed convex cones with non-empty interiors and S_1 and S_2 be non-empty open sets in R^n and R^m, respectively such that $C_1 \times C_2 \subseteq S_1 \times S_2$. Suppose $f = (f_1, f_2, \ldots, f_k) : S_1 \times S_2 \to R^k$ be a vector-valued differentiable function.

Definition 2.3. The function f is said to be K-invex at $u \in S_1$ with respect to $\eta : S_1 \times S_2 \to R^n$ if for all $x \in S_1$ and for fixed $v \in S_2$, we have $\{f_1(x, v) - f_1(u, v) - \eta^T(x, u)\nabla f_1(u, v), \ldots, f_k(x, v) - f_k(u, v) - \eta^T(x, u)\nabla f_k(u, v)\} \in K$.

Now, we generalize the definition of a real-valued G-invex function introduced by Antczak [2] to the vectorial case.

Definition 2.4 ([2])**.** The function f is said to be $K - G_f$-invex (or, $K - G$-invex) at $u \in S_1$ (with respect to η) if there exists a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_k}) : R \to R^k$ such that any of its component

$G_{f_i} : I_{f_i}(S_1 \times S_2) \to R$, where $I_{f_i}(S_1 \times S_2), i = 1, 2, \ldots, k$, is the range of f_i, is a strictly increasing function on its domain and $\eta : S_1 \times S_2 \to R^n$ is a vector-valued function such that, for all $x \in S_1$ for fixed $v \in S_2$

$$\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta^T(x,u)(G'_{f_1}(f_1(u,v))\nabla f_1(u,v)), \ldots,$$
$$G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(u,v)) - \eta^T(x,u)(G'_{f_k}(f_k(u,v))\nabla f_k(u,v))\} \in K.$$

We will now show the existence of the above definition by giving an example.

Example 2.5. Let $k = 2$, $n = 1$, $S_1 = S_2 = R_+$, $C_1 = C_2 = R_+$ and $K = \{(x, y) \in R^2 : y \geqq 0, x \leqq y\}$. Let also $f : S_1 \times S_2 \to R^2$, $G_{f_i} : I_{f_i} \to R$ $(i = 1, 2)$ and $\eta : S_1 \times S_2 \to R$ be defined as:

$$f(x,y) = \{f_1(x,y), f_2(x,y)\},$$

where

$$f_1(x,y) = e^y,\ f_2(x,y) = xe^y, G_{f_1}(t) = t,\ G_{f_2}(t) = t^2$$
$$\text{and } \eta(x,u) = x - u.$$

Next, we will show that the function defined above is $K - G_f$-invex at $u = 0$. Applying the definition of $K - G_f$-invex at $u = 0$, we have

$$\Big\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta^T(x,u)(G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)),$$
$$G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(u,v)) - \eta^T(x,u)(G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v))\Big\}$$
$$= (0, x^2 e^{2v}) \in K$$

Hence, $f = (f_1, f_2)$ is $K - G_f$-invex function at $u = 0$ in S_1 with respect to η.

Definition 2.5. The function f is said to be $K - G_f$-incave (or, $K - G$-incave) at $u \in S_1$ (with respect to ξ) if there exists a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_k}) : R \to R^k$ such that any of its component $G_{f_i} : I_{f_i}(S_1 \times S_2) \to R$, where $I_{f_i}(S_1 \times S_2), i = 1, 2, \ldots, k$, is the range of f_i, is a strictly increasing function on its domain and a vector-valued function $\xi : S_1 \times S_2 \to R^n$ such that, for all $x \in S_1$ and for fixed $v \in S_2$,

$$\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \xi^T(x,u)(G'_{f_1}(f_1(u,v))\nabla f_1(u,v)), \ldots,$$

$$G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(u,v)) - \xi^T(x,u)(G'_{f_k}(f_k(u,v))\nabla f_k(u,v))\} \in -K.$$

Example 2.6. Let $k = 2$, $n = 1$, $S_1 = S_2 = R$, $C_1 = C_2 = R$ and $K = \{(x, y) \in R^2 : y \geqq 0, 2x \leqq 3y\}$, then $-K = \{(x, y) \in R^2 : 2x \geqq 3y, y \leqq 0\}$.

Let $f(x, y) = \{f_1(x, y), f_2(x, y)\}$, where $f_1(x, y) = x^2 \sin^2 y$, $f_2(x, y) = y^2$.

Suppose $G_{f_1}(t) = t,\ G_{f_2}(t) = t^2$ and $\eta(x,u) = xu$, where
$f : S_1 \times S_2 \to R^2$, $G_{f_i} : I_{f_i} \to R$ $(i = 1,2)$ and $\eta : S_1 \times S_2 \to R$.

Now, at $u = 0 \in S_1$, for all $x \in S_1$ and for fixed $v \in S_2$, we have,

$$\Big\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta^T(x,u)(G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)),$$
$$G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(u,v)) - \eta^T(x,u)(G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v))\Big\}$$
$$= (x^2 \sin^2 v, 0) \in -K$$

Hence, $f = (f_1, f_2)$ is $K - G_f$-incave function at $u = 0$ in S_1 with respect to η.

Definition 2.7 ([10])**.** Let D be a compact convex set in R^n. The support function of D is defined by

$$S(x|D) = \max\{x^T y : y \in D\}.$$

The subdifferentiable of $S(x|D)$ is given by

$$\partial S(x|D) = \{z \in D : z^T x = S(x|D)\}.$$

For any set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

3. **Duality model I**

Consider the following pair of Mond-Weir type nondifferentiable multiobjective symmetric dual programs:

Primal Problem (MP)

K-minimize

$$F = \{G_{f_1}(f_1(x,y)) + S(x|D_1) - y^T z_1, \ldots, G_{f_k}(f_k(x,y)) + S(x|D_k) - y^T z_k\}$$

subject to

$$-\Big[\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - z_i)\Big] \in C_2^*, \tag{3.1}$$

$$y^T\Big[\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - z_i)\Big] \geqq 0, \tag{3.2}$$

$$\lambda \in \text{int}K^*, x \in C_1, z_i \in E_i, i = 1,2,\ldots,k. \tag{3.3}$$

Dual Problem (MD)
K-maximize

$$G = \{G_{f_1}(f_1(u,v)) - S(v|E_1) + u^T w_1, \ldots, G_{f_k}(f_k(u,v)) - S(v|E_k) + u^T w_k\}$$

subject to

$$\Big[\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + w_i)\Big] \in C_1^*, \tag{3.4}$$

$$u^T\Big[\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + w_i)\Big] \leqq 0, \tag{3.5}$$

$$\lambda \in \text{int} K^*, v \in C_2, w_i \in D_i, i = 1, 2, \ldots, k, \tag{3.6}$$

where for $i = 1, 2, \ldots, k$,

(i) K^*, C_1^* and C_2^* are the positive polar cones of K, C_1 and C_2, respectively,
(ii) $f_i : S_1 \times S_2 \to R$, $G_f = (G_{f_1}, \ldots, G_{f_k}) : R \to R^k$ such that any of its component $G_{f_i} : I_{f_i}(S_1 \times S_2) \to R$ is a strictly increasing function on its domain are differentiable functions,
(iii) D_i and E_i are compact convex sets in R^n and R^m, respectively, and
(iv) $S(x|D_i)$ and $S(v|E_i)$ are the support functions of D_i and E_i, respectively.

