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ON SUBGROUPS OF TOPOLOGIZED FUNDAMENTAL GROUPS AND GENERALIZED COVERINGS

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ABSTRACT. In this paper, we are interested in studying subgroups of topologized fundamental groups and their influences on generalized covering maps. More precisely, we find some relationships between generalized covering subgroups and the other famous subgroups of the fundamental group equipped with the compact-open topology and the whisker topology. Moreover, we present some conditions under which generalized coverings, semicoverings and coverings are equal.

Keywords: Generalized covering, quasitopological fundamental group, whisker topology, semilocally small generated, homotopically Hausdorff.

MSC(2010): Primary: 57M10; Secondary: 57M12, 57M05, 55Q05.

1. Introduction

1.1. Motivation and history. We recall that a continuous map $p : \tilde{X} \rightarrow X$ is a covering map if every point of X has an open neighborhood which is evenly covered by p . It is well-known that the induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}) \rightarrow \pi_1(X, x)$ is a monomorphism and so $\pi_1(\tilde{X}, \tilde{x}) \cong p_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$. Based on some recent works in [13, 19, 21, 20, 26] there is a chain of some effective subgroups of the fundamental group $\pi_1(X, x)$ as follows:

$$(1.1) \quad \{e\} \leq \pi_1^s(X, x) \leq \pi_1^{sg}(X, x) \leq \overline{\pi_1^{sg}(X, x)} \leq \tilde{\pi}_1^{sp}(X, x) \\ \leq \pi_1^{sp}(X, x) \leq p_*\pi_1(\tilde{X}, \tilde{x}) \leq \pi_1(X, x),$$

where $\pi_1^s(X, x)$ is the subgroup of all small loops at x [26]; $\pi_1^{sg}(X, x)$ is the subgroup of all small generated loops, i.e., the subgroup generated by the set of lollipops $\{[\alpha * \beta * \alpha^{-1}] \mid [\beta] \in \pi_1^s(X, \alpha(1)), \alpha \in P(X, x)\}$, where $P(X, x)$ is the space of all paths with initial point x [26]; $\tilde{\pi}_1^{sp}(X, x)$ is the Spanier group of X , the intersection of the Spanier subgroups relative to open covers of X [12,

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Definition 2.3]; and $\tilde{\pi}_1^{sp}(X, x)$ is the path Spanier group, i.e., the intersection of all path Spanier subgroups $\tilde{\pi}_1(\mathcal{V}, x)$ where \mathcal{V} is a path open cover of X [25, Section 3]. Note that $\pi_1(X, x)$ in the chain (1.1) is equipped with the quotient topology induced by the compact-open topology.

Some authors extended the notion of covering maps and introduced rigid coverings [2], semicoverings [3] and generalized coverings [6, 13]. These generalizations focus on keeping some properties of covering maps and eliminating the evenly covered property. Brazas [3] defined semicoverings by removing the evenly covered property and keeping local homeomorphism and the unique path and homotopy lifting properties. For generalized coverings, the local homeomorphism is replaced with the unique lifting property (see [6, 13]). In each case, one of the interesting problems is to classify covering, semicovering and generalized covering spaces with respect to subgroups of the fundamental group. A subgroup H of the fundamental group $\pi_1(X, x)$ is called a covering, semicovering and generalized covering subgroup if there is a covering, semicovering and generalized covering map $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ such that $H = p_*\pi_1(\tilde{X}, \tilde{x})$, respectively.

Brazas [6, Theorem 2.36] showed that the intersection of any collection of generalized covering subgroups of $\pi_1(X, x)$ is also a generalized covering subgroup. We give another proof for the above fact in Corollary 2.13 which is simpler than that of Brazas and denote the intersection of all generalized covering subgroups of $\pi_1(X, x)$ by $\pi_1^{gc}(X, x)$. In Section 2, we find the location of the subgroup $\pi_1^{gc}(X, x)$ in the chain (1.1). In fact, we prove that $\pi_1^{sg}(X, x) \leq \pi_1^{gc}(X, x) \leq \pi_1^{sg}(X, x)$ for a locally path connected space X . Note that $\pi_1^{sg}(X, x)$ is the topological closure of the subgroup $\pi_1^{sg}(X, x)$ in the quasitopological fundamental group $\pi_1^{qtop}(X, x)$. By the quasitopological fundamental group $\pi_1^{qtop}(X, x)$ we mean the fundamental group $\pi_1(X, x)$ equipped with the quotient topology induced by the compact-open topology on the space of continuous based loops at x which makes $\pi_1^{qtop}(X, x)$ a quasitopological group (see [2, 4]). Moreover, we give some examples in which $\pi_1^{sg}(X, x) \neq \pi_1^{gc}(X, x)$ and $\pi_1^{gc}(X, x) \neq \pi_1^{sp}(X, x)$.

It seems interesting to find equivalent conditions for a subgroup H of the fundamental group $\pi_1(X, x)$ to be a covering, semicovering or generalized covering subgroup. Based on some classical results of Spanier [23] every covering subgroup contains the Spanier group $\pi_1^{sp}(X, x)$. Hence for every subgroup $H \leq \pi_1(X, x)$ if $\pi_1^{sp}(X, x) \cap H \neq \pi_1^{sp}(X, x)$, then H can not be a covering subgroup. If $\pi_1^{sp}(X, x) \cap H = \pi_1^{sp}(X, x)$, then by some results in [12, 21] H is a covering subgroup if and only if H contains an open normal subgroup of $\pi_1^{qtop}(X, x)$. For semicovering subgroups, based on some results in [5, 25], one can state a similar result, i.e., if $\tilde{\pi}_1^{sp}(X, x) \cap H \neq \tilde{\pi}_1^{sp}(X, x)$, then H can not be a semicovering subgroup. For locally path connected spaces, if $\tilde{\pi}_1^{sp}(X, x) \cap H = \tilde{\pi}_1^{sp}(X, x)$, then H is a semicovering subgroup if and only

if H is an open subgroup of $\pi_1^{qtop}(X, x)$. In Section 2, we intend to give a similar result for generalized covering subgroups. In fact, we show that if $\pi_1^{gc}(X, x) \cap H \neq \pi_1^{gc}(X, x)$, then H is not a generalized covering subgroup. If $\pi_1^{gc}(X, x) \cap H = \pi_1^{gc}(X, x)$, then H is a generalized covering subgroup if and only if $H = (p_H)_* \pi_1(\tilde{X}_H, e_H)$, where $p_H : \tilde{X}_H \rightarrow X$ is the end point projection (see Definition 1.3 for the definition of $p_H : \tilde{X}_H \rightarrow X$). Note that Brazas [6, Lemma 5.9] gave two equivalent conditions for the equality $H = (p_H)_* \pi_1(\tilde{X}_H, e_H)$.

It is easy to show that the class of all coverings, semicoverings and generalized coverings over X forms a category denoted by $\text{Cov}(X)$, $\text{SCov}(X)$ and $\text{GCov}(X)$, respectively. By definition, $\text{Cov}(X)$ is a subcategory of $\text{SCov}(X)$. Brazas [6, 5] showed that $\text{Cov}(X) = \text{SCov}(X) = \text{GCov}(X)$ for locally path connected, semi locally simply connected space X . Torabi et al. [24] showed that the first equality can be extended for locally path connected, semilocally small generated spaces, i.e., $\text{Cov}(X) = \text{SCov}(X)$. In Section 2, we show that $\text{SCov}(X) = \text{GCov}(X)$ for locally path connected, semilocally small generated space X .

Spanier [23, p. 82] introduced a different topology on the universal path space \tilde{X} which has been called by Brodskiy et al. [9], the *whisker topology*, denoted by \tilde{X}^{wh} . Note that the topology on the fundamental group $\pi_1(X, x_0)$ inherits as a subspace of the universal path space \tilde{X} with any topology. The fundamental group with the whisker topology is denoted by $\pi_1^{wh}(X, x_0)$. They showed that $\pi_1^{wh}(X, x_0)$ is a topological group when the inverse map is continuous [9, Proposition 4.20]. Although $\pi_1^{wh}(X, x_0)$ is not a quasitopological group, in general, we show that it is a homogenous space (Proposition 3.2). In Section 3, after recalling the whisker topology and presenting some of its properties, we intend to describe its influence on the notion of generalized covering subgroups. Trying to classify the generalized covering subgroups of the fundamental group, we give an example to show that the whisker topology is not suitable for the subject. Moreover, we find some properties of the whisker topology on the quotient space $\pi_1^{wh}(X, x_0)/H$, where H is a subgroup of $\pi_1(X, x_0)$.

In Section 4, we introduce some topological properties in order to preserve the categorical equality between $\text{Cov}(X)$, $\text{SCov}(X)$ and $\text{GCov}(X)$. More precisely, we introduce the notions *semilocally path H -connectedness* and *semilocally H -connectedness* and give their relations to open subgroups of $\pi_1^{qtop}(X, x_0)$ and $\pi_1^{wh}(X, x_0)$ in order to show that $\text{SCov}(X) = \text{GCov}(X)$ if and only if X is semilocally path H -connected, and $\text{GCov}(X) = \text{Cov}(X)$ if and only if X is semilocally H -connected, when $H = \pi_1^{gc}(X, x_0)$.

1.2. Preliminaries. In this section, for convenience of the reader, we intend to give some technical definitions and results which are used in the sequel.

Definition 1.1. Let (X, x_0) be a pointed topological space and H be any subgroup of the fundamental group $\pi_1(X, x_0)$.

