Bulletin of the Iranian Mathematical Society Vol. 30 *No. 2* (*2004*), pp 21-38.

EXPLICIT ABS SOLUTION OF A CLASS OF LINEAR INEQUALITY SYSTEMS AND LP PROBLEMS

H. ESMAEILI*, N. MAHDAVI-AMIRI AND E. SPEDICATO

Communicated by Mohammad Asadzadeh

ABSTRACT. ABS methods have been used extensively for solving linear and nonlinear systems of equations. Here we use them to find explicitly all solutions of a system of m linear inequalities in n variables, $m \leq n$, with full rank matrix. We apply these results to the LP problem with $m \leq n$ inequality constraints, obtaining optimality conditions and an explicit representation of all solutions.

1. Introduction

Let $A \in R^{m,n}$, $m \leq n$, be a full row rank matrix and let $b \in R^m$, $x \in R^n$. Consider the system of linear inequalities

$$Ax \leq b, \quad x \in R^n. \tag{1.1}$$

Given a general system of inequalities $Ax \leq b$, it is called feasible if there exists a vector $x^* \in R^n$ such that $Ax^* \leq b$, and x^* is a feasible solution, otherwise it is called infeasible. It is clear that system (1.1) is feasible if A is full rank, since equation $Ax = c, c \in R^m$ is solvable for any $c \leq b$. The intersection of every finite number

MSC(2000): Primary 15A39; Secondary 90C05
Keywords: Linear inequality, Linear programming, ABS algorithm
Received: 1 September 2004 , Accepted: 31 January 2005

*Corresponding author.

of half spaces $a_i^T x \le b_i$, where $a_i \in R^n$ and $b_i \in R$, $1 \le i \le m$, is a polyhedron in R^n. Thus, the system (1.1) denotes a polyhedron in R^n. Our aim is to define an explicit form for points of the polyhedron (1.1) using the ABS methods. (We should point out that our assumption for A to be full row rank is restrictive, and thus we are only concerned with a subclass of inequality systems here and a linear programming subclass considered later.)

The ABS methods [1] have been used extensively for solving general linear systems of equations and optimization problems. They are a class of direct iteration type methods that, in a finite number of iterations, either determine the general solution of a system of linear equations or establish its inconsistency. ABS methods for solving a linear system of equations are extensively discussed in monograph [2], while monograph [12] considers their application to several problems in optimization, including an ABS reformulation and generalization of the classical simplex method and Karmarkar interior point methods for the LP problem. See also [7] for a recent review of the ABS methods and their various applications as well as some numerical comparisons.

Zhang [10] and Zhao [13] have applied the ABS methods to get a solution of a linear inequalities system. In particular, Zhang [11] has shown that the Huang algorithm in the ABS class can be coupled with a Goldfarb-Idnani active set strategy, see [4], to determine, in a finite number of iterations, the least Euclidean norm of an inequality linear system. Shi [6] has applied the Huang algorithm to generate a sequence of solutions of a linear system such that any limit point is a nonnegative solution.

Here we use the ABS methods to determine feasible points of the polyhedron defined by (1.1), providing an explicit form of the feasible points. We then apply this approach to a class of LP problems.

Spedicato and Abaffy [8] have discussed the solution of the linear programming problem with constraints (1.1). Their method is based on the LU implicit algorithm in the ABS class. Here we solve the problem by getting an explicit form of the points of the polyhedron that is independent of the selection of a particular algorithm in the ABS class. Therefore, our approach is more general.

Section 2 recalls the class of ABS methods and provides some of its properties. Furthermore, in this section we present an algorithm for solving an equality constrained linear programming problem. In section 3, we discuss solving an inequality linear system by providing a representation of the points of a polyhedron of type (1.1). Furthermore, we consider in this section the solution of full rank inequality constrained linear programming problems giving the conditions of optimality and unboundedness.

2. ABS Methods for Solving Linear Systems and Linear Programming

ABS methods have been developed by Abaffy, Broyden and Spedicato [1]. Consider the system of linear equations

$$Ax = b, \tag{2.1}$$

where $A \in R^{m,n}$, $b \in R^m$, $x \in R^n$ and $rank(A) = m$. Let $A = (a_1, \ldots, a_m)^T$, $a_i \in R^n$, $i = 1, \ldots, m$ and $b = (b_1, \ldots, b_m)^T$. Also let $A_i = (a_1, \ldots, a_i)$ and $b^{(i)} = (b_1, \ldots, b_i)^T$.