Remark 3.1. If $D_i = \{0\}$, $E_i = \{0\}$, $f_i = f$, $i = 1, 2, \ldots, k$, and $G_f(t) = t$, then the model (MP) and (MD) reduce to the models discussed in Khurana [14].

Next, we will prove weak, strong and converse duality results between (MP) and (MD).

Theorem 3.2 (Weak duality)**.** *Let $(x, y, \lambda, z_1, z_2, \ldots, z_k)$ and $(u, v, \lambda, w_1, w_2, \ldots, w_k)$ be feasible for (MP) and (MD), respectively. If the following conditions hold:*

(I) *$\{(f_1(\cdot, v)), \ldots, (f_k(\cdot, v))\}$ and $\{(.)^T w_1, \ldots, (\cdot)^T w_k\}$ are $K - G_f$-invex and K-invex, respectively at u with respect to η_1 for fixed v,*
(II) *$\{(f_1(x, \cdot)), \ldots, (f_k(x, \cdot))\}$ and $\{(.)^T z_1, \ldots, (.)^T z_k\}$ are $K - G_f$-incave and K-invex, respectively at y with respect to η_2 for fixed x,*
(III) *$\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$,*

then

$$\begin{aligned}&\Big\{G_{f_1}(f_1(u,v)) - S(v|E_1) + u^T w_1, \ldots, G_{f_k}(f_k(u,v)) - S(v|E_k) + u^T w_k\Big\} \\ &-\Big\{G_{f_1}(f_1(x,y)) + S(x|D_1) - y^T z_1, \ldots, G_{f_k}(f_k(x,y)) + S(x|D_k) - y^T z_k\Big\} \\ &\notin K\backslash\{0\}\end{aligned} \tag{3.7}$$

Proof. The proof is given by contradiction. Let us suppose that (3.7) does not hold. Then

$$\Big\{G_{f_1}(f_1(u,v)) - S(v|E_1) + u^T w_1, \dots, G_{f_k}(f_k(u,v)) - S(v|E_k) + u^T w_k\Big\}$$
$$- \Big\{G_{f_1}(f_1(x,y)) + S(x|D_1) - y^T z_1, \dots, G_{f_k}(f_k(x,y)) + S(x|D_k) - y^T z_k\Big\} \in K\backslash\{0\}.$$

Now, from the fact that $\lambda \in \text{int}K^*$, it follows that

(3.8)
$$\sum_{i=1}^{k} \lambda_i \Big[(G_{f_i}(f_i(x,y)) + S(x|D_i) - y^T z_i) - (G_{f_i}(f_i(u,v)) - S(v|E_i) + u^T w_i)\Big] < 0.$$

Hypothesis (III) and (3.4) imply

$$[\eta_1(x,u) + u]^T \Big(\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + w_i)\Big) \geqq 0.$$

The above inequality together with (3.5) yield

$$\eta_1(x,u)^T \Big(\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + w_i)\Big) \geqq 0. \tag{3.9}$$

Since $\{(f_1(\cdot,v)), \dots, (f_k(\cdot,v))\}$ is $K - G_f$-invex at u with respect to η_1 for fixed v, therefore we have

$$\Big\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta_1^T(x,u)(G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)), \dots,$$
$$G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(u,v)) - \eta_1^T(x,u)(G'_{f_k}(f_k(u,v))\nabla_x f_k(u,v))\Big\} \in K,$$

which using $\lambda \in \text{int}K^*$ yields

(3.10)
$$\sum_{i=1}^{k} \lambda_i (G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v))) \geqq \eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)).$$

Also, by hypothesis (I), we obtain

$$\{x^T w_1 - u^T w_1 - \eta_1^T(x,u) w_1, \dots, x^T w_k - u^T w_k - \eta_1^T(x,u) w_k\} \in K.$$

It follows from $\lambda \in \text{int}K^*$ that

$$\sum_{i=1}^{k} \lambda_i (x^T w_i - u^T w_i) \geqq \eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i w_i. \tag{3.11}$$

Adding (3.10) and (3.11), we get

$$\sum_{i=1}^{k} \lambda_i (G_{f_i}(f_i(x,v)) + x^T w_i - G_{f_i}(f_i(u,v)) - u^T w_i)$$
$$\geqq \eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) + w_i).$$

Further, it follows from (3.9) that

$$\sum_{i=1}^{k} \lambda_i (G_{f_i}(f_i(x,v)) + x^T w_i - G_{f_i}(f_i(u,v)) - u^T w_i) \geqq 0. \tag{3.12}$$

Hypothesis (III) and (3.1) yield

$$[\eta_2(v,y) + y]^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) - z_i) \leqq 0,$$

which together with (3.2) give

$$\eta_2^T(v,y) \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) - z_i) \leqq 0. \tag{3.13}$$

Now, from hypothesis (II), we get

$$\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(x,y)) - \eta_2^T(v,y)(G'_{f_1}(f_1(x,y)) \nabla_y (f_1(x,y)), \ldots,$$
$$G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(x,y)) - \eta_2^T(v,y)(G'_{f_k}(f_k(x,y)) \nabla_y (f_k(x,y)))\} \in -K$$

and

$$(v^T z_1 - y^T z_1 - \eta_2^T(v,y) z_1, \ldots, v^T z_k - y^T z_k - \eta_2^T(v,y) z_k) \in K$$

It follows from $\lambda \in \text{int} K^*$ that

$$\sum_{i=1}^{k} \lambda_i (G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(x,y))) - \eta_2^T(v,y) \sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) \leqq 0 \tag{3.14}$$

and

$$\sum_{i=1}^{k} \lambda_i (v^T z_i - y^T z_i) - \eta_2^T(v,y) \sum_{i=1}^{k} \lambda_i z_i \geqq 0. \tag{3.15}$$

From (3.14) and (3.15), we get

$$\sum_{i=1}^{k} \lambda_i(G_{f_i}(f_i(x,y)) - y^T z_i - G_{f_i}(f_i(x,v)) + v^T z_i)$$

$$\geqq -\eta_2^T(v,y) \sum_{i=1}^{k} \lambda_i(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - z_i).$$

From (3.13), it follows that

$$\sum_{i=1}^{k} \lambda_i(G_{f_i}(f_i(x,y)) - y^T z_i - G_{f_i}(f_i(x,v)) + v^T z_i) \geqq 0. \tag{3.16}$$

Now, on adding (3.12) and (3.16), we obtain

$$\sum_{i=1}^{k} \lambda_i(G_{f_i}(f_i(x,y)) - y^T z_i + v^T z_i - G_{f_i}(f_i(u,v)) + x^T w_i - u^T w_i) \geqq 0.$$

Finally, using $x^T w_i \leqq S(x|D_i)$ and $v^T z_i \leqq S(v|E_i)$, $i = 1, 2, \ldots, k$, we get

$$\sum_{i=1}^{k} \lambda_i \Big[(G_{f_i}(f_i(x,y)) + S(x|D_i) - y^T z_i) - (G_{f_i}(f_i(u,v)) - S(v|E_i) + u^T w_i)\Big]$$
$$\geqq 0,$$

which contradicts (3.8). This ends the proof of the theorem. □

Now, we illustrate the above weak duality theorem by the following example:

3.1. **Numerical illustration.** Let $k = 2$, $n = m = 1$. Let also $S_1 = S_2 = R_+$, $C_1 = C_2 = R_+$ and

$$K = \{(x,y) \in R^2 : x \geqq 0, -x \leqq y \leqq x\}.$$

Then $C_1^* = C_2^* = R_+$ and $K^* = K$ and $-K = \{(x,y) \in R^2 : x \leqq 0, -x \geqq y \geqq x\}$. Let $f(x,y) = \{f_1(x,y), f_2(x,y)\}$, $f : S_1 \times S_2 \to R^2$, where,

$$f_1(x,y) = -\cos^2 x - \sin^2 y \text{ and } f_2(x,y) = -\sin^2 y.$$

Suppose $G_{f_1}(t) = t$, $G_{f_2}(t) = t^2$, where $G_{f_i} : I_{f_i} \to R$ $(i = 1, 2)$, and the functions $\eta_1, \eta_2 : S_1 \times S_2 \to R$ be given by $\eta_1(x,u) = x$, $\eta_2(v,y) = v$. Let $D_1 = [0,1]$, $D_2 = \{0\}$, $E_1 = \{0\}$ and $E_2 = [0,1]$. Then $S(x|D_1) = \dfrac{x + |x|}{2}$, $S(x|D_2) = S(v|E_1) = 0$ and $S(v|E_2) = \dfrac{v + |v|}{2}$. Under these settings, the primal (MP) and dual (MD) reduce to the following problems (EMP) and (EMD):

Primal Problem (EMP). minimize $F(x,y,\lambda,z_1,z_2)=\Big\{-\cos^2 x-\sin^2 y+\dfrac{x+|x|}{2},\sin^4 y-y^Tz_2\Big\}$

subject to

$$\Big[-2\lambda_1\sin y\cos y+\lambda_2(4\sin^3 y\cos y-z_2)\Big]\leqq 0,$$
$$y^T\Big[-2\lambda_1\sin y\cos y+\lambda_2(4\sin^3 y\cos y-z_2)\Big]\geqq 0,$$
$$|\lambda_2|<\lambda_1,\ x\geqq 0,\ z_2\in[0,1].$$

Dual Problem (EMD). maximize $G(u,v,\lambda,w_1,w_2)=\Big\{-\cos^2 u-\sin^2 v+u^Tw_1,\sin^4 v-\dfrac{v+|v|}{2}\Big\}$

subject to

$$\lambda_1(2\sin u\cos u+w_1)\geqq 0,$$
$$u^T\lambda_1(2\sin u\cos u+w_1)\leqq 0,$$
$$|\lambda_2|<\lambda_1,\ v\geqq 0,\ w_1\in[0,1].$$

Now, first we shall show that for the primal (EMP) and dual (EMD), the hypotheses of Theorem 3.2 hold.

(A.1) $\{(f_1(\cdot,v)),(f_2(\cdot,v))\}$ is $K-G_f$-invex at $u=0$ with respect to η_1 for fixed v for all $x\in S_1$, since

$$\Big\{G_{f_1}(f_1(x,v))-G_{f_1}(f_1(u,v))-\eta_1^T(x,u)(G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)),$$
$$G_{f_2}(f_2(x,v))-G_{f_2}(f_2(u,v))-\eta_1^T(x,u)(G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v))\Big\}$$
$$=(1-\cos^2 x,0)\in K$$

and $\{(.)^Tw_1,(.)^Tw_2\}$ is K-invex at $u=0$ with respect to η_1 for fixed v for all $x\in S_1$, since

$$\{x^Tw_1-u^Tw_1-\eta_1^T(x,u)w_1,x^Tw_2-u^Tw_2-\eta_1^T(x,u)w_2\}=(0,0)\in K,$$

(A.2) $\{(f_1(x,.)),(f_2(x,.))\}$ is $K-G_f$-incave at $y=0$ with respect to η_2 for fixed x for all $v\in S_2$, since

$$\Big\{G_{f_1}(f_1(x,v))-G_{f_1}(f_1(x,y))-\eta_2^T(v,y)(G'_{f_1}(f_1(x,y))\nabla_y f_1(x,y)),$$
$$G_{f_2}(f_2(x,v))-G_{f_2}(f_2(x,y))-\eta_2^T(v,y)(G'_{f_2}(f_2(x,y))\nabla_y f_2(x,y))\Big\}$$
$$=(-\sin^2 v,\sin^4 v)\in -K,$$

and $\{(.)^T z_1, (.)^T z_2\}$ is K-invex at $y = 0$ with respect to η_2 for fixed x for all $v \in S_2$, since

$$\{v^T z_1 - y^T z_1 - \eta_2^T(v,y)z_1, v^T z_2 - y^T z_2 - \eta_2^T(v,y)z_2\} = (0,0) \in K$$

(A.3) $\eta_1(x,u) + u = x + u \in C_1, \forall\ x,u \in S_1$, and $\eta_2(v,y) + y = v + y \in C_2, \forall\ v,y \in S_2$.

Any point $(x, 0, \lambda_1, \lambda_2, 0, z_2)$ such that $x \geqq 0$, $|\lambda_2| < \lambda_1$ and $0 \leqq z_2 \leqq 1$ are feasible to (EMP). Also, the points $(0, v, \lambda_1, \lambda_2, w_1, 0)$ such that $v \geqq 0$, $|\lambda_2| < \lambda_1$ and $0 \leqq w_1 \leqq 1$ satisfy the problem (EMD). Now, at these feasible points,

$$\begin{aligned} & G(u,v,\lambda,w_1,w_2) - F(x,y,\lambda,z_1,z_2) \\ &\quad = \ (-1 - \sin^2 v, \sin^4 v - \frac{v+|v|}{2}) - (-\cos^2 x + \frac{x+|x|}{2}, 0) \\ &\quad = \ (\cos^2 x - 1 - \sin^2 v - \frac{x+|x|}{2}, \sin^4 v - \frac{v+|v|}{2}) \\ &\quad = \ (\cos^2 x - 1 - \sin^2 v - x, \sin^4 v - v) \\ &\quad \notin \ K\backslash\{0\} \text{ (since } \cos^2 x - 1 - \sin^2 v - x \leqq 0,\ \forall\ x,\ v \geqq 0). \end{aligned}$$

In particular, the points $(x, y, \lambda_1, \lambda_2, z_1, z_2) = (\frac{\pi}{6}, 0, 1, \frac{1}{2}, 0, \frac{1}{4})$ and $(u, v, \lambda_1, \lambda_2, w_1, w_2) = (0, \frac{\pi}{4}, 1, \frac{1}{2}, \frac{1}{2}, 0)$ are feasible for the problems (EMP) and (EMD), respectively and also

$$G(u,v,\lambda,w_1,w_2) - F(x,y,\lambda,z_1,z_2) = (\frac{-2\pi - 9}{12}, \frac{1-\pi}{4}) \notin K\backslash\{0\}$$

Hence verified. □

Theorem 3.3 (Strong duality)**.** *Let* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k)$ *be an efficient solution of (MP). Fix* $\lambda = \bar{\lambda}$ *in* (MD). *Let*

(I) $\{G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i\}_{i=1}^k$ *be linearly independent;*

(II) *the matrix*
$$\sum_{i=1}^k \bar{\lambda}_i \{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\nabla_y f_i(\bar{x},\bar{y})^T \ + \ G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\}$$
be positive or negative definite;

(III) $R_+^k \subseteq K$.