- (1) The space X is called *homotopically Hausdorff relative to H* if for every path $\alpha \in P(X, x_0)$ and every $g \in \pi_1(X, x_0)$ with $g \notin H$, there is an open neighborhood U of $\alpha(1)$ for which there is no loop $\gamma : (I, \dot{I}) \rightarrow (U, \alpha(1))$ such that $[\alpha * \gamma * \alpha^{-1}] \in Hg$.
- (2) The space X is called *homotopically Hausdorff* if it is homotopically Hausdorff relative to the trivial subgroup $H = 1$.
- (3) The space X is called *homotopically path Hausdorff relative to H* if for every pair of paths $\alpha, \beta \in P(X, x_0)$ with $\alpha(1) = \beta(1)$ and $[\alpha * \beta^{-1}] \notin H$, there is an integer $n \geq 1$ and a sequence of open sets $U_1, U_2, \dots, U_{2^{n-1}}$ with $\alpha([\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}]) \subseteq U_j$ such that if $\gamma \in P(X, x_0)$ is another path satisfying $\gamma([\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}]) \subseteq U_j$ for $1 \leq j \leq 2^{n-1}$ and $\gamma(\frac{j-1}{2^{n-1}}) = \alpha(\frac{j-1}{2^{n-1}})$ for $1 \leq j \leq 2^{n-1}$, then $[\gamma * \beta^{-1}] \notin H$.
- (4) The space X is called *homotopically path Hausdorff* if it is homotopically path Hausdorff relative to the trivial subgroup $H = 1$.

Note that the original definition [12, Definition 2.10] has been stated for only the trivial subgroup.

A non-trivial loop $\alpha : (I, \dot{I}) \rightarrow (X, x)$ is called small if there exists a representative of the homotopy class $[\alpha] \in \pi_1(X, x)$ in every open neighborhood U of x .

Definition 1.2. A non-simply connected space X is called:

- (1) A *small loop space* if for every $x \in X$, any loop $\alpha : (I, \dot{I}) \rightarrow (X, x)$ is small, i.e., $\pi_1(X, x) = \pi_1^s(X, x)$ [26, Definition 2].
- (2) A *semilocally small loop space* if for each $x \in X$, there exists an open neighborhood U of x such that $i_*\pi_1(U, y) = \pi_1^s(X, y)$, for all $y \in U$ [20, Definition 4.1].
- (3) A *small generated space* if $\pi_1(X, x) = \pi_1^{sg}(X, x)$, for every $x \in X$ [24, Definition 2.10].
- (4) A *semilocally small generated space* if for each $x \in X$, there exists an open neighborhood U of x such that $i_*\pi_1(U, x) = \pi_1^{sg}(X, x)$ [24, Definition 3.5].
- (5) A *Spanier space* if $\pi_1(X, x) = \pi_1^{sp}(X, x)$, for an arbitrary point $x \in X$ [19, Definition 2.1].
- (6) A *semilocally Spanier space* if for each $x \in X$, there exists an open neighborhood U of x such that $i_*\pi_1(U, x) = \pi_1^{sp}(X, x)$ [19, Definition 4.1].

In order to find equivalent conditions for a subgroup H of the fundamental group $\pi_1(X, x_0)$ to be a generalized covering subgroup we need the following concepts.

Definition 1.3. Let H be a subgroup of $\pi_1(X, x_0)$ and $P(X, x_0)$ be the path space on X beginning at x_0 . Consider an equivalence relation \sim_H on $P(X, x_0)$ as follows.

$\alpha_1 \sim_H \alpha_2$ if and only if $\alpha_1(1) = \alpha_2(1)$ and $[\alpha_1 * \alpha_2^{-1}] \in H$. The equivalence class of α is denoted by $\langle \alpha \rangle_H$. One can consider the quotient space $\tilde{X}_H = P(X, x_0) / \sim_H$ and the map $p_H : (\tilde{X}_H, e_H) \rightarrow (X, x_0)$ defined by $\langle \alpha \rangle_H \mapsto \alpha(1)$, where e_H is the class of constant path at x_0 .

If $\alpha \in P(X, x_0)$ and U is an open neighborhood of $\alpha(1)$, then a continuation of α in U is a path $\beta = \alpha * \gamma$, where γ is a path in U with $\gamma(0) = \alpha(1)$. Put $N(\langle \alpha \rangle_H, U) = \{\langle \beta \rangle_H \in \tilde{X}_H \mid \beta \text{ is a continuation of } \alpha \text{ in } U\}$. It is well known that the subsets $N(\langle \alpha \rangle_H, U)$ form a basis for a topology on \tilde{X}_H for which the function $p_H : (\tilde{X}_H, e_H) \rightarrow (X, x_0)$ is continuous (see [23, p. 82]). Brodskiy et al. [9] called this topology on \tilde{X}_H the whisker topology.

Note that the map $p_H : (\tilde{X}_H, e_H) \rightarrow (X, x_0)$ has the path lifting property and if X is path connected, then p_H is surjective (see [23, p. 83]).

Remark 1.4. In this paper, we use the categories $\text{Cov}(X)$, $\text{SCov}(X)$ and $\text{GCov}(X)$ the notions such as classical and categorical universal objects of them. It seems necessary to distinguish these notions. For example, it is well-known that the classical universal object in the category $\text{Cov}(X)$ is the simply connected covering of X , if any. On the other hand, a categorical universal covering is a covering $p : \tilde{X} \rightarrow X$ with the property that for every covering $q : \tilde{Y} \rightarrow X$ with a path connected space \tilde{Y} there exists a unique (up to equivalence) covering $f : \tilde{X} \rightarrow \tilde{Y}$ such that $q \circ f = p$. The spaces having the classical universal covering are semilocally simply connected and the spaces having the categorical universal covering are called coverable [21].

Throughout the paper, the topological space X is assumed to be connected and locally path connected.

2. Generalized covering subgroups of the quasitopological fundamental group

Universal generalized covering map was introduced by Fischer and Zastrow [13] and extended to generalized covering maps by Brazas [6]. The definition is based on removing the evenly covered property and keeping the unique lifting property from classical covering maps.

Definition 2.1. A continuous map $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ has the *unique lifting property* (UL for abbreviation) if for every connected, locally path connected space (Y, y_0) and every continuous map $f : (Y, y_0) \rightarrow (X, x_0)$ with $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ for $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique continuous map $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$. If \tilde{X} is a connected, locally path connected space and $p : \tilde{X} \rightarrow X$ has UL property, then p and \tilde{X} are called a generalized covering map and a generalized covering space for X , respectively. Moreover, a continuous map $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ has the *unique path lifting*

property (UPL for abbreviation) if it has the UL property with respect to every path $\alpha : I \rightarrow X$.

Remark 2.2. It is clear that if a map $p : \tilde{X} \rightarrow X$ has UL property, then it has also UPL property, but the converse does not hold, in general. It will be interesting to find maps which support the equality of UL and UPL properties. Brazas [6, Lemma 5.9] showed that the maps $p_H : (\tilde{X}_H, e_H) \rightarrow (X, x_0)$, which introduced in Definition 1.3, have this condition, i.e., for any subgroup $H \leq \pi_1(X, x_0)$ the map $p_H : (\tilde{X}_H, e_H) \rightarrow (X, x_0)$ has UL property if and only if it has UPL property.

Definition 2.3. A subgroup H of the fundamental group $\pi_1(X, x_0)$ is called a generalized covering subgroup of $\pi_1(X, x_0)$ if there is a generalized covering map $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ such that $H = p_*\pi_1(\tilde{X}, \tilde{x}_0)$. Brazas [6, Theorem 2.36] showed that the intersection of any collection of generalized covering subgroups of $\pi_1(X, x_0)$ is also a generalized covering subgroup. We denote the intersection of all generalized covering subgroups of $\pi_1(X, x_0)$ by $\pi_1^{gc}(X, x_0)$.

In the following proposition, we show that $\pi_1^{gc}(X, x_0)$ contains $\pi_1^{sg}(X, x_0)$ as a subgroup of $\pi_1(X, x_0)$.

Proposition 2.4. For a pointed topological space (X, x_0) , we have

$$\pi_1^{sg}(X, x_0) \leq \pi_1^{gc}(X, x_0).$$

Proof. Let g be any generator of $\pi_1^{sg}(X, x_0)$. Then $g = [\alpha * \omega * \alpha^{-1}]$, where α is a path beginning at x_0 and ω is a small loop at $\alpha(1)$. Suppose $g \notin H$ for a generalized covering subgroup H of $\pi_1(X, x_0)$. Since H is a generalized covering subgroup of $\pi_1(X, x_0)$, by [13, Proposition 6.4], X is homotopically Hausdorff relative to H and so there exists an open neighborhood U_g of $\alpha(1)$ such that there is no loop $\gamma : (I, \dot{I}) \rightarrow (U_g, \alpha(1))$ with $[\alpha * \gamma * \alpha^{-1}] \in Hg$. Since ω is a small loop at $\alpha(1)$, there exists $\omega' : (I, \dot{I}) \rightarrow (U_g, \alpha(1))$ such that $[\omega] = [\omega']$. Thus $g = [\alpha * \omega * \alpha^{-1}] = [\alpha * \omega' * \alpha^{-1}]$ which implies that $[\alpha * \omega' * \alpha^{-1}] \in Hg$. This is a contradiction to X being homotopically Hausdorff relative to H , hence $\pi_1^{sg}(X, x_0) \leq H$. Since H is an arbitrary generalized covering subgroup, then $\pi_1^{sg}(X, x_0) \leq \pi_1^{gc}(X, x_0)$. \square

In the case of locally path connected and semilocally simply connected spaces, the subgroups in the chain $(*)$ which are contained in $\pi_1^{sp}(X, x_0)$ are equal since one can show that $\pi_1^{sp}(X, x_0) = 1$. This result comes from the existence of the classical universal covering space. Note that for $\pi_1^{sp}(X, x_0) = 1$, the semilocally simply connectedness is not a necessary condition. For instance, Fischer and Zastrow [13, Example 4.15] showed that the Spanier group of the Hawaiian earring, HE, is trivial which implies the existence of the generalized universal covering space for HE, hence the trivial subgroup is a generalized universal covering subgroup, so $\pi_1^{gc}(HE, x_0) = 1$, where x_0 is the wedge point

(see [6, Theorem 2.36]). Using this fact and the chain (*) it is easy to see that HE is a homotopically Hausdorff space and $\pi_1^{sg}(HE, x_0) = \pi_1^{gc}(HE, x_0) = 1$.