Given $x_1 \in R^n$ arbitrary and $H_1 \in R^{n,n}$, Spedicato's parameter, arbitrary and nonsingular, note that for any $x \in R^n$ we can write $x = x_1 + H_1^T q$ for some $q \in R^n$.

The basic ABS class of methods consists of direct iteration type methods for computing the general solution of (2.1). In the beginning of the ith iteration, $i \geq 1$, the general solution for the first $i-1$ equations is at hand. It is clear that if x_i is a solution for the first $i-1$ equations and if $H_i \in R^{n,n}$, with $rank(H_i) = n - i + 1$, is such that the columns of H_i^T span the null space of A_{i-1}^T, then

$$x = x_i + H_i^T q,$$

with arbitrary $q \in R^n$, forms the general solution of the first $i-1$ equations. That is, with

$$H_i A_{i-1} = 0, \tag{2.2}$$

we have

$$A_{i-1}^T x = b^{(i-1)}. \tag{2.3}$$

Now, since $\mathrm{rank}(H_i) = n - i + 1$ and H_i^T is a spanning matrix for $\mathrm{null}(A_{i-1}^T)$, by assumption (one that is trivially valid for $i = 1$), then if we let

$$p_i = H_i^T z_i, \tag{2.4}$$

with arbitrary $z_i \in R^n$, Broyden's parameter, then $A_{i-1}^T p_i = 0$ and

$$x(\alpha) = x_i - \alpha p_i, \tag{2.5}$$

for any scalar α, solves the first $i-1$ equations. We can set $\alpha = \alpha_i$ so that $x_{i+1} = x(\alpha_i)$ solves the ith equation as well. If we let

$$\alpha_i = \frac{a_i^T x_i - b_i}{a_i^T p_i}, \tag{2.6}$$

with assumption $a_i^T p_i \neq 0$, then

$$x_{i+1} = x_i - \alpha_i p_i \tag{2.7}$$

is a solution for the first i equations. Now, to complete the ABS step, H_i must be updated to H_{i+1} so that $H_{i+1} A_i = 0$. It suffices to let

$$H_{i+1} = H_i - u_i v_i^T \tag{2.8}$$

and to select u_i, v_i so that $H_{i+1} a_j = 0$, $j = 1, \ldots, i$. The updating formula (2.8) for H_i is a rank-one correction to H_i. The matrix H_i is generally known as the *Abaffian*. The ABS methods of the unscaled or basic class define $u_i = H_i a_i$ and $v_i = H_i^T w_i / w_i^T H_i a_i$, where w_i, Abaffy's parameter, is an arbitrary vector satisfying

$$w_i^T H_i a_i \neq 0. \tag{2.9}$$

Thus, the updating formula can be written as below:

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}. \tag{2.10}$$

We can now give the general steps of an ABS algorithm [1,2]. In the algorithm below, r_{i+1} denotes the rank of A_i and hence the rank of H_i equals $n - r_{i+1} + 1$.

The Basic ABS Algorithm for Solving General Linear Systems

(1) Choose $x_1 \in R^n$, arbitrary, and $H_1 \in R^{n,n}$, arbitrary and nonsingular. Let $i = 1$, $r_i = 0$.
(2) Compute $t_i = a_i^T x_i - b_i$ and $s_i = H_i a_i$.
(3) *If* ($s_i = 0$ and $t_i = 0$) *then* let $x_{i+1} = x_i$, $H_{i+1} = H_i$, $r_{i+1} = r_i$ and *go to* step (7) (the ith equation is redundant). *If* ($s_i = 0$ and $t_i \neq 0$) *then* Stop (the ith equation and hence the system is incompatible).
(4) $\{s_i \neq 0\}$ Compute the search direction $p_i = H_i^T z_i$, where $z_i \in R^n$ is an arbitrary vector satisfying $z_i^T H_i a_i = z_i^T s_i \neq 0$. Compute

$$\alpha_i = \frac{t_i}{a_i^T p_i}$$

and let

$$x_{i+1} = x_i - \alpha_i p_i.$$

(5) {Updating H_i} Update H_i to H_{i+1} by

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$

where $w_i \in R^n$ is an arbitrary vector satisfying $w_i^T s_i \neq 0$.
(6) Let $r_{i+1} = r_i + 1$.
(7) *If* $i = m$ *then* Stop (x_{m+1} is a solution) *else* let $i = i + 1$ and *go to* step (2).

Remark 2.1. If the system (2.1) is compatible then the general solution is given by

$$x = x_{m+1} + H_{m+1}^T q, \tag{2.11}$$

where $q \in R^n$ is arbitrary.

Below, we list certain properties of the ABS methods [2]. For simplicity, we assume $\mathrm{rank}(A_i) = i$.

