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ON THE FIXED NUMBER OF GRAPHS

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Abstract. A set of vertices S of a graph G is called a fixing set of G, if only the trivial automorphism of G fixes every vertex in S. The fixing number of a graph is the smallest cardinality of a fixing set. The fixed number of a graph G is the minimum k, such that every k-set of vertices of G is a fixing set of G. A graph G is called a k-fixed graph, if its fixing number and fixed number are both k. In this paper, we study the fixed number of a graph and give a construction of a graph of higher fixed number from a graph of lower fixed number. We find the bound on k in terms of the diameter d of a distance-transitive k-fixed graph.
Keywords: Fixing set, stabilizer, fixing number, fixed number.
MSC(2010): Primary: 05C25; Secondary: 05C60.

1. Introduction

Let $G = (V(G), E(G))$ be a connected graph of order n. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the number of edges that are incident to v in G. The *distance* between two vertices x and y, denoted by $d(x, y)$, is the shortest length of a path between x and y in G. The *eccentricity* of a vertex $x \in V(G)$ is $e(x) = \max_{y \in V(G)} d(x, y)$ and the *diameter* of G is $\max_{x \in V(G)} e(x)$. For a vertex $v \in V(G)$, the *neighborhood* of v, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G.

An *automorphism* of G, $g : V(G) \to V(G)$, is a permutation on $V(G)$ such that $g(u)g(v) \in E(G)$ if and only if $uv \in E(G)$, i.e., the adjacency is preserved under automorphism g. The set of all such permutations for a graph G forms a group under the operation of composition of permutations. It is called the *automorphism group* of G, denoted by $Aut(G)$ which is a subgroup of symmetric group S_n, the group of all permutations on n vertices. A graph G with the trivial automorphism group is called a *rigid* or *asymmetric* graph and such a graph has no symmetries. In this paper, all graphs (unless stated otherwise)

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have non-trivial automorphism group i.e., $Aut(G) \neq \{id\}$. Let $u, v \in V(G)$, we say u is *similar* to v, denoted by $u \sim v$ (or more specifically $u \sim^g v$) if there is an automorphism $g \in Aut(G)$ such that $g(u) = v$. It can be seen that the similarity is an equivalence relation on the vertices of G, and hence, it partitions the vertex set $V(G)$ into disjoint equivalence classes, called orbits of G. The *orbit* of a vertex v is defined as $\mathcal{O}(v) = \{u \in V(G) | u \sim v\}$. The idea of fixing sets was introduced by Erwin and Harary in [4]. They used the following terminology: The *stabilizer* of a vertex $v \in V(G)$ is defined as, $stab(v) = \{f \in Aut(G) | f(v) = v\}$. The *stabilizer* of a set of vertices $F \subseteq V(G)$ is defined as, $stab(F) = \{f \in Aut(G) | f(v) = v \text{ for all } v \in F\} = \cap_{v \in F} stab(v)$. A vertex v is *fixed* by an automorphism $g \in Aut(G)$, if $g \in stab(v)$. A set of vertices F is a *fixing set*, if $stab(F)$ is trivial, i.e., the only automorphism that fixes all vertices of F is the trivial automorphism. The smallest cardinality of a fixing set is called the *fixing number* of G and it is denoted by $fix(G)$. We shall refer a set of vertices $A \subset V(G)$ for which $stab(A) \setminus \{id\} \neq \emptyset$ as a *non-fixing* set. A vertex $v \in V(G)$ is called a *fixed* vertex, if $stab(v) = Aut(G)$. Every graph has a fixing set. Trivially, the set of vertices itself is a fixing set. It is also clear that a set containing all but one vertex is a fixing set. The following theorem gives a relation between orbits and stabilizers.

Theorem 1.1 (Orbit-Stabilizer Theorem). *Let G be a connected graph and $v \in V(G)$,*

$$|Aut(G)| = |\mathcal{O}(v)||stab_{Aut(G)}(v)|.$$

Boutin introduced determining set of a graph in [2]. A set $D \subseteq V(G)$ is said to be a *determining set* for G, if whenever $g, h \in Aut(G)$ so that $g(x) = h(x)$ for all $x \in D$, then $g(v) = h(v)$ for all $v \in V(G)$. The minimum cardinality of a determining set of a graph G, denoted by $Det(G)$, is called the *determining number* of G. The following lemma given in [5] shows the equivalence between definitions of fixing set and determining set.