Example 2.5. To present an example to show that $\pi_1^{sg}(X, x_0)$ may be a proper subgroup of $\pi_1^{gc}(X, x_0)$, consider the space RX described in [27, Definition 7]. By [27, Theorem 16], RX is a metric, path connected, locally path connected and homotopically Hausdorff space and so by [12, 26] $\pi_1^{sg}(X, x_0) = 1$, which does not admit a generalized universal covering space, i.e., $\pi_1^{gc}(X, x_0) \neq 1$.

In the following theorem, using the previous proposition, we find the location of the subgroup $\pi_1^{gc}(X, x_0)$ in the chain (*).

Theorem 2.6. *If (X, x_0) is a locally path connected space, then there exists the following chain of subgroups of $\pi_1(X, x_0)$.*

$$\begin{aligned} \{e\} \leq \pi_1^s(X, x_0) \leq \pi_1^{sg}(X, x_0) \leq \pi_1^{gc}(X, x_0) \leq \overline{\pi_1^{sg}(X, x_0)} \\ \leq \tilde{\pi}_1^{sp}(X, x_0) \leq \pi_1^{sp}(X, x_0) \leq \pi_1(X, x_0). \end{aligned}$$

Proof. Recall from [24, Theorem 2.2] that $\overline{\{e\}} = \overline{\pi_1^{sg}(X, x_0)}$. Using Proposition 2.4, it is enough to show that $\pi_1^{gc}(X, x_0) \leq \overline{\{e\}}$. Recall from [1, Proposition 1.4.13] that the closure of a subgroup from a quasitopological group is also a subgroup. Since X is locally path connected and $\{e\}$ is a closed subgroup of $\pi_1^{qtop}(X, x_0)$, by [7, Theorem 11] $p_H : \tilde{X}_H \rightarrow X$ has *UPL* property for $H = \{e\}$. Now from Remark 2.2, $\overline{\{e\}}$ is a generalized covering subgroup and so it is done. \square

Example 2.7. As an example of a locally path connected space which supports the inequality $\pi_1^{gc}(X, x_0) \neq \pi_1^{sp}(X, x_0)$, consider the space A in [11, Fig. 1]. It consists of a rotated topologist's sine curve about its limiting arc, the "central axis", where this surface tends to, and a system of horizontal arc is attached to them so that they become countable dense of radial cross sections to make the space locally path connected at the central arc. Let a be any point of central axis. Fischer et al. [12, Proposition 3.2] showed that $\pi_1^{sp}(A, a) \neq 1$ and A is a homotopically path Hausdorff space. Also, they proved that every homotopically path Hausdorff space has the generalized universal covering space [12, Proof of Theorem 2.9]. It means that $\pi_1^{gc}(A, a) = 1$.

Brazas [6, Lemma 5.10] gave the following relationship between generalized covering maps and the spaces \tilde{X}_H .

Lemma 2.8. *Suppose $\hat{p} : (\hat{X}, \hat{x}) \rightarrow (X, x_0)$ is a generalized covering map with $\hat{p}_*(\pi_1(\hat{X}, \hat{x})) = H$. Then there is a homeomorphism $h : (\hat{X}, \hat{x}) \rightarrow (\tilde{X}_H, e_H)$ such that $p_H \circ h = \hat{p}$.*

A semicovering map introduced by Brazas in [3, Definition 3.1] as a local homeomorphism with continuous lifting of paths and homotopies. Then he

[5, Definition 2.4] simplified this definition by showing that the property of continuous lifting of homotopies can obtain from the continuous lifting of paths. Further simplification could be done, such as Kowkabi et al. [18, Theorem 3.2] by showing that the continuous lifting of paths can be replaced with the UPL property. Also, this result can be concluded from a new definition of semicoverings which Klevdal [16, Definition 7] presented and showed that these definitions are equivalent. In this paper, we use a continuous surjective local homeomorphism with *UPL* property as the definition of a semicovering maps.

The following lemma which has been proved in [17, Theorem 3.2], presents a relationship between semicoverings and generalized coverings. We give another proof using some properties of the topologized fundamental group.

Lemma 2.9. *Let $p: \tilde{X} \rightarrow X$ be a semicovering map. Then p has UL property.*

Proof. Let $H = p_*\pi_1(\tilde{X}, \tilde{x}) \leq \pi_1(X, x_0)$. Recall from [5, Theorem 3.5] that H is an open subgroup of $\pi_1^{qtop}(X, x_0)$. Since $\pi_1^{qtop}(X, x_0)$ is a quasitopological group [4], H is also a closed subgroup of $\pi_1^{qtop}(X, x_0)$ [1, Theorem 1.3.5]. Therefore, by [7, Theorem 11] $p_H: \tilde{X}_H \rightarrow X$ has UPL property. Now Remark 2.2 guaranties that H is a generalized covering subgroup. Hence it is done. \square

The above lemma states that every semicovering map is a generalized covering map for connected locally path connected spaces. Clearly, the converse does not hold in general (see [13, Example 4.15]). It seems interesting to find some topological properties which guarantee the converse.

Fischer and Zastrow [13, Lemma 2.3] showed that the map $p_H: \tilde{X}_H \rightarrow X$ is open when $H = 1$. One can prove the same result for any subgroup $H \leq \pi_1(X, x_0)$ with a similar proof; “The projection map $p_H: \tilde{X}_H \rightarrow X$ is open if and only if X is locally path connected”. Thus for locally path connected spaces, the projection map p_H is a local homeomorphism if and only if it is locally injective (a map $p: Y \rightarrow X$ is called locally injective if for every $y \in Y$, there exists an open neighborhood U of y such that the restriction of p on U is injective).

On the other hand, every map $p: \tilde{X} \rightarrow X$ with UL property is surjective when X is a path connected space. Hence we have the following proposition.

Proposition 2.10. *For a connected locally path connected space X , a generalized covering map $p: \tilde{X} \rightarrow X$ is a semicovering map if and only if it is locally injective.*

Moreover, Fischer and Zastrow [14, Lemma 5.5] showed that local homeomorphismness is a sufficient condition for the endpoint projection $p_H: \tilde{X}_H \rightarrow X$ to have UPL property. Hence the following result holds.

Proposition 2.11. *For a connected locally path connected space X and any subgroup $H \leq \pi_1(X, x_0)$, the map $p_H: \tilde{X}_H \rightarrow X$ is a semicovering map if and only if it is locally injective.*

In the following lemma, we give necessary and sufficient conditions on a subgroup H of $\pi_1(X, x_0)$ to be a generalized covering subgroup. Note that Fischer and Zastrow [13, Lemma 2.8] stated a result similar to the following for the trivial subgroup $H = 1$. Then Brodskiy et al. [8, Proposition 2.18] extended the one side of the inclusion $(p_H)_*\pi_1(\tilde{X}_H, e_H) \leq H$ for every generalized covering subgroup. In fact, they showed that for a path connected space X , the map $p_H : \tilde{X}_H \rightarrow X$ has UPL property if and only if $(p_H)_*\pi_1(\tilde{X}_H, e_H) \leq H$. The other side is obvious. Recently, Brazas [6, Lemma 5.9] gave two equivalent conditions for the equality $H = (p_H)_*\pi_1(\tilde{X}_H, \tilde{e}_H)$ (see also [22, Corollary 3.2]). Now, we represent the result with another point of view.

Lemma 2.12. *Suppose that H is a subgroup of the fundamental group $\pi_1(X, x_0)$. If $\pi_1^{gc}(X, x_0) \cap H \neq \pi_1^{gc}(X, x_0)$, then H can not be a generalized covering subgroup. If $\pi_1^{gc}(X, x_0) \cap H = \pi_1^{gc}(X, x_0)$, then H is a generalized covering subgroup if and only if $H = (p_H)_*\pi_1(\tilde{X}_H, e_H)$, where $p_H : \tilde{X}_H \rightarrow X$ is the end point projection.*

Proof. The first part is obvious by the definition of $\pi_1^{gc}(X, x_0)$. For the second part, assume that $\pi_1^{gc}(X, x_0) \cap H = \pi_1^{gc}(X, x_0)$. We show that if $(p_H)_*\pi_1(\tilde{X}_H, e_H) \leq H$, then $p_H : \tilde{X}_H \rightarrow X$ has UPL property and use Remark 2.2. Let $\alpha, \beta : (I, 0) \rightarrow (\tilde{X}_H, e_H)$ be two paths with $\alpha(0) = \beta(0) = e_H$ and $p_H \circ \alpha = p_H \circ \beta = f$. For $s \in (0, 1]$ we show that $\alpha(s) = \beta(s)$. Let $\langle \gamma \rangle_H := \alpha(s)$ and $\langle \mu \rangle_H := \beta(s)$, where $\gamma, \mu \in P(X, x_0)$. Since p_H has path lifting property, γ and μ have standard lifts $\tilde{\gamma}, \tilde{\mu} : (I, 0) \rightarrow (\tilde{X}_H, e_H)$, respectively, i.e., $\tilde{\gamma}(t) = \langle \gamma_t \rangle_H$ and $\tilde{\mu}(t) = \langle \mu_t \rangle_H$ for every $t \in I$ and so $\tilde{\gamma}(1) = \langle \gamma_1 \rangle_H = \langle \gamma \rangle_H = \alpha(s)$ and $\tilde{\mu}(1) = \langle \mu_1 \rangle_H = \langle \mu \rangle_H = \beta(s)$. Clearly $[\tilde{\gamma} * \alpha^{-1} |_{[0,s]} * \beta |_{[0,s]} * \tilde{\mu}^{-1}] \in \pi_1(\tilde{X}_H, e_H)$. Then by assumption $(p_H)_*[\tilde{\gamma} * \alpha^{-1} |_{[0,s]} * \beta |_{[0,s]} * \tilde{\mu}^{-1}] \in H$. Therefore $[\gamma * \mu^{-1}] = [\gamma * f^{-1} |_{[0,s]} * f |_{[0,s]} * \mu^{-1}] = [(p_H \circ \tilde{\gamma}) * (p_H \circ \alpha^{-1} |_{[0,s]}) * (p_H \circ \beta |_{[0,s]}) * (p_H \circ \tilde{\mu}^{-1})] \in H$ which implies that $\langle \gamma \rangle_H = \langle \mu \rangle_H$. \square

Brazas [6, Theorem 2.36] using pullbacks showed that the intersection of any collection of generalized covering subgroups of $\pi_1(X, x_0)$ is also a generalized covering subgroup. In the following, using the above theorem, we give another proof which seems simpler than that of Brazas.