Then there exists $\bar{w}_i \in D_i$, $i = 1, 2, \ldots, k$, *such that* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$ *is a feasible solution for* (MD) *and the objective values of* (MD) *and* (MD) *are equal. Moreover, if the hypotheses in Theorem 3.2 hold for all feasible solutions of* (MD) *and* (MD), *then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$ *is an efficient solution for* (MD).

Proof. Given that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k)$ is an efficient solution of (MP). Following the necessary optimality conditions, given by Fritz John [4], there exist

$\alpha \in K^*$, $\beta \in C_2$ and $\mu \in R_+$ such that

$$\begin{aligned} &\Big[\sum_{i=1}^{k} \alpha_i (G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}) + \xi_i) \\ &\quad + (\beta - \mu\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i (G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})\nabla_y f_i(\bar{x},\bar{y})^T \\ &\quad + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{xy} f_i(\bar{x},\bar{y}))\Big]^T (x - \bar{x}) \geqq 0 \text{ for all } x \in C_1, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\sum_{i=1}^{k} (\alpha_i - \mu\bar{\lambda}_i)(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i) \\ &\quad + (\beta - \mu\bar{y})^T \Big[\sum_{i=1}^{k} \bar{\lambda}_i (G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\nabla_y f_i(\bar{x},\bar{y})^T \\ &\quad + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y}))\Big] = 0, \end{aligned} \tag{3.18}$$

$$(\beta - \mu\bar{y})^T \Big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i\Big](\lambda_i - \bar{\lambda}_i) \geqq 0 \quad \text{for all } \lambda \in \text{int}K^*,\ i = 1,2,\ldots,k, \tag{3.19}$$

$$\beta^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i) = 0, \tag{3.20}$$

$$\mu\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i) = 0,$$

$$\alpha_i\bar{y} + \bar{\lambda}_i\beta - \bar{\lambda}_i\mu\bar{y} \in N_{E_i}(\bar{z}_i),\ i = 1,2,\ldots,k, \tag{3.21}$$

$$\xi_i^T\bar{x} = S(\bar{x}|D_i),\ i = 1,2,\ldots,k,$$

$$\xi_i \in D_i,\ i = 1,2,\ldots,k,\ (\alpha,\beta,\mu) \neq 0. \tag{3.22}$$

Inequality (3.19) can be rewritten as

$$(\beta - \mu\bar{y})^T \Big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i\Big] = 0,\ i = 1,2,\ldots,k. \tag{3.23}$$

Post-multiplying the inequality (3.18) by $(\beta - \mu\bar{y})$ and using (3.23), we have

$$\begin{aligned} &(\beta - \mu\bar{y})^T \Big[\sum_{i=1}^{k} \bar{\lambda}_i (G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T \\ &\quad + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y}))\Big](\beta - \mu\bar{y}) = 0. \end{aligned} \tag{3.24}$$

Using hypothesis (II) in (3.24), we get

$$\beta = \mu \bar{y}. \tag{3.25}$$

Substituting $\beta = \mu \bar{y}$ in (3.18), we have

$$\sum_{i=1}^{k} (\alpha_i - \mu \bar{\lambda}_i)(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i) = 0.$$

From hypothesis (I), we obtain

$$\alpha_i = \mu \bar{\lambda}_i, \ i = 1, 2, \ldots, k. \tag{3.26}$$

We now claim that $\alpha_i \neq 0$ for all $i = 1, 2, \ldots, k$. If possible, $\alpha_{t_0} = 0$ for some $i = t_0$, then $\mu \bar{\lambda}_{t_0} = 0$. Since $\bar{\lambda} \in \text{int} K^* \subseteq \text{int} R^k_+$ (by hypothesis (III)), therefore $\bar{\lambda} > 0$ and thus $\mu = 0$. This together with (3.25) yields $\beta = 0$. Therefore, $(\alpha, \beta, \mu) = 0$, a contradiction to (3.22). Hence $\alpha_i \neq 0$, for all i. Also, from the fact that $\alpha \in K^*$ and $K^* \subseteq R^k_+$, it follows that $\alpha_i > 0$, $i = 1, 2, \ldots, k$. Hence, the relation (3.26) implies $\mu > 0$. Now, using (3.25) in (3.17), we obtain

$$\Big[\sum_{i=1}^{k} \alpha_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y}) + \xi_i)\Big]^T (x - \bar{x}) \geqq 0, \ \text{for all } x \in C_1. \tag{3.27}$$

Substituting (3.26) in (3.27) and the fact that $\mu > 0$ give

$$\Big[\sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y}) + \xi_i)\Big]^T (x - \bar{x}) \geqq 0, \ \text{for all } x \in C_1. \tag{3.28}$$

Let $x \in C_1$. Then $\bar{x} + x \in C_1$ as C_1 is a closed convex cone and so from (3.28), it yields

$$x^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y}) + \xi_i) \geqq 0,$$

which implies

$$\sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y}) + \xi_i) \in C_1^*.$$

Now, taking $x = 0$ and $x = 2\bar{x}$ simultaneously in (3.28), we have

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y}) + \xi_i) = 0. \tag{3.29}$$

Also, from the expression (3.25), we get $\bar{y} = \dfrac{\beta}{\mu} \in C_2$ as $\mu > 0$. Again, setting $\xi_i = \bar{w}_i$, $i = 1, 2, \ldots, k$, $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$ satisfies all the constraints of the dual problem and hence $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$ is a feasible solution of the (MD). Further, the expressions (3.21), (3.25) and $\alpha_i > 0$, yield

$$\bar{y} \in N_{E_i}(\bar{z}_i).$$

Again since $E_i,\ i=1,2,\dots,k$ are compact convex sets in R^n, $\bar{y}^T\bar{z}_i = S(\bar{y}|E_i)$. Rewriting the expression (3.29), we obtain

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})) = -\bar{x}^T\xi_i = -S(\bar{x}|D_i). \tag{3.30}$$

Further, from (3.20), (3.25) and $\mu > 0$, we have

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i) = 0,$$

which gives

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})) = \bar{y}^T\bar{z}_i = S(\bar{y}|E_i). \tag{3.31}$$

Therefore, (3.30) and (3.31) together give

$$\Big\{G_{f_1}(f_1(\bar{x},\bar{y})) + S(\bar{x}|D_1) - \bar{y}^T\bar{z}_1, \dots, G_{f_k}(f_k(\bar{x},\bar{y})) + S(\bar{x}|D_k) - \bar{y}^T\bar{z}_k\Big\}$$
$$= \Big\{G_{f_1}(f_1(\bar{x},\bar{y})) - S(\bar{y}|E_1) + \bar{x}^T\xi_1, \dots, G_{f_k}(f_k(\bar{x},\bar{y})) - S(\bar{y}|E_k) + \bar{x}^T\xi_k\Big\},$$

that is, the two objective values coincide.