- $H_i a_i \neq 0$ if and only if a_i is linearly independent of $a_1, \ldots, a_{i-1}$.
- Every row of H_i corresponding to a nonzero component of w_i is linearly dependent on other rows.

- The search directions $p_1, \ldots, p_i$ are linearly independent.
- We have

$$L_i = A_i^T P_i, \tag{2.12}$$

where $P_i = (p_1, \ldots, p_i)$ and L_i is a nonsingular lower triangular matrix.
- The set of directions $p_1, \ldots, p_i$ together with independent columns of H_{i+1}^T form a basis for R^n.
- The matrix $W_i = (w_1, \ldots, w_i)$ has full column rank and

$$\mathrm{Null}(H_{i+1}^T) = \mathrm{Range}(W_i), \tag{2.13}$$

$$\mathrm{Range}(H_{i+1}^T) = \mathrm{Null}(A_i^T). \tag{2.14}$$

- If rows $j_1, \ldots, j_i$ of W_i are linearly independent then the same rows of H_{i+1} are linearly dependent and vice versa.
- If $s_i \neq 0$ then

$$\mathrm{rank}(H_{i+1}) = \mathrm{rank}(H_i) - 1.$$

- The updating formula H_i can be written as:

$$H_{i+1} = H_1 - H_1 A_i (W_i^T H_1 A_i)^{-1} W_i^T H_1, \tag{2.15}$$

where $W_i^T H_1 A_i$ is strongly nonsingular (the determinants of all of its main principal submatrices are nonzero).
- Matrices $S_i = (s_1, \ldots, s_i)$ and $U_i = (u_1, \ldots, u_i)$, where $u_i = H_i^T w_i / w_i^T s_i$, are full rank. Moreover, matrices

$$M_i = S_i^T W_i \tag{2.16}$$

and

$$N_i = U_i^T A_i \tag{2.17}$$

are nonsingular and upper triangular.

In the next section, we deduce a representation of the general solution of an inequality linear system and then consider solving some certain full row rank inequality constrained linear programming problems.

3. **Solving Certain Linear Inequality Systems and Linear Programming Problems**

Consider the inequality linear system

$$Ax \le b, \tag{3.1}$$

where $A = (a_1, \dots, a_m)^T \in R^{m,n}$, $x \in R^n$, $b \in R^m$ and $m \le n$. Suppose $\operatorname{rank}(A) = m$. Our aim is to determine the general solution of (3.1) using the ABS methods. For this end, consider the linear system

$$Ax = y, \quad x \in R^n, \tag{3.2}$$

where $y = (y_1, \dots, y_m)^T \in R^m$ is a parameter vector. Note that x is a solution to (3.1) if and only if it satisfies (3.2) for some $y \le b$. Thus, the solutions of (3.1) can be obtained by solving (3.2) for all parameter vectors y such that $y \le b$.

Now consider a parameter vector y such that $y \le b$. We can solve system (3.2) using the ABS methods. For this task, take $x_1 \in R^n$, $H_1 \in R^{n,n}$ arbitrary and nonsingular. From the ABS properties, x_{i+1}, for each i, $1 \le i \le m$, satisfies the first i equations of (3.2). Since, $y_j \le b_j$, for $j \le i$, then x_{i+1} satisfies the first i inequalities of (3.1) too. Therefore, x_{m+1} is a solution to inequalities (3.1). Note that x_{m+1} is a function of the parameter vector y, i.e. $x_{m+1} = x_{m+1}(y)$. Thus by varying y, $y \le b$, we can obtain an infinite number of solutions to (3.1) of the form $x_{m+1} = x_{m+1}(y)$. However, an explicit form of x_{m+1} as a function of the parameter vector y is not obvious.

Now we want to obtain an explicit form for $x_{m+1} = x_{m+1}(y)$. Let $A_i = (a_1, \dots, a_i)$ and $W_i = (w_1, \dots, w_i)$. The matrix $(W_i^T H_1 A_i)^{-1} W_i^T H_1$ is called a W_i-left inverse of A_i and is denoted by $A_{W_i}^{-1}$. We know, from the ABS properties, that $Q_i = W_i^T H_1 A_i$ is strongly nonsingular and also that

$$H_{i+1} = H_1 - H_1 A_i (W_i^T H_1 A_i)^{-1} W_i^T H_1 = H_1 - H_1 A_i A_{W_i}^{-1}. \tag{3.3}$$

We need the determinant of Q_i, given in the following lemma.

