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Author(s):

D. Jiang

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ON TENSOR PRODUCT L-FUNCTIONS AND LANGLANDS FUNCTORIALITY

D. JIANG

Dedicated to Freydoon Shahidi on the occasion of his 70th birthday

Abstract. In the spirit of the Langlands proposal on *Beyond Endoscopy*, we discuss the explicit relation between the Langlands functorial transfers and automorphic L-functions. It is well-known that the poles of the L-functions have deep impact to the Langlands functoriality. Our discussion also includes the meaning of the central value of the tensor product L-functions in terms of the Langlands functoriality. This leads to the theory of the twisted automorphic descents for cuspidal automorphic representations of general classical groups.
Keywords: Automorphic representations, L-functions, Langlands functoriality, endoscopy, automorphic descent.
MSC(2010): Primary: 22E50; Secondary: 11F70, 11F85, 22E55.

1. Introduction

Let G be a reductive algebraic group defined over a number field F. Denote by $\mathcal{A}_{\rm cusp}(G)$ the set of all equivalence classes of irreducible unitary representations of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of F, occurring in the cuspidal spectrum of G. Let ${}^LG$ be the L-group of G. For any irreducible admissible representations ρ of ${}^LG$ in a complex vector space V_ρ, Langlands [47] defines the automorphic L-function

$$L(s,\pi,\rho) := \prod_\nu L_\nu(s,\pi,\rho) \tag{1.1}$$

for any π belonging to $\mathcal{A}_{\rm cusp}(G)$. Note that at the ramified local places ν of π, the local L-function $L_\nu(s,\pi,\rho)$ should be defined by means of the local Langlands conjecture of G over F_ν. It is a theorem of Langlands that the eulerian product in (1.1) converges absolutely for the real part of s large. It is a conjecture of Langlands [47] that this family of automorphic L-functions

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have meromorphic continuation to the whole complex plane $\mathbb{C}$ and satisfy the functional equation

$$L(s,\pi,\rho)=\epsilon(s,\pi,\rho)\cdot L(1-s,\pi^\vee,\rho).$$

Moreover, $L(s,\pi,\rho)$ has finitely many poles on $s\geq\frac{1}{2}$. When $\pi=\otimes_\nu\pi_\nu$ is of Ramanujan type, i.e. at every local place ν, π_ν is tempered, the possible poles of $L(s,\pi,\rho)$ are at $s=1$. In this case, the Langlands proposal for *Beyond Endoscopy* [49] predicts that the order of the pole at $s=1$ of $L(s,\pi,\rho)$ when ρ runs over all irreducible representations of ${}^LG$ should determine the source of the Langlands functorial transfers of π (Problem 2.1). This is one of the most fundamental problems in the modern theory of automorphic forms.

Inspired by the idea of Langlands, we show in [27] that the Langlands proposal for beyond endoscopy may also detect the structure of endoscopy for cuspidal automorphic representations of Ramanujan type. Our objective of this paper is to explain the idea of [27] on one hand, and on the other hand, to explain the relation of the tensor product L-functions (their poles or central values) with the Langlands functoriality.

Let $\tau\in\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_N)$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_N(\mathbb{A})$. The tensor product L-functions of π and τ are defined by

$$L(s,\pi\times\tau):=\prod_\nu L_\nu(s,\pi\times\tau). \tag{1.2}$$

We will restrict ourselves to the case when G is a classical group defined over F. When G is a classical group defined over F, the theory of endoscopic classification of Arthur [2,54] and [43] shows that the location of the poles and their orders of $L(s,\pi\times\tau)$ determine the endoscopic structure of π, or the global Arthur parameter of π. Hence it is much elementary, based on the endoscopic classification theory, to explain the relation of the poles of the tensor product L-functions with Langlands functoriality. However, it is much more delicate to explain the relation of the central value of the tensor product L-functions with Langlands functoriality. The theory of twisted automorphic descents developed in [31, 38] and [40] is for the first time to address this important relation. As explained below, this important relation is philosophically connected to the global Gan-Gross-Prasad conjecture [14].