Next, we will show that $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\dots,\bar{w}_k)$ is an efficient solution of (MD). On the contrary, assume that $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\dots,\bar{w}_k)$ is not an efficient solution of (MD). Then there exists $(u^*,v^*,\bar{\lambda},w_1^*,w_2^*,\dots,w_k^*)$, a feasible solution for (MP) such that

$$\Big\{G_{f_1}(f_1(u^*,v^*)) - S(v^*|E_1) + {u^*}^T w_1^*, \dots, G_{f_k}(f_k(u^*,v^*)) - S(v^*|E_k) + {u^*}^T w_k^*\Big\}$$
$$- \Big\{G_{f_1}(f_1(\bar{x},\bar{y})) - S(\bar{y}|E_1) + \bar{x}^T\bar{w}_1, \dots, G_{f_k}(f_k(\bar{x},\bar{y})) - S(\bar{y}|E_k) + \bar{x}^T\bar{w}_k\Big\} \in K\backslash\{0\}$$

Finally, using $\bar{x}^T\bar{w}_i = S(\bar{x}|D_i)$ and $\bar{y}^T\bar{z}_i = S(\bar{y}|E_i)$, $i=1,2,\dots,k$, in the above expression, we have

$$\Big\{G_{f_1}(f_1(u^*,v^*)) - S(v^*|E_1) + {u^*}^T w_1^*, \dots, G_{f_k}(f_k(u^*,v^*)) - S(v^*|E_k) + {u^*}^T w_k^*\Big\}$$
$$- \Big\{G_{f_1}(f_1(\bar{x},\bar{y})) + S(\bar{x}|D_1) - \bar{y}^T\bar{z}_1, \dots, G_{f_k}(f_k(\bar{x},\bar{y})) + S(\bar{x}|D_k) - \bar{y}^T\bar{z}_k\Big\}$$
$$\in K\backslash\{0\}$$

which contradicts Theorem 3.2. Hence $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\dots,\bar{w}_k)$ is the efficient solution of (MD). This ends the proof. □

Theorem 3.4 (Converse duality)**.** *Let* $(\bar{u},\bar{v},\bar{\lambda},\bar{w}_1,\bar{w}_2,\dots,\bar{w}_k)$ *be an efficient solution of* (MD)*. Fix* $\lambda = \bar{\lambda}$ *in (MP). Let*

(I) $\{G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v}) - \bar{w}_i\}_{i=1}^k$ *be linearly independent;*

(II) *the matrix*

$$\sum_{i=1}^{k} \bar{\lambda}_i \{G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})\}$$

be positive or negative definite;

(III) $R^k_+ \subseteq K$.

Then there exists $\bar{z}_i \in E_i$, $i = 1, 2, \ldots, k$, *such that* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k)$ *is a feasible solution for* (MP) *and the objective values of* (MP) *and* (MD) *are equal. Moreover, if the hypotheses in Theorem 3.3 hold for all feasible solutions of* (MP) *and* (MD), *then* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k)$ *is an efficient solution for* (MP).

Proof. The proof follows on the lines of Theorem 3.3. □

4. Duality model II

Consider the following pair of Wolfe type nondifferentiable multiobjective symmetric dual programs:

Primal problem (WP). K-minimize $F = \{G_{f_1}(f_1(x,y)) + S(x|D)e_1 -$

$$y^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))e_1, \ldots, G_{f_k}(f_k(x,y)) + S(x|D)e_k -$$

$$y^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))e_k\}$$

subject to

$$-\Big[\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)) - z\Big] \in C_2^*, \tag{4.1}$$

$$\lambda^T e = 1, \tag{4.2}$$

$$\lambda \in \text{int}K^*,\ x \in C_1,\ z \in E. \tag{4.3}$$

Dual Problem (WD). K-maximize $G = \{G_{f_1}(f_1(u,v)) - S(v|E)e_1$

$$- u^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))e_1, \ldots, G_{f_k}(f_k(u,v)) - S(v|E)e_k$$

$$-u^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))e_k\}$$

subject to

$$\Big[\sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)) + w\Big] \in C_1^*, \tag{4.4}$$

$$\lambda^T e = 1,$$

$$\lambda \in \text{int}K^*, \; v \in C_2, \; w \in D,$$

where $e = (e_1, e_2, \ldots, e_k) \in \text{int}K$ is fixed and for $i = 1, 2, \ldots, k$,

(i) K^*, C_1^* and C_2^* are the positive polar cones of K, C_1 and C_2, respectively,
(ii) $f_i : S_1 \times S_2 \to R$, $G_f = (G_{f_1}, \ldots, G_{f_k}) : R \to R^k$ such that any of its component $G_{f_i} : I_{f_i}(S_1 \times S_2) \to R$ is a strictly increasing function on its domain are differentiable functions,
(iii) D and E are compact convex sets in R^n and R^m, respectively, and
(iv) $S(x|D)$ and $S(v|E)$ are the support functions of D and E, respectively.

Remark 4.1. If $D = \{0\}$, $E = \{0\}$, $f_i = f$, $i = 1, 2, \ldots, k$, and $G_f(t) = t$, then (WP) and (WD) become the models discussed in Suneja et al. [18].

Next, we will prove weak, strong and converse duality results between (WP) and (WD).

Theorem 4.2 (Weak duality). *Let (x, y, λ, z) and (u, v, λ, w) be feasible for* (WP) *and* (WP)*, respectively. If the following conditions hold:*

(I) $\{(f_1(\cdot, v)), \ldots, (f_k(\cdot, v))\}$ *and* $\{(\cdot)^T we_1, \ldots, (.)^T we_k\}$ *are* $K-G_f$*-invex and* K*-invex at* u *with respect to* η_1 *for fixed* v*,*
(II) $\{(f_1(x, .)), \ldots, (f_k(x, .))\}$ *and* $\{(.)^T ze_1, \ldots, (.)^T ze_k\}$ $K - G_f$*-incave and* K*-invex at* y *with respect to* η_2 *for fixed* x *and*
(III) $\eta_1(x, u) + u \in C_1$ *and* $\eta_2(v, y) + y \in C_2$*,*

then

$$\begin{aligned}
(4.5) \quad & \Big\{G_{f_1}(f_1(u, v)) - S(v|E)e_1 - u^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u, v)) \nabla_x f_i(u, v)) e_1, \ldots, \\
& G_{f_k}(f_k(u, v)) - S(v|E)e_k - u^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(u, v)) \nabla_x f_i(u, v)) e_k \Big\} \\
& - \Big\{G_{f_1}(f_1(x, y)) + S(x|D)e_1 - y^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x, y)) \nabla_y f_i(x, y)) e_1, \ldots, \\
& G_{f_k}(f_k(x, y)) + S(x|D)e_k - y^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(x, y)) \nabla_y f_i(x, y)) e_k \Big\} \notin K \backslash \{0\}.
\end{aligned}$$

Proof. Contrary to (4.5), suppose that

$$\Big\{G_{f_1}(f_1(u,v)) - S(v|E)e_1 - u^T\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))e_1,\dots,$$
$$G_{f_k}(f_k(u,v)) - S(v|E)e_k - u^T\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))e_k\Big\} -$$
$$\Big\{G_{f_1}(f_1(x,y)) + S(x|D)e_1 - y^T\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))e_1,\dots,$$
$$G_{f_k}(f_k(x,y)) + S(x|D)e_k - y^T\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))e_k\Big\} \in K\backslash\{0\}.$$

Now, (4.2), (4.3) and the above expression imply

(4.6)
$$\Big[\sum_{i=1}^{k}\lambda_i G_{f_i}(f_i(x,y)) + S(x|D) - y^T\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))\Big]$$
$$-\Big[\sum_{i=1}^{k}\lambda_i G_{f_i}(f_i(u,v)) - S(v|E) - u^T\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))\Big] < 0.$$