Lemma 3.1. *For all i, $1 \le i \le m$, we have:*

$$\det(W_i^T H_1 A_i) = w_1^T s_1 \times \cdots \times w_i^T s_i,$$

where $s_j = H_j a_j$, $1 \le j \le i$.

Proof. We proceed by induction. For $i = 1$ we have:

$$\det(W_1^T H_1 A_1) = \det(w_1^T H_1 a_1) = \det(w_1^T s_1) = w_1^T s_1.$$

Now, suppose $\det(W_i^T H_1 A_i) = w_1^T s_1 \times \cdots \times w_i^T s_i$. Then,

$$\begin{aligned}
&W_{i+1}^T H_1 A_{i+1} \\
&= \begin{bmatrix} W_i^T \\ w_{i+1}^T \end{bmatrix} H_1 \begin{bmatrix} A_i & a_{i+1} \end{bmatrix} \\
&= \begin{bmatrix} W_i^T H_1 A_i & W_i^T H_1 a_{i+1} \\ w_{i+1}^T H_1 A_i & w_{i+1}^T H_1 a_{i+1} \end{bmatrix} \\
&= \begin{bmatrix} W_i^T H_1 A_i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_i & (W_i^T H_1 A_i)^{-1} W_i^T H_1 a_{i+1} \\ w_{i+1}^T H_1 A_i & w_{i+1}^T H_1 a_{i+1} \end{bmatrix}
\end{aligned}$$

implying that

$$\begin{aligned}
&\det(W_{i+1}^T H_1 A_{i+1}) \\
&= \det(W_i^T H_1 A_i) \times \left(w_{i+1}^T H_1 a_{i+1} - w_{i+1}^T H_1 A_i (W_i^T H_1 A_i)^{-1} W_i^T H_1 a_{i+1} \right) \\
&= \det(W_i^T H_1 A_i) \times w_{i+1}^T \left(H_1 - H_1 A_i (W_i^T H_1 A_i)^{-1} W_i^T H_1 \right) a_{i+1} \\
&= \det(W_i^T H_1 A_i) \times w_{i+1}^T H_{i+1} a_{i+1} \\
&= \det(W_i^T H_1 A_i) \times w_{i+1}^T s_{i+1} \\
&= w_1^T s_1 \times \cdots \times w_i^T s_i \times w_{i+1}^T s_{i+1}. \quad \square
\end{aligned}$$

Corollary 3.2. *It is possible to take w_i, for all i, such that*

$$\det(W_i^T H_1 A_i) > 0. \quad \square$$

Theorem 3.3 [5]

(a) *The vector* $A_{W_i}^{-T}\bar{y}_i$, *where* $\bar{y}_i = (y_1, \dots, y_i)^T$, *satisfies the first* i *equations of* (3.2).

(b) *If* $w_j = z_j$, *for all* j, $1 \le j \le i$, *then the general solution for the first* i *equations of* (3.2) *is given by*

$$x = A_{W_i}^{-T}\bar{y}_i + (I - A_{W_i}^{-T}A_i^T)x_1 + H_{i+1}^T q, \tag{3.4}$$

where $q \in R^n$ *is arbitrary.* $\square$

If H_1 is an arbitrary nonsingular matrix, then from (3.3) we have that $H_1^{-1}H_{i+1} = I - A_i A_{W_i}^{-1}$. Hence $A_{W_i}^{-T}\bar{y}_i + (I - A_{W_i}^{-T}A_i^T)x_1 = A_{W_i}^{-T}\bar{y}_i + H_{i+1}^T H_1^{-T}x_1$. Therefore, by setting

$$x_{i+1} = A_{W_i}^{-T}\bar{y}_i + H_{i+1}^T H_1^{-T}x_1, \tag{3.5}$$

and noticing that x_{i+1} is a solution of the first i equations of (3.2), then (3.4) is just another form of (2.11).

Remark 3.4. The matrix $A_{W_i}^{-T}$ can be computed by the recurrence

$$\begin{aligned} A_{W_1}^{-T} &= H_1^T w_1 / w_1^T H_1 a_1, \\ A_{W_i}^{-T} &= \left[\left(I - H_i^T w_i a_i^T / a_i^T H_i^T w_i \right) A_{W_{i-1}}^{-T} \qquad H_i^T w_i / a_i^T H_i^T w_i \right] \end{aligned}$$

or, if we take $w_i = z_i$, for all i, by the recurrence

$$\begin{aligned} A_{W_1}^{-T} &= p_1 / a_1^T p_1, \\ A_{W_i}^{-T} &= \left[\left(I - p_i a_i^T / a_i^T p_i \right) A_{W_{i-1}}^{-T} \qquad p_i / a_i^T p_i \right]. \end{aligned}$$

(For a proof see [5]). Thus, the explicit form of the particular solution $x_{i+1} = x_{i+1}(\bar{y}_i)$ of the first i equations of (3.2) is

$$x_{i+1}(\bar{y}_i) = A_{W_i}^{-T}\bar{y}_i + H_{i+1}^T H_1^{-T}x_1, \tag{3.6}$$

where $\bar{y}_i = (y_1, \dots, y_i)^T$.