The paper is orgainzed as follows. Section 2 will review briefly the problem (Problem 2.1) in the Langlands proposal for beyond endoscopy and explain the connection with representation theory and invariant theory of complex algebraic groups. Section 3 is to review some known results about poles of certain automorphic L-functions and the endoscopic transfers with particular examples, and explain the connection with the problem in the Langlands proposal for beyond endoscopy. Sections 4 and 5 are to discuss the relation of the tensor product L-functions and the Langlands functoriality. In Section 4, we discuss the poles of the tensor product L-functions with connection to the structure of the global Arthur parameters, which leads to the (τ, b)-theory as outlined in [28]. The (τ, b)-theory for cuspidal automorphic representations π of classical groups is to detect the occurrence of the simple global Arthur parameter (τ, b) in the global Arthur parameter of π by means of other basic invariants associated to π. This discussion includes the our recent progress in this direction. In Section 5, we explain how the central value of the tensor product L-functions can be used to detect the structure of the generalized Jacquet-Langlands correspondences between pure inner forms of classical groups, via the theory of twisted automorphic descents as recently developed by the author and his collaborators [31,38,39], and [40].

Freydoon Shahidi has made substantial contributions to the modern theory of automorphic forms, in particular, to the theory of automorphic L-functions and the Langlands functoriality. The author wish this paper is proper for the special volume dedicated to him at the occasion of his seventieth birthday. The author would also like to take this opportunity to thank him for his constant encouragement and support.

2. **On the Langlands proposal for beyond endoscopy**

We review briefly a main problem in the Langlands proposal for beyond endoscopy and try to understand the problem from representation theory and invariant theory of complex algebraic reductive groups.

2.1. **The problem.** Let F be a number field, and $\mathbb{A}$ the ring of adeles of F. Take G to be a reductive algebraic group defined over F. Denote by $\mathcal{A}_{\text{cusp}}(G)$ the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$ that occur in the cuspidal spectrum of $G(\mathbb{A})$. More precisely, we take the closed subgroup $G(\mathbb{A})^1$ of $G(\mathbb{A})$ given by

$$G(\mathbb{A})^1 := \{x \in G(\mathbb{A}) \mid |\chi(x)|_{\mathbb{A}} = 1, \forall \chi \in X(G)_F\}, \tag{2.1}$$

where $X(G)_F$ is the group of all F-rational characters of G. We consider the space of all square integrable functions on $G(F)\backslash G(\mathbb{A})^1$, which is denoted by $L^2(G(F)\backslash G(\mathbb{A})^1)$. As right $G(\mathbb{A})^1$-invariant Hilbert subspaces, we have

$$L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A})^1) \subset L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})^1) \subset L^2(G(F)\backslash G(\mathbb{A})^1), \tag{2.2}$$

where $L^2_{\text{disc}}(\cdot)$ denotes the discrete spectrum of $G(\mathbb{A})^1$ and $L^2_{\text{cusp}}(\cdot)$ denotes the cuspidal spectrum of $G(\mathbb{A})^1$. We denote by $\mathcal{A}_2(G)$ the set of irreducible unitary representations of $G(\mathbb{A})$ whose restriction to $G(\mathbb{A})^1$ are irreducible constituents of the discrete spectrum $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})^1)$. A similar definition applies to $\mathcal{A}_{\text{cusp}}(G)$. It is clear that $\mathcal{A}_{\text{cusp}}(G) \subset \mathcal{A}_2(G)$. A $\pi \in \mathcal{A}_{\text{cusp}}(G)$ is called of *Ramanujan type* if in the restricted tensor product decomposition, $\pi = \otimes_\nu \pi_\nu$, all the local components π_ν are tempered. We denote by $\mathcal{A}^r_{\text{cusp}}(G)$ the subset of $\mathcal{A}_{\text{cusp}}(G)$ consisting of all cuspidal representations of $G(\mathbb{A})$ of the Ramanujan type. For $\pi \in \mathcal{A}_{\text{cusp}}(G)$, the partial L-functions $L^S(s, \pi, \rho)$ with ρ running through the finite dimensional complex representations of the L-group ${}^LG$ are well defined for the real part of s large and form a family of analytic invariants of π. It is natural to ask how the family of the analytic invariants $L^S(s,\pi,\rho)$ determine the basic structure of π in the sense of the Langlands functoriality. When $\pi \in \mathcal{A}^r_{\mathrm{cusp}}(G)$, the partial L-functions $L^S(s,\pi,\rho)$ are holomorphic for the real part of s greater than one and may have a pole at $s = 1$. Hence the order of the pole at $s = 1$ of $L^S(s,\pi,\rho)$ becomes sensitive to the structure of π.