Since $\{(f_1(\cdot,v)),\dots,(f_k(\cdot,v))\}$ is $K-G_f$-invex at u with respect to η_1 for fixed v, we have

$$\Big\{G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta_1^T(x,u)(G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)),\dots,$$
$$G_{f_k}(f_k(x,v)) - G_{f_k}(f_k(u,v)) - \eta_1^T(x,u)(G'_{f_k}(f_k(u,v))\nabla_x f_k(u,v))\Big\} \in K.$$

Using $\lambda \in \mathrm{int}K^*$, it follows that

$$\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v))\Big] - \eta_1^T(x,u)\Big[\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))\Big] \geqq 0. \tag{4.7}$$

Since $\{(.)^T we_1,\dots,(.)^T we_k\}$ is K-invex at u with respect to η_1 for fixed v, we have

$$\Big\{x^T we_1 - u^T we_1 - \eta_1^T(x,u)we_1,\dots,x^T we_k - u^T we_k - \eta_1^T(x,u)we_k\Big\} \in K.$$

Using (4.2) and the fact that $\lambda \in \mathrm{int}K^*$, we get

$$x^T w - u^T w \geqq \eta_1^T(x,u)w. \tag{4.8}$$

Adding (4.7) and (4.8), we have

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(x,v))-G_{f_i}(f_i(u,v))\Big]+x^Tw\\ -u^Tw\geqq \eta_1^T(x,u)\Big[\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))+w\Big].\end{aligned}\tag{4.9}$$

Further, hypothesis (III) and (4.4) give

$$[\eta_1(x,u)+u]^T\Big[\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))+w\Big]\geqq 0,$$

which along with (4.9) yields

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(x,v))-G_{f_i}(f_i(u,v))\Big]+x^Tw-u^Tw\\ +u^T\Big[\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v))+w\Big]\geqq 0.\end{aligned}\tag{4.10}$$

Similarly, from hypotheses (II), (III), constraints (4.1)-(4.2) and $\lambda\in \text{int}K^*$, we get

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\Big[G_{f_i}(f_i(x,y))-G_{f_i}(f_i(x,v))\Big]+v^Tz-y^Tz\\ -y^T\Big[\sum_{i=1}^{k}\lambda_i(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y))-z\Big]\geqq 0.\end{aligned}\tag{4.11}$$

Further, adding (4.10) and (4.11), we get

$$\begin{aligned}&\Big\{\sum_{i=1}^{k}\lambda_i(G_{f_i}(f_i(x,y))+x^Tw-y^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)\Big]\Big\}\\ &-\Big\{\sum_{i=1}^{k}\lambda_i(G_{f_i}(f_i(u,v))-v^Tz-u^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)\Big]\Big\}\geqq 0.\end{aligned}$$

Finally, using the fact that $x^Tw\leqq S(x|D)$ and $v^Tz\leqq S(v|E)$, it follows that

$$\begin{aligned}&\Big\{\sum_{i=1}^{k}\lambda_i(G_{f_i}(f_i(x,y))+S(x|D)-y^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)\Big]\Big\}\\ &-\Big\{\sum_{i=1}^{k}\lambda_i(G_{f_i}(f_i(u,v))-S(v|E)-u^T\sum_{i=1}^{k}\lambda_i\Big[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)\Big]\Big\}\geqq 0,\end{aligned}$$

which contradicts (4.6). Hence the result. □

4.1. **Numerical illustration.** Let $k=2$, $n=m=1$, $S_1=S_2=R_+$, $C_1=C_2=R_+$. Then $C_1^*=C_2^*=R_+$. Let also $f: S_1\times S_2\to R^2$, $f(x,y)=\{f_1(x,y),f_2(x,y)\}$, where

$$f_1(x,y)=\cos^2 y \text{ and } f_2(x,y)=\sin^2 x+\cos^2 y.$$

Suppose $G=(G_{f_1},G_{f_2}): R\to R^2$ be defined as:

$$G_{f_1}(t)=t,\ G_{f_2}(t)=2t.$$

Consider $(\eta_1,\eta_2): S_1\times S_2\to R$ as:

$$\eta_1(x,u)=x,\ \eta_2(v,y)=v.$$

Let $K=\{(x,y)\in R^2 : x\geqq 0, y\geqq -x\}$. Then $-K=\{(x,y)\in R^2 : x\leqq 0, y\leqq -x\}$ and $K^*=\{(x,y)\in R_+^2 : x\geqq y\}$. Let $(e_1,e_2)=(1,1)\in \text{int}K$. Let $D=[0,1]$ and $E=\{0\}$. Then $S(x|D)=\dfrac{x+|x|}{2}$ and $S(v|E)=0$.

Under the above defined expressions, the primal-dual pair (WP) and (WD) reduce to the following problems (EWP) and (EWD):

Primal Problem (EWP). Minimize $L(x,y,\lambda,z)=\Big\{\cos^2 y+\dfrac{x+|x|}{2}+y^T(2\lambda_1\sin y\cos y+4\lambda_2\sin y\cos y),$

$$2(\sin^2 x+\cos^2 y)+\frac{x+|x|}{2}+y^T(2\lambda_1\sin y\cos y+4\lambda_2\sin y\cos y)\Big\},$$

subject to

$$\Big[-2\lambda_1\sin y\cos y-4\lambda_2\sin y\cos y\Big]\leqq 0,$$
$$\lambda_1+\lambda_2=1,$$
$$\lambda_1>0,\ \lambda_2>0,\ \lambda_1-\lambda_2>0,\ x\geqq 0.$$

Dual Problem (EWD). Maximize $M(u,v,\lambda,w)=\Big\{\cos^2 v-u^T(4\lambda_2\sin u\cos u), 2(\sin^2 u+\cos^2 v)-u^T(4\lambda_2\sin u\cos u)\Big\}$
subject to

$$4\lambda_2\sin u\cos u+w\geqq 0,$$
$$\lambda_1+\lambda_2=1,$$
$$\lambda_1>0,\ \lambda_2>0,\ \lambda_1-\lambda_2>0,\ v\geqq 0,\ w\in[0,1].$$

First, we shall show that for the primal (EWP) and dual (EWD), the hypotheses of Theorem 4.2 hold.

