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IMPROVED LOGARITHMIC-GEOMETRIC MEAN INEQUALITY AND ITS APPLICATION

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Abstract. In this short note, we present a refinement of the logarithmic-geometric mean inequality. As an application of our result, we obtain an operator inequality associated with the geometric and logarithmic means.
Keywords: Taylor's theorem, logarithmic mean, geometric mean, operator inequality.
MSC(2010): Primary: 47A63; Secondary: 26D07, 26D15.

1. Introduction

The logarithmic mean of positive numbers a, b is defined by

$$L(a,b) = \int_0^1 a^t b^{1-t} dt = \begin{cases} \dfrac{a-b}{\log a - \log b}, & a \neq b \\ a \quad , & a = b \end{cases},$$

which is of interest in chemical engineering, statistics, and thermodynamics [2, 1]. It is well-known that

$$\sqrt{ab} \le L(a,b). \tag{1.1}$$

For two invertible positive operators A and B, the weighted geometric mean $A\#_t B$ is defined by

$$A\#_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}.$$

When $t = 1/2$, we write $A\#B$ for brevity. The relative operator entropy

$$S(A|B) = A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

which was introduced by Fujii and Kamei [5], has been generalized and extended in various directions [3, 6, 7, 9, 10, 8, 11, 12].

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Let A and B be invertible positive operators. It is known [1, p. 230] that

$$A\#B \leq \int_0^1 A\#_t B dt, \tag{1.2}$$

which is an operator version of the inequality (1.1).

In this short note, we obtain a refinement of the inequality (1.1). As an application of our result, we present an improvement of the inequality (1.2).

2. Main results

In this section, we give an operator inequality associated with geometric and logarithmic means. To do this, we need the following lemma [4, Remark 1].

Lemma 2.1. *Let $a > 0$. If $\frac{1}{2} - \frac{\sqrt{3}}{6} \leq x \leq \frac{1}{2} + \frac{\sqrt{3}}{6}$, then*

$$\frac{a^x + a^{1-x}}{2} \leq \int_0^1 a^t dt, \tag{2.1}$$

where the expression $\frac{a^x + a^{1-x}}{2}$ is the Heinz mean of a and $b = 1$.

Theorem 2.2. *Let $a > 0$. Then*

$$\left(1 + \frac{(\log a)^2}{24}\right)\sqrt{a} \leq \int_0^1 a^t dt. \tag{2.2}$$

Proof. Let

$$f(x) = \frac{a^x + a^{1-x}}{2}, \quad \frac{1}{2} - \frac{\sqrt{3}}{6} \leq x \leq \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

It is easy to see that the function f is twice differentiable. Simple calculations show that

$$f'(x) = \frac{a^x - a^{1-x}}{2} \times \log a,$$

and

$$f''(x) = f(x) \times (\log a)^2.$$

So, for any given $x \in \left(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right)$, by Taylor's theorem, there exists $\xi \in \left(x, \frac{1}{2}\right)$ such that

$$f(x) = \sqrt{a} + \frac{(\log a)^2}{2}\frac{a^\xi + a^{1-\xi}}{2}\left(x - \frac{1}{2}\right)^2. \tag{2.3}$$

It follows from (2.1) and (2.3) that

$$\sqrt{a}+\frac{(\log a)^2}{2}\frac{a^{\xi}+a^{1-\xi}}{2}\left(x-\frac{1}{2}\right)^2\leq\int_0^1 a^t dt.$$

So, the arithmetic-geometric mean inequality and the last inequality complete the proof. □

Remark 2.3. Let $a,b>0$. Replacing a by $\frac{a}{b}$ in inequality (2.2) and then multiplying b on both sides, we have

$$\left(1+\frac{(\log a-\log b)^2}{24}\right)\sqrt{ab}\leq\int_0^1 a^t b^{1-t}dt=L(a,b),$$

which is a refinement of the inequality (1.1).

Next, as an application of the inequality (2.2), we present an improvement of the inequality (1.2).

Theorem 2.4. *Let C and B be bounded linear operators. If B is positive and C is invertible, then*

(2.4)

$$\begin{aligned}C^*\left(\left(C^{-1}\right)^*BC^{-1}\right)^{1/2}C &+ \frac{1}{24}K\left(C^{-1}\left(\left(C^{-1}\right)^*BC^{-1}\right)^{1/2}\left(C^{-1}\right)^*\right)K\\ &\leq \int_0^1 C^*\left(\left(C^{-1}\right)^*BC^{-1}\right)^t Cdt,\end{aligned}$$

where

$$K=C^*\log\left(\left(C^{-1}\right)^*BC^{-1}\right)C.$$

Proof. First assume that B is an invertible positive operator. The general case will follow from the special one by a continuity argument. Let

$$T=\left(C^{-1}\right)^*BC^{-1}.$$

By the inequality (2.2), we have

$$T^{1/2}+\frac{1}{24}(\log T)T^{1/2}(\log T)\leq\int_0^1 T^t dt.$$

Multiplying by C^* on the left-hand side and C on the right-hand side, we obtain

$$C^*T^{1/2}C+\frac{1}{24}C^*(\log T)T^{1/2}(\log T)C\leq\int_0^1 C^*T^tCdt,$$

which is equivalent to

$$\begin{aligned}C^*\left(\left(C^{-1}\right)^*BC^{-1}\right)^{1/2}C &+ \frac{1}{24}K\left(C^{-1}\left(\left(C^{-1}\right)^*BC^{-1}\right)^{1/2}\left(C^{-1}\right)^*\right)K\\ &\leq \int_0^1 C^*\left(\left(C^{-1}\right)^*BC^{-1}\right)^t Cdt,\end{aligned}$$

where

$$K = C^* \log\left(\left(C^{-1}\right)^* BC^{-1}\right) C.$$

This completes the proof. □

Remark 2.5. Let A be an invertible positive operator. Putting $C = A^{1/2}$ in the inequality (2.4), we get

$$A\#B + \frac{1}{24} S(A|B) A^{-1} (A\#B) A^{-1} S(A|B) \leq \int_0^1 A\#_t B dt,$$

which is a refinement of the inequality (1.2).

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