Thus, we have proved the following theorem.

Theorem 3.5. *If in the ABS algorithm we take $w_i = z_i$, for all i, then the general solution of the inequality system* (3.1) *is*

$$x = x_{m+1} + H_{m+1}^T q, \quad q \in R^n, \tag{3.7}$$

where

$$x_{m+1} = A_{W_m}^{-T} y + H_{m+1}^T H_1^{-T} x_1 \tag{3.8}$$

and $y \in R^m$, $y \leq b$, is arbitrary. □

The vector $y \in R^m$ is called a feasible parameter vector for (3.1) if $y \leq b$. In what follows, we obtain a relation between the coefficient matrix A, the matrix L in the implicit transformation of A by the ABS algorithm, and the feasible parameter vector y.

From the ABS algorithm we have

$$x_{i+1} = x_i - \frac{a_i^T x_i - y_i}{\delta_i} p_i,$$

where $\delta_i = a_i^T p_i = s_i^T z_i \neq 0$. Since z_i is arbitrary, we can assume $\delta_i > 0$ without loss of generality. Let $\alpha_i = (a_i^T x_i - y_i)/\delta_i$. Then $y_i = a_i^T x_i - \alpha_i \delta_i$. But $y_i \leq b_i$, thus $\alpha_i \geq (a_i^T x_i - b_i)/\delta_i$. Therefore, y_i has the form $y_i = a_i^T x_i - \alpha_i \delta_i$, where $\delta_i = a_i^T p_i > 0$ and $\alpha_i \geq (a_i^T x_i - b_i)/\delta_i$ is arbitrary. Consequently, since

$$x_i = x_1 - \sum_{j=1}^{i-1} \alpha_j p_j$$

we get

$$y_i = a_i^T x_i - \alpha_i \delta_i = a_i^T x_i - \alpha_i a_i^T p_i = a_i^T (x_i - \alpha_i p_i) = a_i^T \left(x_1 - \sum_{j=1}^{i} \alpha_j p_j \right).$$

Since for $j > i$, $a_i^T p_j = 0$, then we can write $y_i = a_i^T (x_1 - P\alpha)$, where $P = (p_1, \ldots, p_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)^T$. Therefore, every feasible vector y can be written as

$$y = A(x_1 - P\alpha) = Ax_1 - L\alpha,$$

where $L = AP$ is a nonsingular lower triangular matrix from (2.12) and $\alpha \in R^m$ satisfies the inequality system

$$L\alpha \geq Ax_1 - b. \tag{3.9}$$

Now we prove that (3.9) is not only a necessary but is also a sufficient condition for y to be a feasible parameter vector.

Theorem 3.6. *Let $x_1 \in R^n$ be an arbitrary initial point for an ABS algorithm with $w_i = z_i$, for all i, and let α be a vector with components α_i as set in the i-th iteration of the ABS algorithm. The vector y is feasible for the inequality system* (3.1) *if and only if*

$$y = A(x_1 - P\alpha) = Ax_1 - L\alpha \tag{3.10}$$

and (3.9) *holds.*

Proof. We have proven above that if y is a feasible parameter vector then it has the form (3.10). Conversely, assume that for some x_1 we have:

$$y = Ax_1 - L\alpha, \quad L\alpha \geq Ax_1 - b.$$

Then

$$y = Ax_1 - L\alpha = Ax_1 - AP\alpha = A(x_1 - P\alpha),$$

implying that $x = x_1 - P\alpha$ satisfies the system (3.2). On the other hand,

$$y = Ax_1 - L\alpha \leq Ax_1 - Ax_1 + b = b.$$

Therefore, y is a feasible parameter vector for the inequalities (3.1). □

Remark 3.7.