R. Langlands formulates the following problem in [49], which addresses the relation between poles of automorphic L-functions and the basic functorial structure of automorphic representations. This functorial structure is carried by a mysterious closed subgroup of the Langlands dual group ${}^LG$ of G, which may be denoted by $\mathcal{H}_\pi$. We note that in [49], this group was denoted by ${}^\lambda H_\pi$. It is not known how to define $\mathcal{H}_\pi$ precisely for a given π. In general, the group $\mathcal{H}_\pi$ may not be the Langlands dual group of a reductive algebraic group over F.

Problem 2.1 (Langlands [49])**.** *Let G be a reductive algebraic group defined over a number field F, and let π be an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$. Assume that π is of Ramanujan type. There exists a closed subgroup $\mathcal{H}_\pi$ of the Langlands dual group ${}^LG$ of G such that for all finite-dimensional complex representations ρ of ${}^LG$, the multiplicity $m_{\mathcal{H}_\pi}(\rho)$ of the trivial representation of $\mathcal{H}_\pi$ occurring in the restriction of ρ to $\mathcal{H}_\pi$ is equal to the order $m_\pi(\rho)$ of the pole at $s = 1$ of the Langlands automorphic L-function $L(s,\pi,\rho)$ associated to the pair (π,ρ).*

We are going to understand this problem in terms of the conjectural Langlands group $\mathcal{L}_F$. As described by Arthur in [2], the theory of automorphic representations suggests the existence of the hypothetical Langlands group $\mathcal{L}_F$, whose complex representations are the global Langlands parameters that should classify the automorphic representations of $G(\mathbb{A})$ up to global L-packets. Assume that a $\pi \in \mathcal{A}_{\mathrm{cusp}}(G)$ has a hypothetical global Langlands parameter

$$\phi \ : \ \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG.$$

Then one may take $\mathcal{H}_\pi$ to be the closure of the image $\phi(\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}))$ in ${}^LG$, as a topological group with the topology given by the semi-direct product of the complex dual group $G^\vee(\mathbb{C})$ and a finite Galois group. In this situation, one may define a function

$$\rho \in R({}^LG) \mapsto m_{\mathcal{H}_\pi}(\rho) \in \mathbb{Z}$$

where $R({}^LG)$ denotes the set of equivalence classes of all irreducible complex representations of ${}^LG$, and $\mathbb{Z}$ is the ring of all integers. One may call this function the *dimension datum* of $\mathcal{H}_\pi$ and denote it by $\mathfrak{D}_{\mathcal{H}_\pi}$. It follows that the dimension data $\mathfrak{D}_{\mathcal{H}_\pi}$ of $\mathcal{H}_\pi$ can be viewed as elements in the product space

$\mathbb{Z}^{R({}^LG)}$ endowed with the product topology. In general, this produces a map from the set of closed subgroups $\mathcal{H}$ of the Langlands dual group ${}^LG$ to the product space $\mathbb{Z}^{R({}^LG)}$ by

$$\mathfrak{D} \ : \ \mathcal{H} \mapsto \mathfrak{D}_{\mathcal{H}}.$$

Hence one would like to understand the structure of the image of the map $\mathfrak{D}$ in the product space $\mathbb{Z}^{R({}^LG)}$ and the structure of the fibers of the map $\mathfrak{D}$. It is easy to check that if two subgroups, $\mathcal{H}_1$ and $\mathcal{H}_2$, of ${}^LG$ are conjugate to each other, then $\mathfrak{D}_{\mathcal{H}_1} = \mathfrak{D}_{\mathcal{H}_2}$. Thus the structure of the fibers of the map $\mathfrak{D}$ is given up to conjugation.