Corollary 2.13. *If $\{H_j/j \in J\}$ is any collection of generalized covering subgroups of $\pi_1(X, x_0)$, then $H = \bigcap_j H_j$ is a generalized covering subgroup.*

Proof. Using Lemma 2.12, it is enough to show that

$$(p_H)_*\pi_1(\tilde{X}_H, e_H) \leq \bigcap_j (p_{H_j})_*\pi_1(\tilde{X}_{H_j}, e_{H_j}) = H.$$

Let $[\alpha] = [p_H \circ \tilde{\alpha}] = (p_H)_*[\tilde{\alpha}] \in (p_H)_*\pi_1(\tilde{X}_H, e_H)$, where $\tilde{\alpha} : I \rightarrow \tilde{X}_H$ is a loop at e_H with $\tilde{\alpha}(t) = \langle \beta_t \rangle_H$. We define $\tilde{\alpha}_j : I \rightarrow \tilde{X}_{H_j}$ by $\tilde{\alpha}_j(t) = \langle \beta_t \rangle_{H_j}$, for

every $j \in J$. It is clear that $\tilde{\alpha}_j$ is a loop at e_{H_j} and $p_H \circ \tilde{\alpha} = p_{H_j} \circ \tilde{\alpha}_j$. Thus $[p_H \circ \tilde{\alpha}] = [p_{H_j} \circ \tilde{\alpha}_j] = [\alpha]$ for every $j \in J$. Hence $(p_H)_* \pi_1(\tilde{X}_H, e_H) \leq H$. \square

The immediate result of the above corollary is that $\pi_1^{gc}(X, x_0)$ is a generalized covering subgroup of $\pi_1(X, x_0)$. Similarly, $\pi_1^{sp}(X, x_0)$ and $\tilde{\pi}_1^{sp}(X, x_0)$ are also generalized covering subgroups. Moreover, one can find easy to check that every pointed topological space has categorical generalized universal covering subgroup (see Remark 1.4).

Recall that for any subgroup H_0 of the fundamental group $\pi_1(X, x_0)$, we have $H_0 \leq (p_{H_0})_* \pi_1(\tilde{X}_{H_0}, e_{H_0})$. Hence, using Lemma 2.12, if H_0 is not a generalized covering subgroup, then $H_0 \neq (p_{H_0})_* \pi_1(\tilde{X}_{H_0}, e_{H_0})$. Put $H_{i+1} = (p_{H_i})_* \pi_1(\tilde{X}_{H_i}, e_{H_i})$ for any natural number $i \in \mathbb{N}$ and use again Lemma 2.12. This process forms an ascending chain of subgroups of the fundamental group. If H_i is a generalized covering subgroup, then by Lemma 2.12, $H_i = H_{i+1}$ which gives the following corollary.

Corollary 2.14. *Let $H_0 \leq \pi_1(X, x_0)$ and define $H_{i+1} = (p_{H_i})_* \pi_1(\tilde{X}_{H_i}, e_{H_i})$ for any natural number $i \in \mathbb{N}$. Then $\{H_i\}_{i=0}^\infty$ is an ascending chain of subgroups of $\pi_1(X, x_0)$. Moreover, H_i is a generalized covering subgroup of $\pi_1(X, x_0)$ if and only if $H_j = H_i$ for every $j \geq i$.*

It is well-known that if H is an open subgroup of a quasitopological group and $H \leq K$, then K is also an open subgroup. In general, the result for H does not hold when K is an open subgroup.

Proposition 2.15. *Let X be a locally path connected space and K be an open subgroup of $\pi_1^{qtop}(X, x_0)$ and $H \leq K$ such that $[K : H] < \infty$. Then H is an open subgroup of $\pi_1^{qtop}(X, x_0)$ if and only if X is homotopically Hausdorff relative to H .*

Proof. Recall from [26, Theorem 4.1] that there is a path open cover \mathcal{V} of X such that $\tilde{\pi}(\mathcal{V}, x_0) \leq K$. Since $H \leq K$ and $[K : H] < \infty$, one can write K as a disjoint finite union of cosets of H , i.e., $K = H \cup Hg_1 \cup \dots \cup Hg_n$, where $g_i \notin H$ for all $1 \leq j \leq n$. Since X is homotopically Hausdorff relative to H , for each $j = 1, \dots, n$ and for every path $\alpha \in P(X, x_0)$ there is an open neighborhood U_j of $\alpha(1) \in X$ such that $[\alpha * i_* \pi_1(U_j, \alpha(1)) * \alpha^{-1}] \cap Hg_j = \emptyset$. Fix $\alpha \in P(X, x_0)$ and put $U = V_\alpha \cap (\bigcap_{j=1}^n U_j)$, where $V_\alpha \in \mathcal{V}$ is the open neighborhood of $\alpha(1)$. Clearly $\alpha(1) \in U \neq \emptyset$ and $[\alpha * i_* \pi_1(U, \alpha(1)) * \alpha^{-1}] \leq H$. Now one can construct a path open cover \mathcal{U} of x_0 such that $\tilde{\pi}(\mathcal{U}, x_0) \leq H$, where $\tilde{\pi}(\mathcal{U}, x_0)$ is the path Spanier subgroup of \mathcal{U} (see [25] for the definition). [25, Theorem 4.1] implies that H is an open subgroup of $\pi_1^{qtop}(X, x_0)$. Conversely, since X is locally path connected and H is an open subgroup of $\pi_1^{qtop}(X, x_0)$, then H is a semicovering subgroup [3, Theorem 3.5]. Therefore, the result comes from Lemma 2.9. \square

Let X be a connected, locally path connected space and H be a subgroup of $\pi_1(X, x_0)$. Then H is a semicovering subgroup if and only if it is open in $\pi_1^{qtop}(X, x_0)$ (see [5, Theorem 3.5]). Using this fact and Proposition 2.15, we have the following results for semicovering subgroups.

Corollary 2.16. *Let X be a connected, locally path connected space and $K \leq \pi_1(X, x_0)$ be a semicovering subgroup. If $H \leq K$ with $[K : H] < \infty$, then the following statements are equivalent.*

- (1) X is homotopically Hausdorff relative to H .
- (2) H is a generalized covering subgroup.
- (3) X is homotopically path Hausdorff relative to H .
- (4) H is a semicovering subgroup.

Corollary 2.17. *Let X be a connected, locally path connected space and $[\pi_1(X, x_0) : H] < \infty$, then the following statements are equivalent.*

- (1) X is homotopically Hausdorff relative to H .
- (2) H is a generalized covering subgroup.
- (3) X is homotopically path Hausdorff relative to H .
- (4) H is a semicovering subgroup.

Remark 2.18. The class of all coverings, semicoverings and generalized coverings on X forms a category denoted by $\text{Cov}(X)$, $\text{SCov}(X)$ and $\text{GCov}(X)$, respectively. A morphism between two objects $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow X$ is a map $h : \tilde{Y} \rightarrow \tilde{X}$ such that $p \circ h = q$. It is easy to see that $\text{Cov}(X)$ is a subcategory of $\text{SCov}(X)$. Moreover, Lemma 2.9 implies the following:

$$\text{Cov}(X) \leq \text{SCov}(X) \leq \text{GCov}(X).$$

Existence of universal objects in the above categories is considered by some people. Pakdaman et al. [21, Corollary 3.13] showed that the Spanier covering is the universal object in the category of $\text{Cov}(X)$ for a locally path connected space X . Brazas presented a similar result for the category $\text{SCov}(X)$. He introduced the universal object of the category $\text{SCov}(X)$ in [3, Definition 3.1] and gave properties which support the existence of it [3, Corollary 7.21]. In each case the universal object may not exist, in general. However, in the category $\text{GCov}(X)$ the categorical universal object exists, in general, since it is easy to see from Corollary 2.13 that $\pi_1^{qc}(X, x_0)$ is a generalized covering subgroup. It means that the map $p_H : \tilde{X}_H \rightarrow X$ is a generalized covering, when $H = \pi_1^{qc}(X, x_0)$. Therefore, we have the following result.

Corollary 2.19. *$\tilde{X}_{\pi_1^{qc}(X, x_0)}$ is the categorical universal object in the category $\text{GCov}(X)$.*

Using [3, Corollary 7.2] and [6, Example 5.4] one can conclude easily that $\text{Cov}(X) = \text{SCov}(X) = \text{GCov}(X)$ for a locally path connected and semilocally simply connected space X . Torabi et al. [24] extended the first equality for

locally path connected and semilocally small generated spaces, i.e., $\text{Cov}(X) = \text{SCov}(X)$. Now in the following, we extend the result to generalized coverings.