(1) We can deduce every feasible parameter vector y from a feasible parameter vector y^*. To show this, suppose that y^* is a feasible parameter vector ($y^* = Ax_1 - L\alpha^*$, where $L\alpha^* \geq Ax_1 - b$). If $y = y^* - L\beta$, where $L\beta \geq y^* - b$, then y is a feasible parameter vector since firstly, $b \geq y^* - L\beta = y$ and secondly, the system $Ax = y$ is compatible because $y^* \in \text{Range}(A)$ by (3.10) and $L\beta = AP\beta \in \text{Range}(A)$. Conversely, if y is a feasible parameter vector, then

$$y = Ax_1 - L\alpha = y^* + L\alpha^* - L\alpha = y^* - L(\alpha - \alpha^*) = y^* - L\beta,$$

for $\beta = \alpha - \alpha^*$, and

$$L\beta = L\alpha - L\alpha^* \geq Ax_1 - b - L\alpha^* = y^* - b.$$

(2) If $b \geq 0$, then $y^* = 0$ is a feasible parameter vector. In this case, the feasible parameter vectors y are in the form $y = L\beta$, where $L\beta \leq b$.

Now, let

$$d = |\det(W_m^T H_1 A_m)| = |\det(W_m^T H_1 A^T)|$$

and

$$M = d\, A_{W_m}^{-T}.$$

For convenience, let $H = H_{m+1}$. By Theorem 3.6 and (2.14), i.e. $\mathrm{Null}(A) = \mathrm{Range}(H^T)$, we can prove the following theorem.

Theorem 3.8. *Let $A_{W_m}^{-T}$ be the right W_m-inverse of A obtained from application of an ABS algorithm with $w_i = z_i$ for all i, to A, H be the resulting Abaffian and*

$$M = |\det(W_m^T H_1 A^T)|\, A_{W_m}^{-T} = d\, A_{W_m}^{-T}.$$

Then the following statement holds:

$$\begin{aligned} &\{x \in R^n \mid Ax \leq b\} = \\ &\{A_{W_m}^{-T} b - M\gamma - H^T q \mid q \in R^n,\ \gamma \in R_+^m\}. \end{aligned} \tag{3.11}$$

Proof. Firstly we show that

$$\{x \in R^n \mid Ax \leq b\} = \{x_1 - P\alpha - H^T q \mid q \in R^n,\ \alpha \in R^m,\ L\alpha \geq Ax_1 - b\}.$$

For this end, suppose $x \in R^n$ has the form $x = x_1 - P\alpha - H^T q$, where $q \in R^n$ and $\alpha \in R^m$ is such that $L\alpha \geq Ax_1 - b$. Then we have $Ax = Ax_1 - L\alpha \leq b$.

Conversely, assume x^* satisfies $Ax \leq b$. Let $y^* = Ax^*$. Thus the system $Ax = y^*$ is compatible. Hence, by using the ABS algorithm, we can write $x^* = x_1 - P\alpha^* - H^T q^*$, for some $\alpha^* \in R^m$ and $q^* \in R^n$. Thus, $y^* = Ax^* = Ax_1 - L\alpha^*$. Since $y^* \leq b$, then α^* satisfies $Ax_1 - L\alpha^* \leq b$.

Now, let $w_i = z_i$, for all i. Then, by (3.4), we have that

$$
\begin{aligned}
&\{x \in R^n \mid Ax \leq b\} \\
&= \{x_1 - P\alpha - H^T q \mid q \in R^n,\ \alpha \in R^m,\ L\alpha \geq Ax_1 - b\} \\
&= \{x_1 - P\alpha - H^T q \mid q \in R^n,\ \alpha \in R^m, \\
&\qquad L\alpha = AP\alpha = Ax_1 - b + d\gamma,\ \gamma \in R^m_+\} \\
&= \{x_1 - P\alpha - H^T q \mid q \in R^n,\ \alpha \in R^m, \\
&\qquad P\alpha = A^{-T}_{W_m}(Ax_1 - b + d\gamma) + (I - A^{-T}_{W_m}A)x_1 + H^T q,\ \gamma \in R^m_+\} \\
&= \{x_1 - A^{-T}_{W_m}(Ax_1 - b + d\gamma) - (I - A^{-T}_{W_m}A)x_1 - H^T q \mid q \in R^n,\ \gamma \in R^m_+\} \\
&= \{A^{-T}_{W_m}b - M\gamma - H^T q \mid q \in R^n,\ \gamma \in R^m_+\}. \quad \square
\end{aligned}
$$