Lemma 1.2 ([5]). *A set of vertices is a fixing set if and only if it is a determining set.*

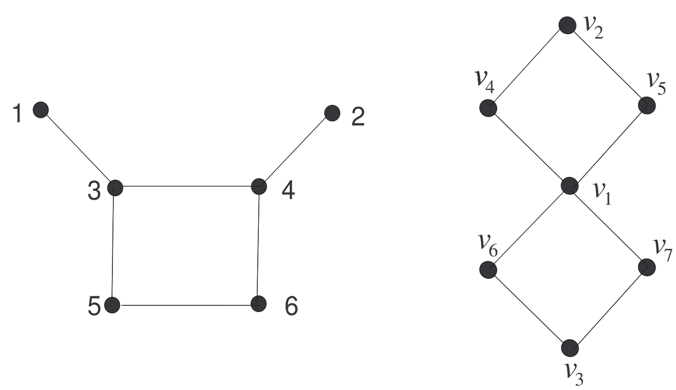
Thus, notions of the fixing number and the determining number of a graph G are same.

Jannesari and Omoomi have discussed the properties of resolving graphs and randomly k-dimensional graphs in [7] and [6], which were based on the well-known graph notions resolving number and metric dimension. In this paper, we define the fixed number of a graph, fixing graph and k-fixed graphs. We discuss the properties of these graphs in the context of fixing sets and the fixing number.

The *fixed number* of a graph G, $fxd(G)$, is the minimum k such that every *k-set* of vertices is a fixing set of G. It may be noted that $0 \leq fix(G) \leq fxd(G) \leq n - 1$. A graph is said to be a *k-fixed graph*, if $fix(G) = fxd(G) = k$. In this paper, the fixed number k, remains in the focus of our attention. A path graph of even order is a 1-fixed graph. Similarly, a cyclic graph of odd order is a 2-fixed graph. We give a construction of a graph with $fxd(G) = r + 1$ from a graph with $fxd(G) = r$ in Theorem 2.8. Also, a characterization of k-fixed graphs is given in Theorem 3.7.

2. The fixed number

Consider the graph G_1 depicted in Figure 1. It is clear that $Aut(G) = \{e, (12)(34)(56)\}$. Also, $stab(v) = \{id\}$ for all $v \in V(G)$. Thus, $\{v\}$ for each



(a) Graph G_1 (a) Graph G_2

FIGURE 1.

$v \in V(G)$ forms a fixing set for G. Hence, $fix(G) = fxd(G) = 1$ and G is 1-fixed graph. Thus, we have the following proposition immediately from the definition of fixing set.

Proposition 2.1. *Let G be a connected graph and $fxd(G) = 1$, then*

(i) $|\mathcal{O}(v)| = |Aut(G)|$ *for all* $v \in V(G)$.

(ii) *G does not have fixed vertices.*

Proof. (i) Since, $|stab(v)| = 1$ for all $v \in V(G)$, therefore the result follows from Theorem 1.1. (ii) As $stab(v) = Aut(G)$ for a fixed vertex $v \in V(G)$, therefore $\{v\}$ does not form a fixing set for G. □

The problem of 'finding the minimum k such that every k-subset of vertices of G is a fixing set of G' is equivalent to the problem of 'finding the maximum

r such that there exist an r-subset of vertices of G which is not a fixing set of G'. Thus, the largest cardinality of a non-fixing set in a graph G helps in finding the fixed number of G. We can see $r = 0$ and $r = 5$ for the graphs G_1 and G_2 in Figure 1, respectively. Now, consider the graph G_2 in Figure 1. Here, $A = \{v_1, v_2, v_3, v_4, v_5\}$ is a non-fixing set with the largest cardinality and $g = (v_6v_7) \in stab(A)$ is the only non-trivial automorphism in $stab(A)$. Thus, there exist a set $B = \{v_6, v_7\} \subset V(G) \setminus A$ such that $v_6 \sim^g v_7$. In fact, for each non-fixing set A and each non-trivial automorphism $g \in stab(A)$, there exist at least one set $B \subset V(G) \setminus A$ such that $u \sim^g v$ for all distinct $u, v \in B$. Thus, we have the following remark about non-fixing sets.

Remark 2.2. Let G be a graph of order n.