If G is F-split, one may replace the Langlands dual group ${}^LG$ by the complex dual group $G^\vee(\mathbb{C})$. The dimension datum problems turn to be problems in representations and invariants of complex Lie groups and hence can be reformulated in terms of the representations of compact Lie groups. We may abuse the notation and assume that $G^\vee$ is a compact semisimple Lie group here. In [49], Langlands suggests that for a given dimension datum $\underline{n} \in \mathbb{Z}^{R(G^\vee)}$, the fiber $\mathfrak{D}^{-1}(\underline{n})$ consists of finitely many conjugate classes of closed subgroups of $G^\vee$. The dimension datum problems have been investigated by M. Larsen and R. Pink [50] and by Song Wang [63] and [64], and more systematically by Jinpeng An, Jiu-Kang Yu and Jun Yu [1] and by Jun Yu [65] and [66]. It is important to point out that [1, Theorem 3.4] provides a criterion for when a dimension datum $\mathfrak{D}_{\mathcal{H}}$ is an isolated point in the image $\mathfrak{I}(\mathfrak{D})$ of the map $\mathfrak{D}$ inside $\mathbb{Z}^{R(G^\vee)}$. This isolation property of the dimension datum $\mathfrak{D}_{\mathcal{H}}$ is equivalent to the property that the dimension datum $\mathfrak{D}_{\mathcal{H}}$ is completely determined by a finite subset of irreducible representations in $R(G^\vee)$. In such a circumstance, dimension datum problem turns to be a finite problem in representations and invariants of the compact Lie group $G^\vee$, which will be important for the understanding of the Langlands functoriality in such a situation.

2.2. **The source of functoriality and observable groups.** The source group $\mathcal{H}_\pi$ of the functoriality may also be understood in terms of the notion of *observable subgroups* of $G^\vee(\mathbb{C})$ when G is assumed to be F-split. We recall our discussion of this issue from [27, Section 4].

We may assume that the Langlands conjecture holds, that is, *all Langlands automorphic L-functions $L(s,\pi,\rho)$ have meromorphic continuation to $\mathbb{C}$ and satisfy the standard functional equation.*

For $\pi \in \mathcal{A}^r_{\mathrm{cusp}}(G)$, we define

$$\mathcal{R}_\pi =: \{\rho \in R(G^\vee(\mathbb{C})) \mid m_\pi(\rho) \geq 1\}, \tag{2.3}$$

and

$$\mathcal{N}_\pi =: \{m_\pi(\rho) \mid \rho \in \mathcal{R}_\pi\}. \tag{2.4}$$

It is clear that $\mathcal{N}_\pi$ is closely related to the dimension data $\mathfrak{D}_{\mathcal{H}_\pi}$, with the assumption of the existence of the algebraic subgroup $\mathcal{H}_\pi$ of $G^\vee$. Assume that

Problem 2.1 has an affirmative solution, and one has

$$m_{\mathcal{H}_\pi}(\rho) = m_\pi(\rho) \tag{2.5}$$

for all $\rho \in R(G^\vee(\mathbb{C}))$. For any $\rho \in \mathcal{R}_\pi$, denote by V_ρ the space of the representation ρ of $G^\vee$. It follows that

$$m_\pi(\rho) = m_{\mathcal{H}_\pi}(\rho) = \dim_{\mathbb{C}} V_\rho^{\mathcal{H}_\pi} \in \mathcal{N}_\pi.$$

Take linearly independent vectors $v_1, \cdots, v_{m_\pi(\rho)} \in V_\rho^{\mathcal{H}_\pi}$ and denote by $G^\vee_{v_j}$ the stabilizer of v_j, i.e.

$$G^\vee_{v_j} = \{g \in G^\vee(\mathbb{C}) \mid \rho(g)(v_j) = v_j\}.$$

By [24, Theorem 1.2], these groups are observable subgroups of $G^\vee(\mathbb{C})$. In general, one may call an algebraic subgroup $\mathcal{A}$ of $G^\vee(\mathbb{C})$ *observable* if $G^\vee(\mathbb{C})/\mathcal{A}$ is quasi-affine following the definition in [6] or [56, Page 172]). Following [24, Chapter 1], one has the following definition.

Definition 2.2. Let $\mathcal{A}$ be an algebraic subgroup of $G^\vee(\mathbb{C})$. Define

$$\mathcal{A}'' = \{g \in G^\vee(\mathbb{C}) \mid f(xg) = f(x) \text{ for all } f \in \mathbb{C}[G^\vee(\mathbb{C})]^{\mathcal{A}}\}$$

where $\mathbb{C}[G^\vee(\mathbb{C})]^{\mathcal{A}}$ denotes the $\mathcal{A}$-invariants in $\mathbb{C}[G^\vee(\mathbb{C})]$. Then the algebraic subgroup $\mathcal{A}''$ containing $\mathcal{A}$ is called the *observable hull* of $\mathcal{A}$. If $\mathcal{A} = \mathcal{A}''$, then $\mathcal{A}$ is called *observable*.