(B.1) $\{(f_1(\cdot,v)),(f_2(\cdot,v))\}$ is $K-G_f$-invex at $u=0$ with respect to η_1 for fixed v for all $x\in S_1$, since

$$\Big\{G_{f_1}(f_1(x,v))-G_{f_1}(f_1(u,v))-\eta_1^T(x,u)(G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)),$$
$$G_{f_2}(f_2(x,v))-G_{f_2}(f_2(u,v))-\eta_1^T(x,u)(G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v))\Big\}$$
$$=(0,2\sin^2 x)\in K,$$

and $\{(.)^T we_1,(.)^T we_2\}$ is K-invex at $u=0$ with respect to η_1 for fixed v for all $x\in S_1$, since
$(x^Tw-u^Tw-\eta_1^T(x,u)w, x^Tw-u^Tw-\eta_1^T(x,u)w)=(0,0)\in K.$

(B.2) $\{(f_1(x,.)),(f_2(x,.))\}$ is $K-G_f$-incave at $y=0$ with respect to η_2 for fixed x for all $v\in S_2$, since

$$\Big\{G_{f_1}(f_1(x,v))-G_{f_1}(f_1(x,y))-\eta_2^T(v,y)(G'_{f_1}(f_1(x,y))\nabla_y f_1(x,y)),G_{f_2}(f_2(x,v))$$
$$-G_{f_2}(f_2(x,y))-\eta_2^T(v,y)(G'_{f_2}(f_2(x,y))\nabla_y f_2(x,y))\Big\}$$
$$=(\cos^2 v-1,2(\cos^2 v-1))\in -K,$$

and $\{(.)^T ze_1,(.)^T ze_2\}$ is K-invex at $y=0$ with respect to η_2 for fixed x for all $v\in S_2$, since

$$(v^Tz-y^Tz-\eta_2^T(v,y)z, v^Tz-y^Tz-\eta_2^T(v,y)z)=(0,0)\in K.$$

(B.3) $\eta_1(x,u)+u=x+u\in C_1,\forall\, x,u\in S_1$ and $\eta_2(v,y)+y=v+y\in C_2,\forall\, v,y\in S_2$. The points $(x,0,\lambda_1,\lambda_2,z)$ s.t. $x\geqq 0$, $\lambda_1+\lambda_2=1$ with $\lambda_1>0,\ \lambda_2>0$ and $\lambda_1-\lambda_2>0$ are feasible to (EWP). Also, the points $(0,v,\lambda_1,\lambda_2,w)$ s.t. $v\geqq 0$, $\lambda_1+\lambda_2=1$ with $\lambda_1>0,\ \lambda_2>0$ and $\lambda_1-\lambda_2>0$ satisfy (EWD).
Now, at these feasible points,

$$\begin{aligned}M(u,v,\lambda,w)-L(x,y,\lambda,z)&=(\cos^2 v,2\cos^2 v)-(1+\frac{x+|x|}{2},2(\sin^2 x+1)+\frac{x+|x|}{2})\\&=(\cos^2 v,2\cos^2 v)-(1+x,2(\sin^2 x+1)+x)\\&=(\cos^2 v-1-x,2\cos^2 v-2(\sin^2 x+1)-x)\\&\notin K\backslash\{0\}(\text{since } \cos^2 v-1-x\leqq 0,\ \forall\ x,\ v\geqq 0).\end{aligned}$$

In particular, the points $(x,y,\lambda_1,\lambda_2,z)=(\frac{\pi}{6},0,\frac{3}{4},\frac{1}{4},0)$ and $(u,v,\lambda_1,\lambda_2,w)=(0,\frac{\pi}{3},\frac{3}{4},\frac{1}{4},1)$ are feasible for the problems (EWP) and (EWD), respectively and also

$$M(u,v,\lambda,w)-L(x,y,\lambda,z)=\Big(\frac{-9-2\pi}{12},\frac{-12-\pi}{6}\Big)\notin K\backslash\{0\}$$

Hence verified. □

Theorem 4.3 (Strong duality)**.** *Let* $(\bar{x},\bar{y},\bar{\lambda},\bar{z})$ *be an efficient solution of* (WP). *Fix* $\lambda=\bar{\lambda}$ *in* (WD). *Let*

(I) *the vectors* $\{G'_{f_1}(f_1(\bar{x},\bar{y}))\nabla_y f_1(\bar{x},\bar{y}),\ldots,G'_{f_k}(f_k(\bar{x},\bar{y}))\nabla_y f_k(\bar{x},\bar{y})\}$ *be linearly independent;*

(II) *the matrix*
$$\sum_{i=1}^{k}\bar{\lambda}_i\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\nabla_y f_i(\bar{x},\bar{y})^T \;+\; G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\}$$
be positive or negative definite;

Then, there exists $\bar{w}\in D$, *such that* $(\bar{x},\bar{y},\bar{\lambda},\bar{w})$ *is a feasible solution for* (WD) *and the objective values of* (WD) *and* (WD) *are equal. Furthermore, if the hypotheses in Theorem 4.2 hold for all feasible solutions of* (WD) *and* (WD), *then* $(\bar{x},\bar{y},\bar{\lambda},\bar{w})$ *is an efficient solution for* (WD).

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{z})$ is an efficient solution of (WD). Hence, according to the Fritz John optimality condition [4], there exist $\alpha\in K^*$, $\beta\in C_2$ and $\eta\in R$ such that

$$\begin{aligned}&\Big[\sum_{i=1}^{k}\alpha_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))+(\alpha^Te)\bar{\gamma}\\&\quad+(\beta-(\alpha^Te)\bar{y})\Big\{\sum_{i=1}^{k}\bar{\lambda}_i(G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})\nabla_y f_i(\bar{x},\bar{y})^T\\&\quad+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{xy} f_i(\bar{x},\bar{y}))\Big\}\Big]^T(x-\bar{x})\geqq 0,\ \text{for all } x\in C_1,\end{aligned}\tag{4.12}$$

$$\begin{aligned}&\sum_{i=1}^{k}(\alpha_i-(\alpha^Te)\bar{\lambda}_i)(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))\\&\quad+(\beta-(\alpha^Te)\bar{y})^T\Big[\sum_{i=1}^{k}\bar{\lambda}_i\Big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\nabla_y f_i(\bar{x},\bar{y})^T\\&\quad+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\Big\}\Big]=0,\end{aligned}\tag{4.13}$$

$$\Big[[(\beta-(\alpha^Te)\bar{y})^TG'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})]+\eta e_i\Big](\lambda_i-\bar{\lambda}_i)\geqq 0,\tag{4.14}$$
$$\text{for all } \lambda\in \text{int}K^*,\ i=1,2,\ldots,k,$$

$$\beta^T\Big[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))-\bar{z}\Big]=0,\tag{4.15}$$

$$\eta^T(\bar{\lambda}^Te-1)=0,$$

$$\beta\in N_E(\bar{z}),\tag{4.16}$$

$$\bar{\gamma}\in D,\ \bar{\gamma}^T\bar{x}=S(\bar{x}|D),$$

$$(\alpha, \beta, \eta) \neq 0. \tag{4.17}$$

Inequality (4.14) can be re-written as

$$[(\beta - (\alpha^T e)\bar{y})^T G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})] + \eta e_i = 0. \tag{4.18}$$

Multiplying (4.18) by $(\alpha^i - (\alpha^T e)\bar{\lambda}^i)$, $i = 1, 2, \dots, k$, summing for all i, and using $\lambda^T e = 1$, we obtain

$$(\beta - (\alpha^T e)\bar{y})^T \Big[\sum_{i=1}^{k} (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}))\Big](\alpha^i - (\alpha^T e)\bar{\lambda}^i) = 0. \tag{4.19}$$

Again, multiplying (4.13) by $(\beta - (\alpha^T e)\bar{y})^T$ and using (4.19), we get

$$(\beta - (\alpha^T e)\bar{y})^T \Big[\sum_{i=1}^{k} \bar{\lambda}_i \Big\{ G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})\nabla_y f_i(\bar{x}, \bar{y})^T$$
$$+ G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y}) \Big\}\Big](\beta - (\alpha^T e)\bar{y}) = 0.$$