(i) If r $(0 \leq r \leq n-2)$ be the largest cardinality of a non-fixing subset of G, then $fxd(G) = r + 1$.

(ii) Let A be a non-fixing set of G. For each non-trivial $g \in stab(A)$ there exist at least one set $B \subset V(G) \setminus A$ such that $u \sim^g v$ for all distinct $u, v \in B$.

Proposition 2.3. *Let G be a graph and $u, v \in V(G)$ such that $N(v)\backslash\{u\} = N(u)\backslash\{v\}$. Let F be a fixing set of G, then either u or v is in F.*

Proof. Let $u, v \in V(G)$ such that $N(v)\backslash\{u\} = N(u)\backslash\{v\}$. Suppose on contrary, both u and v are not in F. As u and v have common neighbors and $u, v \notin F$, so there exists an automorphism $g \in Aut(G)$ such that $g \in stab(F)$ and $g(u) = v$. Hence, $stab(F)$ has a non-trivial automorphism, a contradiction. □

Theorem 2.4. *Let G be a connected graph of order n. Then,*
$fxd(G) = n - 1$ if and only if $N(v)\backslash\{u\} = N(u)\backslash\{v\}$ for some $u, v \in V(G)$.

Proof. Let $u, v \in V(G)$ such that $N(v)\backslash\{u\} = N(u)\backslash\{v\}$. Suppose on contrary that $fxd(G) \leq n-2$, then $V(G)\backslash\{u, v\}$ is a fixing set for G. But, by Proposition 2.3, every fixing set contains either u or v. This contradiction implies that, $fxd(G) = n - 1$.

Conversely, let $fxd(G) = n - 1$. Then, there exists a non-fixing subset T of $V(G)$ with $|T| = n - 2$. Assume $T = V(G) \setminus \{u, v\}$ for some $u, v \in V(G)$. Our claim is that u, v are those vertices of G for which $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Suppose on contrary $N(u) \setminus \{v\} \neq N(v) \setminus \{u\}$, then there exists a vertex $w \in T$ such that w is adjacent to one of the vertices u or v. Without loss of generality, let w be adjacent to u but not adjacent to v. Let a non-trivial automorphism $g \in stab(T)$ (such a non-trivial automorphism exists because T is not a fixing set). Since g is non-trivial and $V(G) \setminus T = \{u, v\}$, $g(u) = v$. But u cannot map to v under g, because $g \in stab(w)$ and w is adjacent with u and not adjacent to v. Hence, g also fixes u and v, i.e., $g \in stab\{u, v\}$ and consequently g becomes trivial. Hence, $stab(T)$ is trivial, a contradiction. Thus, $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. □

The following theorem given in [3] is useful for the proof of Corollary 2.6.

Theorem 2.5 ([3]). *Let G be a connected graph of order n. Then $fix(G) = n - 1$ if and only if $G = K_n$.*

Corollary 2.6. *Let G be a graph of order n and $G \neq K_n$. If G is $(n-1)$-fixed graph, then for each pair of distinct vertices $u, v \in V(G)$, $N(u)\backslash\{v\} \neq N(v)\backslash\{u\}$.*

Proof. Let $N(u)\backslash\{v\} = N(v)\backslash\{u\}$ for some $u, v \in V(G)$. Then by Theorem 2.4, $fxd(G) = n - 1$. Since $G \neq K_n$, therefore by Theorem 2.5, $fix(G) \neq n - 1 = fxd(G)$. Hence, G is not $(n-1)$-fixed. □

The fixing polynomial, $F(G, x) = \sum_{i=fix(G)}^{n} \alpha_i x^i$, of a graph G of order n is a generating function of sequence $\{\alpha_i\}$ $(fix(G) \leq i \leq n)$, where α_i is the number of fixing subsets of G with the cardinality i. For more detail about fixing polynomial, see [9] where we discussed properties of fixing polynomial and found it for different families of graphs. For example $F(C_3, x) = x^3 + 3x^2$, where C_3 is the cyclic graph of order 3.