By [24, Theorem 2.1], an algebraic subgroup $\mathcal{A}$ of $G^\vee(\mathbb{C})$ is observable if and only if there is a finite-dimensional complex representation ρ of $G^\vee(\mathbb{C})$ and a vector $v \in V_\rho$ such that $\mathcal{A}$ is the stabilizer of v in $G^\vee(\mathbb{C})$. Hence the above two definitions are equivalent. Since $G^\vee_{v_j}$ are observable, it follows that $\mathcal{A}_\rho := \cap_{j=1}^{m_\pi(\rho)} G^\vee_{v_j}$ is observable. It is clear that $\mathcal{H}_\pi \subset \mathcal{A}_\rho$ for all $\rho \in \mathcal{R}_\pi$, and hence we have $\mathcal{H}_\pi \subset \cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$. The following is proved in [27].

Proposition 2.3 ([27, Proposition 4.4])**.** *With notation as above, $\cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$ is the observable hull of $\mathcal{H}_\pi$.*

In general, the algebraic subgroup $\mathcal{H}_\pi$ may not be observable, i.e.

$$\mathcal{H}_\pi = \cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$$

may not hold. However, in [27], we prove

Proposition 2.4 ([27, Proposition 4.5])**.** *Set $\mathcal{A}_\pi = \cap_{\rho\in\mathcal{R}_\pi}\mathcal{A}_\rho$. For any $\rho \in R(G^\vee(\mathbb{C}))$, the following holds:*

$$V_\rho^{\mathcal{A}_\pi} = V_\rho^{\mathcal{H}_\pi}.$$

In particular, $m_{\mathcal{A}_\pi}(\rho) = m_{\mathcal{H}_\pi}(\rho)$ is true for all $\rho \in R(G^\vee(\mathbb{C}))$.

In the next section, we consider the situation of endoscopic transfers in the framework of the Langlands proposal for beyond endoscopy, which yields interesting examples of isolated dimension data.

3. **Poles of certain L-functions and endoscopy**

We will restrict ourselves here to the case of F-split classical groups G and explain how the endoscopy structures of cuspidal automorphic representations of $G(\mathbb{A})$ can be detected by the information on poles of certain L-functions following the formulation of the Langlands proposal for beyond endoscopy.

3.1. **Endoscopic classification of discrete spectrum.** Let G_n be the F-split special orthogonal group SO_{2n+1}, the symplectic group Sp_{2n}, or the F-quasisplit even special orthogonal group SO_{2n}. We may follow [2, Chapter 1] and also [28, Section 2] for the notation. Recall from the work of Arthur [2] the endoscopic classification of the discrete spectrum of G_n.

Define $N = N_{G_n^\vee}$ to be $2n$ if G_n is either SO_{2n+1} or SO_{2n}, and to be $2n+1$ if $G_n = \mathrm{Sp}_{2n}$. The set of global Arthur parameters for G_n is denoted by $\widetilde{\Psi}_2(G_n)$. In order to explicate the structure of the parameters in $\widetilde{\Psi}_2(G_n)$, we first recall from [2] the description of the self-dual, elliptic, global Arthur parameters for GL_N, which is denoted by $\widetilde{\Psi}_{\mathrm{ell}}(N)$. We refer to [2] for detailed discussion about general global Arthur parameters. The elements of $\widetilde{\Psi}_{\mathrm{ell}}(N)$ are denoted by ψ^N, which have the form

$$\psi^N = \psi_1^{N_1} \boxplus \cdots \boxplus \psi_r^{N_r} \tag{3.1}$$

with $N = \sum_{i=1}^r N_i$. The formal summands $\psi_i^{N_i}$ are *simple* parameters of the form

$$\psi_i^{N_i} = \mu_i \boxtimes \nu_i$$

with $N_i = a_i b_i$, where $\mu_i = \tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ and ν_i is a b_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Following the notation used in our previous paper [28], we also denote