Remark 3.9. In place of the inequality system (3.1), we may consider the equivalent equality system

$$Ax + u = b,$$

where $u \in R^m_+$ is a nonnegative vector. By writing the above system as $Ax = b - u$ and letting $x_1 = 0$, the general solution is

$$x = A^{-T}_{W_m}(b - u) - H^T q, \quad q \in R^n$$

by (3.4). Now, as before, let

$$d = |\det(W_m^T H_1 A^T)|$$

and

$$M = d\ A^{-T}_{W_m}.$$

By letting $u = d\gamma$, where $\gamma \in R^m_+$, we have

$$x = A^{-T}_{W_m}b - M\gamma - H^T q.$$

ABS Solution of a Certain Linear Programming Problem

Now consider the linear programming problem, with $x \in R^n$, $b \in R^m$, $m \leq n$, and $\text{rank}(A) = m$:

$$\max\ c^T x \quad : \quad Ax \leq b. \tag{3.12}$$

From Theorem 3.8, the above problem is equivalent to

$$\min\ (c^T M\gamma + c^T H^T q) \quad : \quad \gamma \in R^m_+,\ q \in R^n. \tag{3.13}$$

Now, let $\bar{c} = M^T c$ and

$$I^+ = \{i \mid \bar{c}_i > 0\},$$

$$I^- = \{i \mid \bar{c}_i < 0\},$$

$$I^0 = \{i \mid \bar{c}_i = 0\}.$$

Theorem 3.10. *Let the ABS algorithm be applied to A as in Theorem 3.8. Let $x^* = A_{W_m}^{-T} b$. Then:*

(a) *If $Hc \neq 0$, then problem (3.12) is unbounded and has no solution.*
(b) *If $Hc = 0$ and $I^- \neq \phi$, then problem (3.12) is unbounded and has no solution.*
(c) *If $Hc = 0$ and $I^- = \phi$, then there are an infinite number of optimal solutions for (3.12) having the form*

$$x = x^* - H^T q - \sum_{j \in I^0} \gamma_j M e_j,$$

where $q \in R^n$ and $\gamma_j \geq 0$, $j \in I^0$, are arbitrary and e_j is the j-th unit vector in R^m.

Proof.

(a) Considering (3.13), Hc not being a zero vector, it is obvious that q can be set appropriately to obtain an arbitrarily small value for the objective function.

Now let $Hc = 0$. Using the notation as above, problem (3.12) can be written as

$$\min z = \sum_{j \in I^-} \bar{c}_j \gamma_j + \sum_{j \in I^+} \bar{c}_j \gamma_j + \sum_{j \in I^0} 0 \gamma_j \quad : \quad \gamma_j \geq 0 \quad \forall j.$$

(b) Since $I^- \neq \phi$ then $k \in I^-$ exists. Letting $\gamma = t e_k$, then $z \to -\infty$, when $t \to \infty$. Hence (3.12) is unbounded and has no solution.
(c) In this case, $z = \sum_{j \in I^+} \bar{c}_j \gamma_j + \sum_{j \in I^0} 0 \gamma_j$. The minimizers of z are written as $\gamma = \sum_{j \in I^0} \gamma_j e_j$, where $\gamma_j \geq 0$, $j \in I^0$, is arbitrary.

Thus, the optimal solutions to (3.12) have the form

$$x = x^* - H^T q - \sum_{j \in I^0} \gamma_j M e_j,$$

where $q \in R^n$ and $\gamma_j \geq 0$, $j \in I^0$, are arbitrary. $\square$

Remark 3.11. According to the ABS properties, $HA^T = 0$ and hence $\text{Range}(A^T) = \text{Null}(H)$. The condition $Hc = 0$ is then equivalent to the Kuhn-Tucker condition $c = A^T u$, for some u. Since A^T has full column rank, then u is unique. On the other hand, the vector $\frac{1}{d}\bar{c}$ satisfies the system $A^T u = c$, because, the rows of $A^{-1}_{W_m}$ being linearly independent, the solution of $A^T u = c$ is equivalent to the solution of $A^{-1}_{W_m} A^T u = A^{-1}_{W_m} c$. Since $A^{-1}_{W_m} A^T = I$ then, $u = A^{-1}_{W_m} c = \frac{1}{d}\bar{c}$. Thus, when the w_i are chosen so that $d > 0$, $\bar{c}$ has the same sign as the Lagrange multipliers, componentwise. If $\bar{c} \ngeq 0$ then the problem is unbounded and has no solution. In this case, we realize that $u_i < 0$, for all $i \in I^-$, and $u_i \geq 0$, for all $i \notin I^-$. If $\bar{c} \geq 0$ then the problem has infinitely many optimal solutions. Here, as expected for optimality, $u_i \geq 0$, for all i.

Using the above results, the following algorithm is suggested for solving the linear programming problem (3.12).