Theorem 2.7. *Let G be a k-fixed graph of order n. Then,*

$$F(G, x) = \sum_{i=k}^{n} \binom{n}{i} x^i.$$

Proof. Since $fix(G) = fxd(G) = k$ and superset of a fixing set is also a fixing set, each subset of $V(G)$ with the cardinality i $(k \leq i \leq n)$ is a fixing set. Hence, $\alpha_i = \binom{n}{i}$ for each i, $(k \leq i \leq n)$. □

Theorem 2.8. *Let G be a graph of order n and $fxd(G) = r$. We can construct a graph G' of order $n+1$, from G such that $fxd(G') = r + 1$.*

Proof. Since $fxd(G) = r$, G has a non-fixing set A with the largest cardinality $|A| = r - 1$. By Remark 2.2(ii), for each non-trivial $g \in stab(A)$, there exist at least one set $B \subset V(G) \setminus A$ such that $u \sim^g v$ for all distinct $u, v \in B$. Consider $B = \{v_1, v_2, \ldots, v_l\}$. Take a $K_1 = \{x\}$ and join x with $v_1, v_2, \ldots, v_l$ by edges $xv_1, xv_2, \ldots, xv_l$. We call the new graph G'. This completes the construction of G'. We shall now find a non-fixing subset of G' with the largest cardinality. Since, $v_i \sim^g v_j$ $(i \neq j, 1 \leq i, j \leq l)$ in G and x is adjacent to $v_1, v_2, \ldots, v_l$ in G'. Therefore, we can find a $g' \in Aut(G')$ such that

$$g'(u) = \begin{cases} x & \text{if } u = x, \\ g(u) & \text{if } u \neq x \end{cases}$$

in G'. Clearly, $g' \in stab(x) \cap stab(A) = stab(\{x\} \cup A)$ and $v_i \sim^{g'} v_j$ $(i \neq j, 1 \leq i, j \leq l)$ in G'. Since, g' is non-trivial and A is a non-fixing set of G with the largest cardinality, $A \cup \{x\}$ is a non-fixing set of G' with the largest cardinality. Hence, by Remark 2.2(i), $fxd(G') = |A \cup \{x\}| + 1 = r + 1$. □

The following lemma is useful for finding the fixing number of a tree.

Lemma 2.9 ([4]). *Let T be a tree and $F \subset V(T)$, then F fixes T if and only if F fixes the end vertices of T.*

Theorem 2.10. *For every integers p and q with $2 \leq p \leq q$, there exists a graph G with $fix(G) = p$ and $fxd(G) = q$.*

Proof. For $p = q$, $G = K_{p+1}$ will have the desired property. So we consider $2 \leq p < q$. Consider a graph G obtained from a path $w_1, w_2, \dots, w_{q-p}$. Add $p+1$ vertices $u_1, u_2, \dots, u_{p+1}$ and $p+1$ edges $w_1u_1, w_1u_2, \dots, w_1u_{p+1}$ with w_1. Thus, $|V(G)| = q+1$. Consider the set $F \subset V(G)$, $F = \{u_1, u_2, \dots, u_p\}$, then F fixes the set of end vertices $\{u_1, u_2, \dots, u_p, u_{p+1}\}$ of G. As G is a tree and w_{p-q} is a fixed end vertex, therefore F fixes G by Lemma 2.9. Since F is a fixing set of G with the minimum cardinality, $fix(G) = |F| = p$. Also, $fxd(G) = q$ because $U = \{w_1, w_2, \dots, w_{q-p}, u_1, u_2, \dots, u_{p-1}\}$ is the largest non-fixing set with the cardinality $q - 1$. □

3. The fixing graph

Let G be a connected graph. The set of fixed vertices of G has no contribution in constructing the fixing sets of G, therefore we define a vertex set $S(G) = \{v \in V(G) : v \sim u \text{ for some } u(\neq v) \in V(G)\}$ (set of all vertices of G which are more than one vertex in their orbits). Also consider $V_s(G) = \{(u, v) : u \sim v \ (u \neq v) \text{ and } u, v \in V(G)\}$. If G is an asymmetric graph, then assume that $V_s(G) = \emptyset$. Let $x \in V(G)$, an arbitrary automorphism $g \in stab(x)$ is said to *fix a pair* $(u, v) \in V_s(G)$, if $u \not\sim^g v$. If $(u, v) \notin V_s(G)$, then $u \not\sim v$, and hence, question of fixing pair (u, v) by a $g \in stab(x)$, has no sense. In this section, we use r and s to denote $|S(G)|$ and $|V_s(G)|$ respectively. It is clear that $r \leq n$ and $\frac{r}{2} \leq s \leq \binom{r}{2} \leq \binom{n}{2}$ where s attains its lower bound in the later inequality in the case, when r is even and the pair (u, v) is only fixed by automorphisms in $stab\{u, v\}$ for all $(u, v) \in V_s(G)$. Consider the graph G_2 in Figure 1 where $r = 6$ and $s = 7$. G_2 has a fixed vertex v_1, $S(G_2) = \{v_2, v_3, v_4, v_5, v_6, v_7\}$ and $V_s(G_2) = \{(v_2, v_3), (v_4, v_5), (v_4, v_6), (v_4, v_7), (v_5, v_6), (v_5, v_7), (v_6, v_7)\}$. Since superset of a fixing set is also a fixing set, we are interested in a fixing set with the minimum cardinality. The following remarks tell us the relation between a fixing set F and $S(G)$.