Applying hypothesis (II), we have

$$\beta = (\alpha^T e)\bar{y}. \tag{4.20}$$

Using (4.20) in (4.18), we get $\eta = 0$, as $e = (e_1, e_2, \dots, e_k) \in \text{int}K$ implies $e \neq 0$. Now, (4.13) and (4.20) together gives

$$\sum_{i=1}^{k} (\alpha_i - (\alpha^T e)\bar{\lambda}_i)(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})) = 0.$$

From hypothesis (I), we have

$$\alpha_i = (\alpha^T e)\bar{\lambda}_i, \; i = 1, 2, \dots, k. \tag{4.21}$$

If $\alpha = 0$, then $\alpha^T e = 0$ and hence (4.20) gives $\beta = 0$, which is a contradiction to $(\alpha, \beta, \eta) \neq 0$. Thus $\alpha^T e > 0$ as $0 \neq \alpha \in K^*$ and $e \in \text{int}K$. Hence $\bar{y} = \dfrac{\beta}{\alpha^T e} \in C_2$. Substituting (4.20), (4.21) and using the fact that $\alpha^T e > 0$ in (4.12), we obtain

$$\Big[\sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y})) + \bar{\gamma}\Big]^T (x - \bar{x}) \geqq 0, \text{ for all } x \in C_1. \tag{4.22}$$

Since C_1 is a closed convex cone, therefore x, $\bar{x} \in C_1$ implies $x + \bar{x} \in C_1$ and hence from (4.22), we have

$$x^T \Big[\sum_{i=1}^{k} \bar{\lambda}_i (G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y})) + \bar{\gamma}\Big] \geqq 0, \text{ for all } x \in C_1,$$

which implies

$$\Big[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))+\bar{\gamma}\Big]\in C_1^*.$$

Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{w}=\bar{\gamma})$ is a feasible solution for (WD). Considering $x=0$ and $x=2\bar{x}$, in (4.22), yields

$$\bar{x}^T\Big[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))+\bar{\gamma}\Big]=0,$$

which further reduces to

$$\bar{x}^T\Big[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))\Big]=-\bar{x}^T\bar{\gamma}=-S(\bar{x}|D). \tag{4.23}$$

From (4.16) and (4.20) we have, $(\alpha^T e)\bar{y}\in N_E(\bar{z})$. Since $\alpha^T e>0$, $\bar{y}\in N_E(\bar{z})$.

Now, expressions (4.15), (4.20) and the fact that $\alpha^T e>0$ yield

$$\bar{y}^T\Big[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))-\bar{z}\Big]=0,$$

which implies

$$\bar{y}^T\Big[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))\Big]=\bar{y}^T\bar{z}=S(\bar{y}|E). \tag{4.24}$$

Using (4.23) and (4.24), we obtain

$$\begin{aligned}&\Big\{G_{f_1}(f_1(\bar{x},\bar{y}))+S(\bar{x}|D)e_1-\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))e_1,\ldots,\\ &G_{f_k}(f_k(\bar{x},\bar{y}))+S(\bar{x}|D)e_k-\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))e_k\Big\}\\ &=\Big\{G_{f_1}(f_1(\bar{x},\bar{y}))-S(\bar{y}|E)e_1-\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))e_1,\ldots,\\ &G_{f_k}(f_k(\bar{x},\bar{y}))-S(\bar{y}|E)e_k-\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))e_k\Big\}.\end{aligned}$$

Hence, the two objective functions have equal values. Now, let $(\bar{x},\bar{y},\bar{\lambda},\bar{w})$ be not an efficient solution of (WD), then there exists $(\hat{x},\hat{y},\bar{\lambda},\hat{w})$ which is feasible for (WD) such that

$$\Big\{G_{f_1}(f_1(\hat{x},\hat{y})) - S(\hat{y}|E)e_1 - \hat{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\hat{x},\hat{y}))\nabla_x f_i(\hat{x},\hat{y}))e_1,\ldots,$$
$$G_{f_k}(f_k(\hat{x},\hat{y})) - S(\hat{y}|E)e_k - \hat{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\hat{x},\hat{y}))\nabla_x f_i(\hat{x},\hat{y}))e_k\Big\} -$$
$$\Big\{G_{f_1}(f_1(\bar{x},\bar{y})) - S(\bar{y}|E)e_1 - \bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))e_1,\ldots,$$
$$G_{f_k}(f_k(\bar{x},\bar{y})) - S(\bar{y}|E)e_k - \bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}))e_k\Big\} \in K\backslash\{0\},$$

which from (4.23) and (4.24) yield

$$\Big\{G_{f_1}(f_1(\hat{x},\hat{y})) - S(\hat{y}|E)e_1 - \hat{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\hat{x},\hat{y}))\nabla_x f_i(\hat{x},\hat{y}))e_1,\ldots,$$
$$G_{f_k}(f_k(\hat{x},\hat{y})) - S(\hat{y}|E)e_k - \hat{x}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\hat{x},\hat{y}))\nabla_x f_i(\hat{x},\hat{y}))e_k\Big\} -$$
$$\Big\{G_{f_1}(f_1(\bar{x},\bar{y})) + S(\bar{x}|D)e_1 - \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))e_1,\ldots,$$
$$G_{f_k}(f_k(\bar{x},\bar{y})) + S(\bar{x}|D)e_k - \bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}))e_k\Big\} \in K\backslash\{0\},$$

which is a contradiction to Theorem 4.2. Hence, $(\bar{x},\bar{y},\bar{\lambda},\bar{w})$ is the efficient solution of (WD). □

Theorem 4.4 (Converse duality). *Let $(\bar{u},\bar{v},\bar{\lambda},\bar{w})$ be an efficient solution of* (WD)*. Fix $\lambda=\bar{\lambda}$ in (WP). Let*

(I) *the vectors $\{G'_{f_1}(f_1(\bar{u},\bar{v}))\nabla_x f_1(\bar{u},\bar{v}),\ldots,G'_{f_k}(f_k(\bar{u},\bar{v}))\nabla_x f_k(\bar{u},\bar{v})\}$ be linearly independent;*

(II) *the matrix*
$$\sum_{i=1}^{k}\bar{\lambda}_i\{G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})\nabla_x f_i(\bar{u},\bar{v})^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})\}$$
be positive or negative definite;

Then there exists $\bar{z}\in E$, such that $(\bar{u},\bar{v},\bar{\lambda},\bar{z})$ is a feasible solution for (WP) and the objective values of (WP) and (WD) are equal. Futhermore, if the hypotheses in Theorem 4.2 hold for all feasible solutions of (WP) and (WD), then $(\bar{u},\bar{v},\bar{\lambda},\bar{z})$ is an efficient solution for (WP).

Proof. The proof follows on the lines of Theorem 4.2. □

5. **Conclusions**

In this paper, we have considered the G_f-invex functions over cones and examples which justify the definitions have been illustrated. Two types of dual models-Mond-Weir and Wolfe type multiobjective symmetric dual programs have been formulated. It is to be remarked that the functions which are taken in the primal-dual programs are not differentiable. Considering these nondifferentiable dual programs, we have discussed the corresponding duality relations. Numerical examples which illustrates the weak duality results of Mond-Weir and Wolfe type models have also been depicted in the paper. These results can be further extended to second order nondifferentiable symmetric dual programs and in the fractional programming case also. Several results appearing in the literature comes out as special cases.

Acknowledgements

The authors sincerely thank the referee for the recommendation and the interesting suggestions. The first author is supported by the Ministry of Human Resource and Development, New Delhi, India, for financial support, to carry out this work.

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