An Algorithm for Solving the Linear Programming Problem (3.12)

(1) Apply an ABS algorithm to the coefficient matrix A and compute $A^{-T}_{W_m}$ and H. Compute

$$d = |\det(W_m^T H_1 A^T)|,$$

$$M = d\ A^{-T}_{W_m}.$$

(2) *If* $Hc \neq 0$ *then* Stop (the problem is unbounded and hence has no solution).

(3) Let $\bar{c} = M^T c$ and form the following sets

$$I^+ = \{i \mid \bar{c}_i > 0\},$$

$$I^- = \{i \mid \bar{c}_i < 0\},$$

$$I^0 = \{i \mid \bar{c}_i = 0\}.$$

(4) *If* $I^- \neq \phi$ *then* Stop (the problem is unbounded and hence has no solution).

(5) ($I^- = \phi$) Compute

$$x^* = A^{-T}_{W_m} b.$$

The optimal solutions have the form

$$x = x^* - H^T q - \sum_{j \in I^0} \gamma_j M e_j,$$

where $q \in R^n$, $\gamma_j \geq 0$, $j \in I^0$, are arbitrary numbers and e_j denotes the j-th unit vector in R^m. Stop.

Conclusion

We have used the ABS algorithms for linear real systems to solve full rank linear inequalities and linear programming problems where the number of inequalities is less than or equal to the number of variables. We obtained the conditions of both optimality and unboundedness in the context of ABS algorithms. In another paper [3] similar results were also obtained for the integer inequality and ILP problem. It is left to future work the extension to the case where the number of inequalities is larger than the number of variables, a case quite different than the one considered here in its geometry and algorithmic approach.

Acknowledgment

This work has been partly supported by the Research Council of Sharif University of Technology and by GNIM. The authors are grateful to the referees for their constructive comments.

References

[1] J. Abaffy, C. G. Broyden and E. Spedicato, A class of direct methods for linear equations, *Numer. Math.* **45** (1984), 361-376.

[2] J. Abaffy and E. Spedicato, *ABS Projection Algorithms: Mathematical Techniques for Linear and Nonlinear Equations*, Ellis Horwood, Chichester, 1989.

[3] E. Esmaeili, N. Mahdavi-Amiri and E. Spedicato, ABS solutions of a class of linear inequalities and integer LP problems, *Optim. Methods Softw.* **16** (2001), 179-192.

[4] D. Goldfarb and A. Idnani, A numerically stable dual method for solving strictly convex quadratic programming, *Math. Prog.* **27** (1983), 1-33.

[5] S. Li, On a variational characterization of the ABS algorithms, *Communication at the Second International Conference on ABS Algorithms*, Beijing, June 1995.

[6] G. Shi, An ABS algorithm for generating nonnegative solutions of linear systems, *Proceedings of the First International Conference on ABS Algorithms*, Luoyang, September 1991, E. Spedicato (editor), University of Bergamo, 54-57.

[7] E. Spedicato, E. Bodon, A. Del Popolo and N. Mahdavi-Amiri, ABS methods and ABSPACK for linear systems and optimization: A review, *4OR. Quarterly Journal of the Belgian, French and Italian Operations Research Societies* **1** (2003), 51-66.

[8] E. Spedicato and J. Abaffy, On the use of the ABS algorithms for some linear programming problems, Preprint, University of Bergamo, 1987.

[9] E. Spedicato and M. Zhu, Reformulation of the ABS algorithm via full rank Abaffian, *Proceedings EAMA Conference*, Seville, (1997), 396-403.

[10] L. Zhang, A method for finding a feasible point of inequalities, *Proceedings of the First International Conference on ABS Algorithms*, Luoyang, September 1991, E. Spedicato (editor), University of Bergamo, 131-137.

[11] L. Zhang, An algorithm for the least Euclidean norm solution of a linear system of inequalities via the Huang ABS algorithm and the Goldfarb-Idnani strategy, Report 95/2, University of Bergamo, 1995.

[12] L. Zhang, X. Xia and E. Feng, *Introduction to ABS Methods for Optimization* (in Chinese), Dalian Technology Univeersity Press, Dalian, 1998.

[13] J. Zhao, ABS algorithms for solving linear inequalities, Report 91/21, University of Bergamo, 1991.

H. Esmaeili
Department of Mathematics
Bu-Ali Sina University
Hamedan, Iran
e-mail:esmaeili@basu.ac.ir

N. Mahdavi-Amiri
Department of Mathematical Sciences
Sharif University of Technology
Tehran, Iran
e-mail:nezamm@sharif.edu

E. Spedicato
Department of Mathematics, Statistics and Computer Science
University of Bergamo
Bergamo, Italy
e-mail:emilio.spedicato@unibg.it