Remark 3.1. Let G be a graph. A set $F \subset V(G)$ is a fixing set of G with the minimum cardinality, if $F \subset S(G)$ and an arbitrary $g \in stab(F)$ fixes $S(G)$.

The *Fixing Graph*, $D(G)$, of a graph G is a bipartite graph with bipartition $(S(G), V_s(G))$. A vertex $x \in S(G)$ is adjacent to a pair $(u, v) \in V_s(G)$, if $u \not\sim^g v$ for $g \in stab(x)$. Let $F \subseteq S(G)$, then $N_{D(G)}(F) = \{(x, y) \in V_s(G) | \ x \not\sim^g y \text{ for } g \in stab(F)\}$. In the fixing graph, $D(G)$, the minimum cardinality of a subset F

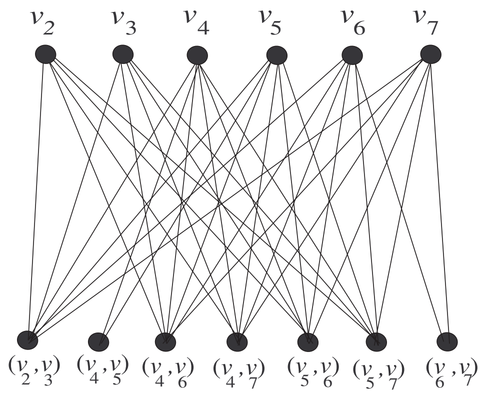


FIGURE 2. The fixing graph of G_2

of $S(G)$ such that $N_{D(G)}(F) = V_s(G)$ is the fixing number of G. Figure 2 shows the fixing graph of graph G_2 given in Figure 1. Since, $N_{D(G_2)}(v) \neq V_s(G_2)$ for all $v \in V(G_2)$ and $N_{D(G_2)}\{v_4, v_6\} = V_s(G_2)$, $\{v_4, v_6\}$ is a fixing set of G_2 with the minimum cardinality, and hence, $fix(G_2) = 2$.

Remark 3.2. Let G be graph and $F \subset S(G)$ be a fixing set of G, then $N_{D(G)}(F) = V_s(G)$.

Also, $\{v_1, v_2, v_3, v_4, v_5\}$ is a non-fixing set of G_2 with the largest cardinality. In fact, every non-fixing set with the largest cardinality must have fixed vertex v_1. Therefore, we have the following proposition.

Proposition 3.3. *Let G be a graph and A be a non-fixing subset of G with the largest cardinality. Then, A contains all fixed vertices of G.*

Proof. Let $x \in V(G)$ be an arbitrary fixed vertex of G. Suppose on contrary $x \notin A$. Then $stab(A \cup \{x\}) = stab(A) \cap stab(x) = stab(A) \cap Aut(G) = stab(A) \neq \{id\}$ (A is non-fixing set). Consequently, $A \cup \{x\}$ is a non-fixing set with the largest cardinality, a contradiction. □

Let t be the minimum number such that $1 \leq t \leq r$ and every t-subset F of $S(G)$ has $N_{D(G)}(F) = V_s(G)$, then t is helpful in finding the fixed number of a graph G. The following theorem gives a way of finding the fixed number of a graph using its fixing graph.