$$\psi_i^{N_i} = (\tau_i, b_i)$$

for $i = 1, 2, \cdots, r$. The global parameter ψ^N is called *elliptic* if the decomposition of ψ^N into the simple parameters is of multiplicity free, i.e. $\psi_i^{N_i}$ and $\psi_j^{N_j}$ are not equivalent if $i \neq j$ in the sense that either τ_i is not equivalent to τ_j or $b_i \neq b_j$. The global parameter ψ^N is called *self-dual* if each simple parameter $\psi_i^{N_i}$ occurs in the decomposition of ψ^N is self-dual in the sense that τ_i is self-dual. A global parameter ψ^N in $\widetilde{\Psi}_{\mathrm{ell}}(N)$ is called *generic* if $b_i = 1$ for $i = 1, 2, \cdots, r$. The set of the generic, self-dual, elliptic, global Arthur parameters for GL_N is denoted by $\widetilde{\Phi}_{\mathrm{ell}}(N)$. Hence elements ϕ in $\widetilde{\Phi}_{\mathrm{ell}}(N)$ are of the form:

$$\phi^N = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1). \tag{3.2}$$

The sets of simple parameters are denoted by $\widetilde{\Psi}_{\mathrm{sim}}(N)$ and $\widetilde{\Phi}_{\mathrm{sim}}(N)$, respectively. It is clear that the set $\widetilde{\Phi}_{\mathrm{sim}}(N)$ is in one-to-one correspondence with the set of equivalence classes of the self-dual, irreducible cuspidal automorphic representations of $\mathrm{GL}_N(\mathbb{A})$. A self-dual $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_a)$ is called *of symplectic type* if the (partial) exterior square L-function $L^S(s, \tau, \wedge^2)$ has a (simple) pole at $s = 1$; otherwise, τ is called *of orthogonal type*. In the latter case, the (partial) symmetric square L-function $L^S(s, \tau, \mathrm{sym}^2)$ has a (simple) pole at $s = 1$.

From [2, Section 1.4], for any parameter ψ^N in $\widetilde{\Psi}_{\mathrm{ell}}(N)$, there is a twisted elliptic endoscopic datum $(G, s, \xi) \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ such that the set of the global parameters $\widetilde{\Psi}_2(G) = \widetilde{\Psi}_2(G, \xi)$ can be identified as as a subset of $\widetilde{\Psi}_{\mathrm{ell}}(N)$. We refer to [2, Section 1.4] for more constructive description of the parameters in $\widetilde{\Psi}_2(G)$. The elements of $\widetilde{\Psi}_2(G_n)$ are of the form

$$\psi = (\tau_1, b_1) \boxplus \cdots \boxplus (\tau_r, b_r). \tag{3.3}$$

Here $N = N_1 + \cdots + N_r$ and $N_i = a_i \cdot b_i$, and $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$ and b_i represents the b_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Note that each simple parameter $\psi_i = (\tau_i, b_i)$ belongs to $\widetilde{\Psi}_2(G_{n_i})$ with $n_i = [\frac{N_i}{2}]$, for $i = 1, 2, \cdots, r$; and for $i \neq j$, ψ_i is not equivalent to ψ_j. The parity for τ_i and b_i is discussed as above. The subset of generic elliptic global Arthur parameters in $\widetilde{\Psi}_2(G_n)$ is denoted by $\widetilde{\Phi}_2(G_n)$, whose elements are in the form of (3.1).

Theorem 3.1 ([2])**.** *For any $\pi \in \mathcal{A}_2(G_n)$, there is a global Arthur parameter $\psi \in \widetilde{\Psi}_2(G_n)$, such that π belongs to the global Arthur packet, $\widetilde{\Pi}_\psi(G_n)$, attached to the global Arthur parameters ψ. Moreover, if $\pi \in \mathcal{A}^r_{\mathrm{cusp}}(G_n)$ is of Ramanujan type, then $\pi \in \widetilde{\Pi}_\phi(G_n)$ with a generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n)$.*

If we denote by $\mathcal{A}^{\widetilde{\Phi}_2}_{\mathrm{cusp}}(G_n)$ the subset of $\mathcal{A}_{\mathrm{cusp}}(G_n)$ consisting of all cuspidal representations of $G_n(\mathbb{A})$ with generic global Arthur parameters, then one has

$$\mathcal{A}^r_{\mathrm{cusp}}(G_n) \subset \mathcal{A}^{\widetilde{\Phi}_2}_{\mathrm{cusp}}(G_n).$$

The *Generalized Ramanujan Conjecture* asserts that the equality

$$\mathcal{A}^r_{\mathrm{cusp}}(G_n) = \mathcal{A}^{\widetilde{\Phi}_2}_{\mathrm{cusp}}(G_n)$$

holds. We note that for the problem (Problem 2.1) in the Lanlands proposal for beyond endoscopy and the discussion in the rest of this paper can be formulated for cuspidal automorphic representations in the set $\mathcal{A}^{\widetilde{\Phi}_2}_{\mathrm{cusp}}(G_n)$.