Proposition 2.20. *If X is a connected locally path connected and semilocally small generated space, then*

$$\text{Cov}(X) = \text{SCov}(X) = \text{GCov}(X).$$

Proof. Since X is semilocally small generated, there exists an open cover \mathcal{U} of X such that $\pi(\mathcal{U}, x) = \pi_1^{sg}(X, x) \leq \pi_1^{gc}(X, x)$. If H is a generalized covering subgroup of $\pi_1(X, x)$, then $\pi(\mathcal{U}, x) \leq \pi_1^{gc}(X, x) \leq H$ which implies that H is a covering subgroup of $\pi_1(X, x)$ [23, Theorem 13]. Hence $\text{GCov}(X) \leq \text{Cov}(X)$. The converse holds by Remark 2.18. \square

The following corollaries seem interesting which are consequences of main results of this section.

Corollary 2.21. *If $f : Y \rightarrow X$ has UL property and X is semilocally small generated, then f is a local homeomorphism.*

Proof. Recall from Definition 2.1 that $f : Y \rightarrow X$ is a generalized covering map. Hence Proposition 2.20 implies that $f : Y \rightarrow X$ is a semicovering map. \square

Corollary 2.22. *If $H \leq \pi_1(X, x_0)$, the endpoint projection map $p_H : \tilde{X}_H \rightarrow X$ has UPL property and X is semilocally small generated, then p_H is a covering map.*

Proof. Using Remark 2.2, $p_H : \tilde{X}_H \rightarrow X$ has UL property. Hence the result holds by Proposition 2.20. \square

3. On the Whisker Topology and Generalized Covering Maps

For any pointed topological space (X, x_0) the whisker topology on the set \tilde{X}_e (as shown in Definition 1.3) can be induced on $\pi_1(X, x_0)$ by the bijection $f : \pi_1(X, x_0) \rightarrow p_e^{-1}(x_0)$ defined by $[\alpha] \mapsto \langle \alpha \rangle_e$, where e is the trivial subgroup of $\pi_1(X, x_0)$. The fundamental group with the whisker topology is denoted by $\pi_1^{wh}(X, x_0)$. Fischer and Zastrow [13, Lemma 2.1] have shown that the whisker topology is finer than the inherited topology from the compact-open topology on $\pi_1(X, x_0)$ which is denoted by $\pi_1^{qtop}(X, x_0)$. In this section, we intend to present some properties of $\pi_1^{wh}(X, x_0)$ in comparison with $\pi_1^{qtop}(X, x_0)$ in order to classify generalized covering subgroups.

At first, it seems necessary to characterize open subsets and open subgroups of $\pi_1^{wh}(X, x_0)$.

Lemma 3.1. *The collection $\beta = \{[\alpha]i_*\pi_1(U, x_0) \mid [\alpha] \in \pi_1(X, x_0), \text{ where } U \text{ is an open neighborhood of } x_0\}$ forms a basis for the whisker topology on $\pi_1(X, x_0)$. Moreover, these basis elements are also closed.*

Proof. Let

$$\beta' = \{N(\langle \gamma \rangle_e, U) \cap p^{-1}(x_0) \mid \langle \gamma \rangle_e \in P(X, x_0) \text{ and } U \text{ is an open subset of } X\}.$$

Note that if $x_0 \notin U$, then $N(\langle \gamma \rangle_e, U) \cap p^{-1}(x_0) = \emptyset$. By definition it is easy to see that $\beta \subseteq \beta'$. Let S be an open subset of $\pi_1^{wh}(X, x_0)$ and $[\alpha] \in S \subseteq \pi_1(X, x_0)$. Then there exists a basis element $N(\langle \mu \rangle_e, U) \cap p^{-1}(x_0) \subseteq S$ which contains $[\alpha]$. We show that $[\alpha]i_*\pi_1(U, x_0) \subseteq S$.

Since $[\alpha] \in N(\langle \mu \rangle_e, U) \cap p^{-1}(x_0)$, there exists $[\eta] \in i_*\pi_1(U, \mu(1))$ such that $\alpha \simeq \mu * \eta$. Now for any $[\xi] \in i_*\pi_1(U, x_0)$ we have:

$$\alpha * \xi \simeq (\mu * \eta) * \xi \simeq \mu * (\eta * \xi).$$

Therefore $[\alpha * \xi] = [\mu * (\eta * \xi)] \in N(\langle \mu \rangle_e, U) \cap p^{-1}(x_0)$. Hence $[\alpha]i_*\pi_1(U, x_0) \subseteq N(\langle \mu \rangle_e, U) \cap p^{-1}(x_0) \subseteq S$.

Moreover, if $[\beta]$ belongs to $[\alpha]i_*\pi_1(U, x_0)$ for any $[\beta] \in \pi_1(X, x_0)$ and some open neighborhood U of x_0 , then it is easy to show that $[\beta]i_*\pi_1(U, x_0) = [\alpha]i_*\pi_1(U, x_0)$ and vice versa. If $[\beta] \notin [\alpha]i_*\pi_1(U, x_0)$, then $[\beta]i_*\pi_1(U, x_0) \cap [\alpha]i_*\pi_1(U, x_0) = \emptyset$. Therefore $[\alpha]i_*\pi_1(U, x_0)$ is closed in $\pi_1^{wh}(X, x_0)$. \square

Despite $\pi_1^{wh}(X, x_0)$ is not a topological group in general (it is not even a quasitopological group, for example, in $\pi_1^{wh}(HE, x_0)$ right translations are not continuous. Let g_1 be the greater circle in HE and $r_{g_1} : \pi_1^{wh}(HE, x_0) \rightarrow \pi_1^{wh}(HE, x_0)$ with $[g] \mapsto [g * g_1]$ be the relative right translation map. Put U is the open neighborhood of x_0 with radius $\frac{1}{3}$. Suppose that r_{g_1} is continuous, then there exists an open neighborhood V of x_0 such that for every $[g] \in i_*\pi_1(V, x_0)$ we have $[g_1^{-1} * g * g_1] \in i_*\pi_1(U, x_0)$. Recall from [10, Theorem 2.5] that it is a contradiction), it has some properties of topological groups. The following propositions state some of them. Recall that a non-empty topological space X is called a G -space, for a group G , if it is equipped with an action of G on X . A *homogeneous space* is a G -space on which G acts transitively.

Proposition 3.2. $\pi_1^{wh}(X, x_0)$ is a homogenous space.

Proof. Clearly $\pi_1^{wh}(X, x_0)$ acts on itself by translation. To show transitivity of this action, it is enough to show that the left translations are homeomorphism [1, Corollary 1.3.6]. Let $\varphi_\alpha : \pi_1^{wh}(X, x_0) \rightarrow \pi_1^{wh}(X, x_0)$ with $[\beta] \mapsto [\alpha * \beta]$, for any $\alpha \in \pi_1^{wh}(X, x_0)$, be a left translation. Clearly φ_α is a bijection. We show that φ_α is continuous. Let U be an open neighborhood of x_0 in X and $[\alpha * \beta]i_*\pi_1(U, x_0)$ be an open basis neighborhood of $[\alpha * \beta]$ in $\pi_1^{wh}(X, x_0)$. It is clear that $[\beta]i_*\pi_1(U, x_0)$ is an open basis neighborhood of $[\beta]$ and $\varphi_\alpha([\beta]i_*\pi_1(U, x_0)) \subseteq [\alpha * \beta]i_*\pi_1(U, x_0)$. \square

An immediate consequence of the above proposition is that every open subgroup of $\pi_1^{wh}(X, x_0)$ is also closed. Recall that a topological space is called *totally separated* if for every pair of disjoint points there exists a clopen subset

which contains one of the points and doesn't contain the other. The following proposition states separation axioms for $\pi_1^{wh}(X, x_0)$.

Proposition 3.3. *For a connected locally path connected space X , the following statements are equivalent:*

- (i) $\pi_1^{wh}(X, x_0)$ is T_0 .
- (ii) $\pi_1^{wh}(X, x_0)$ is T_1 .
- (iii) $\pi_1^{wh}(X, x_0)$ is T_2 .
- (iv) $\pi_1^{wh}(X, x_0)$ is T_3 ($T_3 = \text{regular} + T_1$).
- (v) $\pi_1^s(X, x_0) = 1$.
- (vi) $\pi_1^{wh}(X, x_0)$ is totally separated.

Moreover, $\pi_1^{wh}(X, x_0)$ is regular, in general.

Proof. 1) \Rightarrow 2). Let $[g], [f] \in \pi_1^{wh}(X, x_0)$ and U be an open neighborhood of x_0 in X such that $[g] \notin [f]i_*\pi_1(U, x_0)$. We show that $[f] \notin [g]i_*\pi_1(U, x_0)$. By contrary suppose $[f] \in [g]i_*\pi_1(U, x_0)$, then there exists a loop γ at x_0 with $\gamma(I) \subseteq U$ such that $[f] = [g * \gamma]$ and so $[g] = [f * \gamma^{-1}]$. Thus, it shows that $[g] \in [f]i_*\pi_1(U, x_0)$ which is a contradiction.

2) \Rightarrow 3). Let $[g], [f] \in \pi_1^{wh}(X, x_0)$ and U, V be open neighborhoods of x_0 in X such that $[g] \notin [f]i_*\pi_1(V, x_0)$ and $[f] \notin [g]i_*\pi_1(U, x_0)$. Put $W = U \cap V$ and suppose that $[g]i_*\pi_1(W, x_0) \cap [f]i_*\pi_1(W, x_0) \neq \emptyset$. Thus, there exist $[\mu_1], [\mu_2] \in i_*\pi_1(W, x_0)$ with $[f * \mu_1] = [g * \mu_2]$. Hence $[f] = [g * \mu_2 * \mu_1^{-1}] \in [g]i_*\pi_1(W, x_0)$ which contradicts to the hypothesis. Therefore, $[g]i_*\pi_1(W, x_0) \cap [f]i_*\pi_1(W, x_0) = \emptyset$.