Theorem 3.4. *Let G be a graph of order n and t $(1 \leq t \leq r)$ be the minimum number such that every subset of $S(G)$ with the cardinality t, has neighborhood $V_s(G)$ in $D(G)$. Then,*

$$fxd(G) = t + |V(G) \setminus S(G)|.$$

Proof. We find a non-fixing subset T of $V(G)$ with the largest cardinality. By Proposition 3.3, T contains the set of fixed vertices $V(G) \setminus S(G)$. Moreover, by hypothesis, there is a subset U of $S(G)$ with the cardinality $t-1$, such that $N_{D(G)}(U) \neq V_s(G)$. Then, U is a non-fixing set of G, and hence, $\{V(G) \setminus S(G)\} \cup U$ is a non-fixing set. Also, $\{V(G) \setminus S(G)\} \cup U$ is a non-fixing set of G with the largest cardinality, because by hypothesis, a subset of $S(G)$ with the cardinality t, forms a fixing set of G. Further $\{V(G) \setminus S(G)\} \cap U = \emptyset$. Hence, by Remark 2.2(i),

$$fxd(G) = |V(G) \setminus S(G)| + |U| + 1 = |V(G) \setminus S(G)| + t.$$

□

In [8], we found an upper bound on the cardinality of the edge set $E(D(G))$ of the fixing graph $D(G)$ of a graph G.

Proposition 3.5 ([8])**.** *Let G be a k-fixed graph of order n, then*

$$|E(D(G))| \leq n(\binom{n}{2} - k + 1). \tag{3.1}$$

Now, we find a lower bound on $|E(D(G))|$.

Proposition 3.6. *Let G be a k-fixed graph of order n, then*

$$(\frac{r}{2})(r-k+1) \leq |E(D(G))|.$$

Proof. Let $z \in V_s(G)$ and A be a set of the vertices of $S(G)$ which are not adjacent to z. Since $N_{D(G)}(A) \neq V_s(G)$, A is a non-fixing set of G. Our claim is $deg_{D(G)}(z) \geq r-k+1$. Suppose $deg_{D(G)}(z) \leq r-k$, then $|A| \geq k$, which contradicts that $fxd(G) = k$ (A is non-fixing set with $|A| \geq k$). Thus, $deg_{D(G)}(z) \geq r-k+1$ and consequently,

$$(\frac{r}{2})(r-k+1) \leq s(r-k+1) \leq |E(D(G))|. \tag{3.2}$$

□

Thus, on combining (3.1) and (3.2) we get

$$(\frac{r}{2})(r-k+1) \leq |E(D(G))| \leq n(\binom{n}{2} - k + 1). \tag{3.3}$$

Theorem 3.7. *If G is a k-fixed graph and $|S(G)| = r$, then either $k \leq 3$ or $k \geq r-1$.*

Proof. For each $R \subseteq S(G)$, let $\overline{N}_{D(G)}(R) = V_s(G) \backslash N_{D(G)}(R)$. We claim that, if $R, T \subseteq S(G)$ with $|R| = |T| = k-1$ and $R \neq T$, then $\overline{N}_{D(G)}(R) \cap \overline{N}_{D(G)}(T) = \emptyset$. Otherwise, there exists a pair $\{y, z\} \in \overline{N}_{D(G)}(R) \cap \overline{N}_{D(G)}(T)$. Therefore, $\{y, z\} \notin N_{D(G)}(R \cup T)$, and hence, $R \cup T$ is not a fixing set of G. Since,

$R \neq T$, $|R \cup T| > |T| = k-1$, which contradicts that $fxd(G) = k$. Thus, $\overline{N}_{D(G)}(R) \cap \overline{N}_{D(G)}(T) = \emptyset$.

Since $fix(G) = k$, for each $R \subseteq S(G)$ with $|R| = k-1$, $\overline{N}_{D(G)}(R) \neq \emptyset$. Now, let $\Omega = \{R \subseteq S(G) : |R| = k-1\}$. Therefore,

$$|\bigcup_{R\in\Omega} \overline{N}_{D(G)}(R)| = \sum_{R\in\Omega} |\overline{N}_{D(G)}(R)| \geq \sum_{R\in\Omega} 1 = \binom{r}{k-1}.$$