3) \Rightarrow 5). Let $1 \neq [g] \in \pi_1^{wh}(X, x_0)$, then there exist open neighborhoods U, V of x_0 in X such that $[g]i_*\pi_1(U, x_0) \cap i_*\pi_1(V, x_0) = \emptyset$. Thus $[g] \notin i_*\pi_1(V, x_0)$. It implies that g is not a small loop at x_0 . Therefore $\pi_1^s(X, x_0) = 1$.

5) \Rightarrow 1). Let $[g], [f] \in \pi_1^{wh}(X, x_0)$ and $[g] \neq [f]$. Thus $[f * g^{-1}] \neq 1$. Since $\pi_1^s(X, x_0) = 1$, $[f * g^{-1}]$ is not a small loop at x_0 . Therefore, there exists an open neighborhood V of x_0 in X such that $[f * g^{-1}] \notin i_*\pi_1(V, x_0)$, i.e., $[f] \notin [g]i_*\pi_1(V, x_0)$.

1) \leftrightarrow 6). By Lemma 3.1 for every $[g] \in \pi_1(X, x_0)$ and each open neighborhood V of x_0 in X , $[g]i_*\pi_1(V, x_0)$ is a clopen subset of $\pi_1^{wh}(X, x_0)$. Hence if $\pi_1^{wh}(X, x_0)$ is T_0 , then it is totally separated. The converse holds, in general. To show that 4) is equivalent to the other statements, it is enough to show that $\pi_1^{wh}(X, x_0)$ is regular, in general. Let A be a closed subset of $\pi_1^{wh}(X, x_0)$ and $[g] \notin A$. By definition of closed subsets, there exists an open neighborhood V of x_0 in X such that $[g]i_*\pi_1(V, x_0) \cap A = \emptyset$. Define

$$Ai_*\pi_1(V, x_0) = \cup\{[f]i_*\pi_1(V, x_0) : [f] \in A\}.$$

Clearly $Ai_*\pi_1(V, x_0)$ is an open subset of $\pi_1^{wh}(X, x_0)$ which contains A . It is enough to show that $[g]i_*\pi_1(V, x_0) \cap (Ai_*\pi_1(V, x_0)) = \emptyset$. Suppose that there

exist $[\mu_1], [\mu_2] \in i_*\pi_1(V, x_0)$ and $[f] \in A$ such that $[g * \mu_1] = [f * \mu_2]$. Then $[f] = [g * \mu_1 * \mu_2^{-1}] \in [g]i_*\pi_1(V, x_0)$ a contradiction. \square

It is well-known that the fibers of a covering map are homeomorphic. This fact may not be true in the case of generalized covering maps. As an example, consider the Hawaiian earring that has generalized universal covering map [13, Example 4.15].

Let $x_0 \in HE$ be the wedge point of all circles in HE . Clearly HE is semilocally simply connected at each of its points but does not at $x_0 \in HE$. Then $\pi_1^{wh}(HE, x_0)$ is not discrete and $\pi_1^{wh}(HE, x)$ is discrete in each other point $x \in HE$ [9, Proposition 4.21]. This shows that the fibers $p^{-1}(x_0)$ and $p^{-1}(x)$ are not homeomorphic in the whisker topology. Fischer and Zastrow also showed that for generalized universal covering maps, fibers have the same cardinality. In the following proposition, we extend the result for any subgroup of $\pi_1(X, x_0)$.

Proposition 3.4. *For a connected, locally path connected space X , let $H \leq \pi_1(X, x_0)$ and $p_H : \tilde{X}_H \rightarrow X$ be the endpoint projection ($p_H(\langle \alpha \rangle_H) = \alpha(1)$). Then all fibers over each points of X have the same cardinality.*

Proof. Let $x, y \in X$ and define $\psi : (p_H)^{-1}(x) \rightarrow (p_H)^{-1}(y)$ by $\langle \alpha \rangle_H \mapsto \langle \alpha * \gamma \rangle_H$, where $\gamma : I \rightarrow X$ is a path from x to y . It is easy to show that ψ is a well-defined bijection. \square

Corollary 3.5. *Let $H \leq \pi_1(X, x_0)$, then $|p_H^{-1}(x_0)| = [\pi_1(X, x_0) : H]$.*

Virk and Zastrow [27] presented an example of a space X which is homotopically Hausdorff but doesn't admit a generalized universal covering. In the following theorem, we show that a sufficient condition on a homotopically Hausdorff relative to H space to admit H as a generalized covering subgroup is that H to be a dense subgroup in $\pi_1^{wh}(X, x_0)$.

Theorem 3.6. *Let X be homotopically Hausdorff relative to H and H be a dense subgroup of $\pi_1^{wh}(X, x_0)$. Then $p_H : \tilde{X}_H \rightarrow X$ has UPL property.*

Proof. Suppose $p_H : \tilde{X}_H \rightarrow X$ doesn't have UPL property and let $\alpha \in P(X, x_0)$ be a path with a lift $\beta : I \rightarrow \tilde{X}_H$, $\beta(t) = \langle \beta_t \rangle_H$, $\beta(0) = \tilde{x}_H = \tilde{\alpha}(0)$ distinct from its standard lift $\tilde{\alpha}(t) = \langle \alpha_t \rangle_H$. Then there is $t_0 \in (0, 1]$ such that $\beta(t_0) \neq \tilde{\alpha}(t_0)$. Thus $\langle \alpha_{t_0} \rangle_H \neq \langle \beta_{t_0} \rangle_H$ and so $[\alpha_{t_0} * \beta_{t_0}^{-1}] \notin H$. Since X is homotopically Hausdorff relative to H , there is an open neighborhood U of x_0 such that $i_*\pi_1(U, x_0) \cap [\alpha_{t_0} * \beta_{t_0}^{-1}]H = \emptyset$ which implies that $[\beta_{t_0} * \alpha_{t_0}^{-1}]i_*\pi_1(U, x_0) \cap H = \emptyset$. This contradicts to the density of H in $\pi_1^{wh}(X, x_0)$. \square

Corollary 3.7. *Let H be a dense subgroup of $\pi_1^{wh}(X, x_0)$, then the map $p_H : \tilde{X}_H \rightarrow X$ is a generalized covering map (or equivalently H is a generalized covering subgroup) if and only if X is homotopically Hausdorff relative to H .*

Torabi et al. [24, Theorem 2.2] showed that the closure of the trivial subgroup of $\pi_1^{qtop}(X, x_0)$ contains $\pi_1^{sg}(X, x_0)$. In the whisker topology, this closure is equal to $\pi_1^s(X, x_0)$.

Proposition 3.8. *The closure of the trivial element in $\pi_1^{wh}(X, x_0)$ equals $\pi_1^s(X, x_0)$.*

Proof. If for any $[f] \in \pi_1^{wh}(X, x_0)$ and each neighborhood V of x_0 in X , $1 \in [f]i_*\pi_1(V, x_0)$, then $[f]i_*\pi_1(V, x_0) = i_*\pi_1(V, x_0)$. Thus the closure of the trivial subgroup of the fundamental group equipped with the whisker topology equals with the intersection of all $i_*\pi_1(V, x_0)$, where V is an open neighborhood of x_0 in X . Clearly, this intersection only contains small loops at x_0 . \square

Proposition 3.9. *If X is a homotopically Hausdorff relative to H space, then H is a closed subgroup of $\pi_1^{wh}(X, x_0)$.*

Proof. Let $[\beta] \in \pi_1(X, x_0) \setminus H$ so $[\beta^{-1}] \in \pi_1(X, x_0) \setminus H$ and put α the constant path at $x_0 \in X$. Since X is homotopically Hausdorff relative to H , there exists an open neighborhood U of x_0 in X such that for every $[\gamma] \in \pi_1(U, x_0)$, $[\gamma] \notin [\beta^{-1}]H$ and so $[\beta * \gamma] \notin H$. Hence $[\beta]i_*\pi_1(U, x_0) \cap H = \emptyset$ which implies that H is a closed subgroup of $\pi_1^{wh}(X, x_0)$. \square

As a consequence of the above proposition, we can state the following corollary using this fact that if $p_H : \tilde{X}_H \rightarrow X$ is a generalized covering map, then X is homotopically Hausdorff relative to H [13, Proposition 6.4].

Corollary 3.10. *If $H \leq \pi_1(X, x_0)$ is a generalized covering subgroup, then H is a closed subgroup of $\pi_1^{wh}(X, x_0)$.*

If $H \leq \pi_1^{qtop}(X, x_0)$ is a closed subgroup, then $p_H : \tilde{X}_H \rightarrow X$ has UPL property [7, Theorem 11]. This result will be changed when the fundamental group equipped with the whisker topology.

Question 3.11. *If $H \leq \pi_1(X, x_0)$ is a closed subgroup of $\pi_1^{wh}(X, x_0)$, then when dose $p_H : \tilde{X}_H \rightarrow X$ have the UPL property?*

As mentioned before, in both the whisker topology and the compact-open topology on the fundamental group every open subgroups are also closed, but the converse may not hold, in general. Hence an open subgroup may illustrate some properties which does not hold for the closed one. Therefore, a question may be raised naturally that if H is an open subgroup of $\pi_1^{wh}(X, x_0)$, then whether $p_H : \tilde{X}_H \rightarrow X$ has the UPL property or not? By the following example we show that there is an open (also closed) normal subgroup of the fundamental group equipped with the whisker topology that is not a generalized covering subgroup.

Example 3.12. Consider the *Harmonic Archipelago*, HA (see [26]). Let $a \in HA$ be the canonical based point and put $b \neq a$ be another point. Using [9, Proposition 4.21], $\pi_1^{wh}(HA, b)$ is discrete. Thus the trivial subgroup of $\pi_1^{wh}(HA, b)$ is an open and closed subgroup. It is clearly a normal subgroup.

On the other hand, $\pi_1^{sg}(HA, b) = \pi_1(HA, b)$ (see [24, Remark 2.11]). Then by Proposition 2.4, $\pi_1^{gc}(HA, b) = \pi_1(HA, b)$. Hence there is only the trivial generalized covering which is also a covering. Therefore $\pi_1^{gc}(HA, b) \neq 1$ and so (HA, b) has no categorical generalized universal covering. Thus the trivial subgroup is not a generalized covering subgroup.

For every subgroup $H \leq \pi_1^{wh}(X, x_0)$, there is a homeomorphism $h : p_H^{-1}(x_0) \rightarrow \pi_1^{wh}(X, x_0)/H$ when $\pi_1^{wh}(X, x_0)/H$ is as quotient space of $\pi_1^{wh}(X, x_0)$ with the quotient topology. Let $h : \langle \alpha \rangle_H \mapsto [\alpha]H$, then the following diagram commutes when s is defined by $\langle \alpha \rangle_e \mapsto \langle \alpha \rangle_H$. Clearly s is a quotient map and f is a homeomorphism. Since q is a quotient map, h is a homeomorphism.

$$\begin{array}{ccc} p^{-1}(x_0) & \xrightarrow{f} & \pi_1^{wh}(X, x_0) \\ s \downarrow & & \downarrow q \\ p_H^{-1}(x_0) & \xrightarrow{h} & \pi_1^{wh}(X, x_0)/H. \end{array}$$

Clearly, for every normal subgroup $H \trianglelefteq \pi_1^{wh}(X, x_0)$, the fiber $p_H^{-1}(x_0)$ will be a group by the binary operation $\psi : (p_H^{-1}(x_0))^{wh} \times (p_H^{-1}(x_0))^{wh} \rightarrow ((p_H^{-1}(x_0))^{wh})$ defined by $\psi(\langle \alpha \rangle_H, \langle \beta \rangle_H) = \langle \alpha * \beta \rangle_H$. In general, there is a useful action for any subgroup $H \leq \pi_1(X, x_0)$ that may help to explain the structure of $p_H^{-1}(x_0)$. The map $*_H : p_H^{-1}(x_0) \times \pi_1(X, x_0) \rightarrow p_H^{-1}(x_0)$ defined by $*_H(\langle \alpha \rangle_H, [\gamma]) = \langle \alpha * \gamma \rangle_H$ is a group action with the following properties:

- (1) If $\tilde{x} = \langle \alpha \rangle_H \in p_H^{-1}(x_0)$, then the stabilizer of \tilde{x} is $[\alpha^{-1}H\alpha]$.
- (2) $O(\langle e_0 \rangle_H) = \langle e_0 \rangle_H * \pi_1(X, x_0) = p_H^{-1}(x_0)$, where $O(\langle \alpha \rangle_H)$ is the orbit of $\langle \alpha \rangle_H$.

Part 1 holds since if $\pi_1(X, x_0)_{\tilde{x}} =: \{[g] \in \pi_1(X, x_0) \mid \tilde{x} *_H [g] = \tilde{x}\}$ is the stabilizer of $\tilde{x} \in p_H^{-1}(x_0)$, then $\pi_1(X, x_0)_{\tilde{x}} = \{[g] \in \pi_1(X, x_0) \mid \langle \alpha * g \rangle_H = \langle \alpha \rangle_H\} = \{[g] \in \pi_1(X, x_0) \mid [\alpha * g * \alpha^{-1}] \in H\}$. Therefore, $\pi_1(X, x_0)_{\tilde{x}} = [\alpha^{-1}H\alpha]$, where $\langle \alpha \rangle_H = \tilde{x}$. The second part is a trivial consequence of part 1. This action easily implies another simple proof for $|p_H^{-1}(x_0)| = [\pi_1(X, x_0) : H]$, because $|p_H^{-1}(x_0)| = |o(e_H)| = [\pi_1(X, x_0) : \pi_1(X, x_0)_{e_H}] = [\pi_1(X, x_0) : H]$.

In the following corollary parts 1, 2 and 3 are immediate consequences of Proposition 3.4. Part 4 is a consequences of the above argument.

Corollary 3.13. Let $p_H : \tilde{X}_H \rightarrow X$ be a generalized covering map, then

- (1) For every pair $\tilde{x}_1, \tilde{x}_0 \in p_H^{-1}(x_0)$, the subgroups $(p_H)_* \pi_1(\tilde{X}_H, \tilde{x}_0)$ and $(p_H)_* \pi_1(\tilde{X}_H, \tilde{x}_1)$ are conjugate subgroups of $\pi_1(X, x_0)$.
- (2) $|p_H^{-1}(x_0)| = [\pi_1(X, x_0) : (p_H)_* \pi_1(X_H, \langle e_0 \rangle_H)]$.
- (3) If $K \leq \pi_1(X, x_0)$ is conjugate to $(p_H)_* \pi_1(\tilde{X}_H, \tilde{x})$ for $\tilde{x} = \langle \alpha \rangle_H \in p_H^{-1}(x_0)$, then $K = (p_H)_* \pi_1(\tilde{X}_H, \tilde{y})$ for some $\tilde{y} = \langle \alpha * \gamma \rangle_H$, where $(p_H)_* \pi_1(\tilde{X}_H, \tilde{x}) = [\gamma]K[\gamma^{-1}]$.
- (4) $|O(\langle \alpha \rangle_H)| = [\pi_1(X, x_0) : [\alpha^{-1}H\alpha]]$.

Similar to the trivial case $H = 1$, there is a bijection $\varphi : \langle \alpha \rangle_H *_H i_* \pi_1(U, x_0) \rightarrow (\langle \alpha \rangle_H, U) \cap p_H^{-1}(x_0)$ defined by $(\langle \alpha \rangle_H *_H [\gamma]) = \langle \alpha * \gamma \rangle_H$, for every $H \leq \pi_1(X, x_0)$, any $\langle \alpha \rangle_H \in p_H^{-1}(x_0)$ and every open neighborhood U of x_0 , where $p_H : \tilde{X}_H \rightarrow X$ is the endpoint projection map. The above action extends Proposition 3.2 for every normal subgroup H . The following corollary is a direct consequence of the assertion from the previous page since $p_H^{-1}(x_0)$ is homeomorphic to a quotient of $\pi_1^{wh}(X, x_0)$.

Corollary 3.14. *Let H be a normal subgroup of $\pi_1(X, x_0)$, then $p_H^{-1}(x_0)$ is a homogenous space.*

4. Semilocally (Path) Connectedness with Respect to a Subgroup

It is well-known that semilocally simply connectedness plays an important role in the covering space theory. For example, a locally path connected space has universal covering if and only if it is semilocally simply connected. Recall that a space X is called semilocally simply connected at $x \in X$ if there exists an open neighborhood U of x such that all loops in U at x are nullhomotopic in X . Moreover, a space is called semilocally simply connected if it is semilocally simply connected at each of its points. In this section, we extend these concepts to any subgroup of the fundamental group and introduce their topological equivalences.

Definition 4.1. Let X be a topological space and H be a subgroup of $\pi_1(X, x_0)$. Then we define the following notions.

- (i) X is called *semilocally H -connected at $x_0 \in X$* if there exists an open neighborhood U of x_0 with $i_* \pi_1(U, x_0) \leq H$.
- (ii) X is called *semilocally path H -connected* if for every path α beginning at x_0 there exists an open neighborhood U_α of $\alpha(1)$ with $i_* \pi_1(U_\alpha, \alpha(1)) \leq [\alpha^{-1}H\alpha]$, where $[\alpha^{-1}H\alpha] = \{[\alpha^{-1}\gamma\alpha] \mid [\gamma] \in H\}$.
- (iii) X is called *semilocally H -connected* if for every $x \in X$ and for every path α from x_0 to x , the space X is semilocally $[\alpha^{-1}H\alpha]$ -connected at $x \in X$.

Note that in the case of trivial subgroup $H = e$ both of semilocally path H -connected and semilocally H -connected properties coincide with the semilocally simply connected property.

Theorem 4.2. *Let $H \leq \pi_1(X, x_0)$, then X is semilocally H -connected at $x_0 \in X$ if and only if H is an open subgroup of $\pi_1^{wh}(X, x_0)$.*

Proof. Let X be semilocally H -connected at $x_0 \in X$, i.e., there exists an open neighborhood U of x_0 such that $i_*\pi_1(U, x_0) \leq H$. Thus for every $[\alpha] \in H$, $[\alpha]i_*\pi_1(U, x_0) \leq H$. By Lemma 3.1, H is an open subgroup of $\pi_1^{wh}(X, x_0)$. Conversely, let $[\alpha] \in H$. Since H is an open subgroup of $\pi_1^{wh}(X, x_0)$ then there exists an open neighborhood U of x_0 such that $[\alpha]i_*\pi_1(U, x_0) \leq H$. Therefore, $i_*\pi_1(U, x_0) \leq [\alpha^{-1}]H = H$. Hence X is semilocally H -connected at $x_0 \in X$. \square

Naturally, semilocally H -connectedness at a point depends on the choice of the point. For instance, the Hawaiian earring (HE, a) is semilocally simply connected at $x \in HE$, where x is any non-based point, $x \neq a$, and clearly it is not semilocally simply connected at $a \in HE$.

We recall from [25] that the Spanier group $\pi(\mathcal{U}, x_0)$ with respect to an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ is defined to be the subgroup of $\pi_1(X, x_0)$ which contains all homotopy classes having representatives of the following type:

$$\prod_{j=1}^n \alpha_j \beta_j \alpha_j^{-1},$$

where α_j 's are arbitrary path starting at x_0 and each β_j is a loop inside of the open set $U_j \in \mathcal{U}$.