On the other hand, $\bigcup_{R\in\Omega} \overline{N}_{D(G)}(R) \subseteq V_s(G)$. Hence, $|\bigcup_{R\in\Omega} \overline{N}_{D(G)}(R)| \leq s \leq \binom{r}{2}$. Consequently, $\binom{r}{k-1} \leq \binom{r}{2}$. If $r \leq 4$, then $k \leq 3$. Now, let $r \geq 5$. Thus, $2 \leq \frac{r+1}{2}$. We know that for each $a, b \leq \frac{n+1}{2}$, $\binom{r}{a} \leq \binom{r}{b}$ if and only if $a \leq b$. Therefore, if $k-1 \leq \frac{r+1}{2}$, then $k-1 \leq 2$, which implies $k \leq 3$. If $k-1 \geq \frac{r+1}{2}$, then $r-k+1 \leq \frac{r+1}{2}$. Since, $\binom{r}{r-k+1} = \binom{r}{k-1}$, we have $\binom{r}{r-k+1} \leq \binom{r}{2}$ and consequently, $r-k+1 \leq 2$, which yields $k \geq r-1$. □

4. The distance-transitive graph

We now study the fixed number in a class of graphs known as the distance-transitive graphs. A graph G is called distance-transitive, if $u, v, x, y \in V(G)$ satisfying $d(u, v) = d(x, y)$, then there exist an automorphism $g \in Aut(G)$ such that $u \sim^g x$ and $v \sim^g y$. For example, the complete graph K_n, the cyclic graph C_n, the Petersen graph, the Johnson graph etc, are distance-transitive. For more about distance-transitive graphs see [1]. In this section, we use the terminology as described in Section 3 related to the fixing graph $D(G)$ of a graph G. The following proposition given in [1] tells that the distance transitive graph does not have fixed vertices.

Proposition 4.1 ([1]). *A distance-transitive graph is vertex transitive.*

Thus, if G is a distance-transitive graph, then $S(G) = V(G)$, $r = n$ and $V_s(G)$ consists of all $\binom{n}{2}$ pairs of vertices of G (i.e., $s = \binom{n}{2}$).

Corollary 4.2. *Let G be a distance-transitive graph of order n. If G is k-fixed, then either $k \leq 3$ or $k \geq n-1$.*

Proof. Since $r = n$ for a distance-transitive graph, the result follows from Theorem 3.7. □

Moreover, an expression for bounds on $|E(D(G))|$ of a distance-transitive and k-fixed graph G can be obtained by putting $r = n$ and $s = \binom{n}{2}$ in (3.2) and use the result in (3.3), we get

$$\binom{n}{2}(n-k+1) \leq |E(D(G))| \leq n(\binom{n}{2} - k + 1). \tag{4.1}$$

The following two results given in [7] are useful in our later work.

Observation 4.3 ([7]). Let $n_1, \dots, n_r$ and n be positive integers, with $\sum_{i=1}^{r} n_i = n$. Then, $\sum_{i=1}^{r} \binom{n_i}{2}$ is minimum if and only if $|n_i - n_j| \leq 1$, for each $1 \leq i, j \leq r$.

Lemma 4.4 ([7]). *Let $n, p_1, p_2, q_1, q_2, r_1$ and r_2 be positive integers, such that $n = p_i q_i + r_i$ and $r_i < p_i$, for $1 \leq i \leq 2$. If $p_1 < p_2$, then $(p_1 - r_1)\binom{q_1}{2} + r_1\binom{q_1+1}{2} \geq (p_2 - r_2)\binom{q_2}{2} + r_2\binom{q_2+1}{2}$.*

We define a partition of $V(G)$ with respect to $v \in V(G)$, into the distance classes $\Psi_i(v)$ $(1 \leq i \leq e(v))$ defined as: $\Psi_i(v) = \{x \in V(G) |\, d(v, x) = i\}$.

Proposition 4.5. *Let G be a distance-transitive graph and $v, x, y \in V(G)$. Then $x, y \in \Psi_i(v)$ for some i $(1 \leq i \leq e(v))$ if and only if v is non-adjacent to the pair $(x, y) \in V_s(G)$ in $D(G)$.*

Proof. Let $x, y \in \Psi_i(v)$ for some i $(1 \leq i \leq e(v))$, then $d(v, x) = d(v, y) = i$ and by definition of distance-transitive graph, there exists an automorphism $g \in Aut(G)$ such that $v \sim^g v$ and $x \sim^g y$. Thus, $x \sim^g y$ by an automorphism $g \in stab(v)$ and consequently, the pair (x, y) is not adjacent to v in $D(G)$.