Also, we recall from [25, Definition 3.1] that the path Spanier group $\tilde{\pi}(\mathcal{V}, x_0)$ with respect to a path open cover $\mathcal{V} = \{V_\alpha \mid \alpha \in P(X, x_0)\}$ of X such that $\alpha(1) \in V_\alpha$ for every $\alpha \in P(X, x_0)$, is defined to be the subgroup of $\pi_1(X, x_0)$ which contains all homotopy classes having representatives of the following type:

$$\prod_{j=1}^n \alpha_j \beta_j \alpha_j^{-1},$$

where α_j 's are arbitrary path starting at x_0 and each β_j is a loop inside of the open set V_{α_j} for all $j \in \{1, 2, \dots, n\}$.

Let X be semilocally path H -connected, then by the definition, for every path $\alpha \in p(X, x_0)$, there exists an open neighborhood V_α which for every loop $\gamma : (I, \dot{I}) \rightarrow (V_\alpha, \alpha(1))$, $[\gamma] \in [\alpha^{-1}H\alpha]$ if and only if $[\alpha^{-1}\gamma\alpha] \in H$. Put $\mathcal{V} = \{V_\alpha \mid \alpha \in P(X, x_0)\}$ and construct $\tilde{\pi}(\mathcal{V}, x_0)$ such as the above. It implies easily that $\tilde{\pi}(\mathcal{V}, x_0) \leq H$. On the other hand, Torabi et al. [25, Theorem 4.1] showed that H is an open subgroup of $\pi_1^{qtop}(X, x_0)$ if and only if there exists a path open cover \mathcal{V} of x_0 such that $\tilde{\pi}(\mathcal{V}, x_0) \leq H$. The result can be seen in the following proposition.

Proposition 4.3. *A connected, locally path connected space X is semilocally path H -connected for $H \leq \pi_1(X, x_0)$ if and only if H is an open subgroup of $\pi_1^{qtop}(X, x_0)$.*

Corollary 4.4. *For a connected, locally path connected space X , the categorical equality $\text{SCov}(X) = \text{GCov}(X)$ holds if and only if X is semilocally path $\pi_1^{gc}(X, x_0)$ -connected.*

Proof. Assume that the categorical equality $\text{SCov}(X) = \text{GCov}(X)$ holds, then every generalized covering subgroup and so $\pi_1^{gc}(X, x_0)$ is a semicovering subgroup. By [25, Theorem 4.1] there is a path open cover \mathcal{V} of X such that $\tilde{\pi}(\mathcal{V}, x_0) \leq \pi_1^{gc}(X, x_0)$. Using Proposition 4.3, since $\tilde{\pi}(\mathcal{V}, x_0)$ is open, X is semilocally path $\pi_1^{gc}(X, x_0)$ -connected. By a similar argument, one can prove the converse. \square

Corollary 4.5. *If X is homotopically Hausdorff relative to H and the index of H in $\pi_1(X, x_0)$ is finite, then X is semilocally path H -connected.*

Proof. Since the index of H in $\pi_1(X, x_0)$ is finite, by Proposition 2.15, H is an open subgroup of $\pi_1^{qtop}(X, x_0)$. The result follows by Proposition 4.3. \square

The following proposition states same result for semilocally H -connected spaces.

Proposition 4.6. *A connected, locally path connected space X is semilocally H -connected for $H \leq \pi_1(X, x_0)$ if and only if H is a covering subgroup of $\pi_1(X, x_0)$.*

Proof. Remember that H is a covering subgroup of $\pi_1(X, x_0)$ if and only if there is an open cover \mathcal{U} of X such that the Spanier subgroup $\pi(\mathcal{U}, x_0)$ of $\pi_1(X, x_0)$ is a subgroup of H (see [23]). Let $H \leq \pi_1(X, x_0)$ and assume that X is semilocally H -connected space, then for every $x \in X$ and for any path $\alpha : x_0 \mapsto x$, there is an open neighborhood U at x such that $i_*\pi_1(U, x) \leq [\alpha^{-1}H\alpha]$, equivalently, $[\alpha]i_*\pi_1(U, x)[\alpha^{-1}] \leq H$. Put $\mathcal{U} = \{U \mid x \in X, \alpha : x_0 \mapsto x\}$. It is clear that \mathcal{U} is an open cover of X and $\pi(\mathcal{U}, x_0) \leq H$. Similarly, one can prove the converse. \square

Corollary 4.7. *For a connected, locally path connected space X , the categorical equality $\text{GCov}(X) = \text{Cov}(X)$ holds if and only if X is semilocally $\pi_1^{gc}(X, x_0)$ -connected.*

Proof. Assume that X is semilocally $\pi_1^{gc}(X, x_0)$ -connected. Using Proposition 4.6, $\pi_1^{gc}(X, x_0)$ is a covering subgroup of $\pi_1(X, x_0)$. Let H be a generalized covering subgroup of $\pi_1(X, x_0)$, then clearly $\pi_1^{gc}(X, x_0) \leq H$. By [21, Proposition 3.3], H is a covering subgroup. Therefore $\text{GCov}(X) \leq \text{Cov}(X)$. The equality holds from Remark 2.18. Conversely, if $\text{GCov}(X) = \text{Cov}(X)$, then clearly $\pi_1^{gc}(X, x_0)$ is a covering subgroup of $\pi_1(X, x_0)$ and so by Proposition 4.6 X is semilocally $\pi_1^{gc}(X, x_0)$ -connected. \square

Corollary 4.8. *If X is a connected, locally path connected and semi locally $\pi_1^{gc}(X, x_0)$ -connected space, then $\pi_1^{gc}(X, x_0) = \pi_1^{sp}(X, x_0)$ and X is coverable in the sense of [21].*

Corollary 4.9. *Every semilocally H -connected space is semilocally path H -connected.*

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REFERENCES

- [1] A. Arhangel'skii and M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, 1, Atlantis Press, Paris, 2008.
- [2] D. Biss, The topological fundamental group and generalized covering spaces, *Topology Appl.* **124** (2002) 355–371.
- [3] J. Brazas, Semicoverings: a generalization of covering space theory, *Homology Homotopy Appl.* **14** (2012) 33–63.
- [4] J. Brazas, The fundamental group as topological group, *Topology Appl.* **160** (2013) 170–188.
- [5] J. Brazas, Semicoverings, coverings, overlays and open subgroups of the quasitopological fundamental group, *Topology Proc.* **44** (2014) 285–313.
- [6] J. Brazas, Generalized covering space theories, *Theory Appl. Categ.* **30** (2015) 1132–1162.
- [7] J. Brazas and P. Fabel, On fundamental group with the quotient topology, *J. Homotopy Relat. Struct.* **10** (2015) 71–91.
- [8] N. Brodskiy, J. Dydak, B. Labuz and A. Mitra, Covering maps for locally path connected spaces, *Fund. Math.* **10** (2008) 13–46.
- [9] N. Brodskiy, J. Dydak, B. Labuz and A. Mitra, Topological and uniform structures on universal covering spaces, Arxiv:1206.0071v1 [math.AT].
- [10] J.W. Cannon and G. Conner, The combinatorial structure of the Hawaiian earring group, *Topology Appl.* **106** (2000) 225–271.
- [11] G. Conner, M. Meilstrup, D. Repovš, A. Zastrow and M. Zeljko, On small homotopies of loops, *Topology Appl.* **158** (2011) 397–408.
- [12] H. Fischer, D. Repovš, Z. Virk and A. Zastrow, On semilocally simply connected spaces, *Topology Appl.* **155** (2008) 1089–1097.
- [13] H. Fischer and A. Zastrow, Generalized universal covering spaces and the shape group, *Fund. Math.* **197** (2007) 167–196.
- [14] H. Fischer and A. Zastrow, A core-free semicovering of the Hawaiian Earring, *Topology Appl.* **160** (2013) 1957–1967.
- [15] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, 2002.
- [16] C. Klevdal, A Galois Correspondence with Generalized Covering Spaces, Undergraduate Honors Theses, Paper 956, University of Colorado, Boulder, 2015.
- [17] M. Kowkabi, H. Torabi and B. Mashayekhy, On the category of local homeomorphisms with the unique path lifting property, in: Proc. 24th Iranian Algebra Seminar, p. 96, Kharazmi Univ. 2014.
- [18] M. Kowkabi, H. Torabi and B. Mashayekhy, On semicovering maps, in: 8th Seminar on Geometry and Topology, Amirkabir University of Technology, 2015.
- [19] B. Mashayekhy, A. Pakdaman and H. Torabi, Spanier spaces and covering theory of non-homotopically path Hausdorff spaces, *Georgian Math. J.* **20** (2013) 303–317.

- [20] A. Pakdaman, H. Torabi and B. Mashayekhy, Small loop spaces and covering theory of non-homotopically Hausdorff spaces, *Topology Appl.* **158** (2011) 803–809.
- [21] A. Pakdaman, H. Torabi and B. Mashayekhy, On the existence of categorical universal coverings, *Ital. J. Pure Appl. Math.* **37** (2017) 289–300.
- [22] S.Z. Pashaei, M. Abdullahi Rashid, B. Mashayekhy and H. Torabi, On generalized covering subgroups of a fundamental group, in: Proc. 46th Annual Iranian Mathematics Conference, pp. 964–966, Yazd University, 2015.
- [23] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
- [24] H. Torabi, A. Pakdaman and B. Mashayekhy, Topological fundamental groups and small generated coverings, *Math. Slovaca*, **65** (2015) 1153–1164.
- [25] H. Torabi, A. Pakdaman and B. Mashayekhy, On the Spanier groups and covering and semicovering spaces, Arxiv:1207.4394v1 [math.AT].
- [26] Z. Virk, Small loop spaces, *Topology Appl.* **157** (2010) 451–455.
- [27] Z. Virk and A. Zastrow, A homotopically Hausdorff space which does not admit a generalized universal covering space, *Topology Appl.* **160** (2013) 656–666.

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