Conversely, suppose v is non-adjacent to pair $(x, y) \in V_s(G)$, then $x \sim^g y$ by an arbitrary $g \in stab(v)$. Since g is an isometry, $d(v, x) = d(g(v), g(x)) = d(v, y) = i$ (say). Thus, x, y are in the same distance class $\Psi_i(v)$. □

Proposition 4.6. *Let G be a distance-transitive graph of order n. If G is k-fixed, then for each $v \in V(G)$, $\deg_{D(G)}(v) = \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}$.*

Proof. By Propositon 4.5, the only pairs $(x, y) \in V_s(G)$ which are non-adjacent to $v \in V(G)$ are those in which both x, y belong to the same distance class $\Psi_i(v)$ for each i $(1 \leq i \leq e(v))$. So the number of such pairs in $V_s(G)$ which are not adjacent to v is $\sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}$. Therefore, $deg_{D(G)}(v) = \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}$ □

Thus, an expression for $|E(D(G))|$ can be obtained using Proposition 4.6, (4.2)

$$|E(D(G))| = \sum_{v \in V(G)} [\binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}] = n\binom{n}{2} - \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}$$

From (4.1) and (4.2) we obtain

$$n(k-1) \leq \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2} \leq \binom{n}{2}(k-1). \tag{4.3}$$

Theorem 4.7. *Let G be a distance-transitive graph of order n and diameter d. If G is k-fixed, then $k \geq \frac{n-1}{d}$.*

Proof. Note that, for each $v \in V(G)$, $|\bigcup_{i=1}^{e(v)} \Psi_i(v)| = n-1$. For $v \in V(G)$, let $n-1 = q(v)e(v) + r(v)$, where $0 \leq r(v) < e(v)$. Then, by Observation 4.3, $\sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}$ is minimum if and only if $|\,|\Psi_i(v)| - |\Psi_j(v)|\,| \leq 1$, where $1 \leq i, j \leq e(v)$. This condition will be satisfied, if there are $r(v)$ distance classes having $q(v)+1$ vertices and $e(v) - r(v)$ distance classes having $q(v)$ vertices. Thus, the number of the pairs of vertices in $\Psi_i(v)$ having $q(v)+1$ vertices is $r(v)\binom{q(v)+1}{2}$ and the number of the pairs of vertices in $\Psi_i(v)$ having $q(v)$ vertices is $(e(v)-r(v))\binom{q(v)}{2}$. Thus,

$$(4.4) \qquad (e(v)-r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2} \leq \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2}.$$

Let $w \in V(G)$ with $e(w) = d$, $r(w) = r$, and $q(w) = q$, then $n-1 = qd + r$. Since, for each $v \in V(G)$, $e(v) \leq e(w)$, by Lemma 4.4,
$(d-r)\binom{q}{2} + r\binom{q+1}{2} \leq (e(v)-r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2}$. Therefore,

$$n[(d-r)\binom{q}{2} + r\binom{q+1}{2}] \leq \sum_{v \in V(G)} [(e(v)-r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2}].$$

Thus, by relation (4.3) and (4.4)

$$n[(d-r)\binom{q}{2} + r\binom{q+1}{2}] \leq \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Psi_i(v)|}{2} \leq \binom{n}{2}(k-1).$$

Hence, $q[(d-r)(q-1)+r(q+1)] \leq (n-1)(k-1)$, which implies, $q[(r-d)+(d-r)q+r(q+1)] \leq (n-1)(k-1)$. Therefore, $q(r-d)+q(n-1) \leq (n-1)(k-1)$. Since, $q = \lfloor \frac{n-1}{d} \rfloor$, we have

$$k-1 \geq q + q\frac{r-d}{n-1} = q + \frac{qr}{n-1} - \frac{qd}{n-1} = q + \frac{qr}{n-1} - \frac{\lfloor \frac{n-1}{d} \rfloor d}{n-1} \geq q + \frac{qr}{n-1} - 1.$$

Thus, $k \geq \lfloor \frac{n-1}{d} \rfloor + \frac{qr}{n-1}$. Note that, $\frac{qr}{n-1} \geq 0$. If $\frac{qr}{n-1} > 0$, then $k \geq \lceil \frac{n-1}{d} \rceil$, since k is an integer. If $\frac{qr}{n-1} = 0$, then $r = 0$ and consequently, d divides $n-1$. Thus, $\lfloor \frac{n-1}{d} \rfloor = \lceil \frac{n-1}{d} \rceil$. Therefore, $k \geq \lceil \frac{n-1}{d} \rceil \geq \frac{n-1}{d}$. □

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