3.2. **Twisted endoscopy for GL_N.** We recall the relation between the (twisted) endoscopic structure [45] and [2] of irreducible unitary cuspidal automorphic representation τ of $\mathrm{GL}_N(\mathbb{A})$ and poles of certain types of automorphic L-functions. We illustrate this relation for the exterior square L-functions $L(s, \tau, \wedge^2)$. We recall from [27, Theorem 2.2].

Theorem 3.2. *Let τ be an irreducible, unitary, self-dual, cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$.*

(1) *The exterior square L-function $L(s, \tau, \wedge^2)$ is holomorphic for the real part of s greater than one, as in [44, Theorem 3.1].*
(2) *$L(s, \tau, \wedge^2)$ has at most a simple pole at $s = 1$.*
(3) *$L(s, \tau, \wedge^2)$ has a simple pole at $s = 1$ if and only if τ is a Langlands functorial lifting from an irreducible generic cuspidal automorphic representation π of $\mathrm{SO}_{2n+1}(\mathbb{A})$.*
(4) *Let S be a finite set of local places of F including all archimedean places. The complete exterior square L-function $L(s, \tau, \Lambda^2)$ has a simple pole at $s = 1$ if and only if the partial exterior square L-function $L^S(s, \tau, \Lambda^2)$ has a simple pole at $s = 1$.*
(5) *The partial exterior square L-function $L^S(s, \tau, \Lambda^2)$ has a simple pole at $s = 1$ if and only if τ has a nontrivial Shalika model, as proved in [26].*

We refer to [27, p. 7] for the discussion of the proof of Theorem 3.2. We note that the relation between the pole at $s = 1$ of the exterior square L-function and the existence of the (twisted) endoscopic transfer is closely related to the period condition and also note that the complete exterior square L-function discussed in Theorem 3.2 is given by the Langlands-Shahidi method [58]. The Langlands functorial transfer from SO_{2n+1} to GL_{2n} was first established by the work of J. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi in [11], at almost all local places. It is proved in [33, Theorem 6.1] and [34, Theorem 5.1], and in [12, Theorem 7.1] that the transfer from π to $\tau(\pi)$ is Langlands functorial at all local places. By [22] and also [23], the image of the Langlands functorial transfer from all irreducible generic cuspidal automorphic representations σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ is completely characterized.

Theorem 3.3 ([22, Theorem A])**.** *An irreducible unitary automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ is the weak Langlands functorial transfer of an irreducible generic unitary cuspidal automorphic representation π of $\mathrm{SO}_{2n+1}(\mathbb{A})$ if and only if τ is equivalent to the following isobaric representation*

$$\tau \cong \tau_1 \boxplus \cdots \boxplus \tau_r \tag{3.4}$$

where $\tau_j \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{2n_j})$ for $j = 1, 2, \cdots, r$, with the properties that

(a) *$n = \sum_{j=1}^{r} n_j$ is a partition of n with $n_j > 0$;*
(b) *$\tau_i \ncong \tau_j$ if $i \neq j$;*
(c) *the partial exterior square L-function $L^S(s, \tau_j, \Lambda^2)$ has a pole at $s = 1$.*

We remark that the above theorems work well for all F-quasisplit classical groups and are now established for all irreducible cuspidal automorphic representations with generic global Arthur parameters through the theory of endoscopic classifications of the discrete spectrum [2,54], and [43] and the work of Mœglin [51].

3.3. **Endoscopy for classical groups.** We discuss how to formulate the characterization of endoscopic structure of members in the set $\mathcal{A}_{\mathrm{cusp}}^{\widetilde{\Phi}_2}(G_n)$ along the idea from the Langlands proposal for beyond endoscopy, as in Problem 2.1.

For simplicity, we consider in this section G_n to be an F-split classical group of type SO_{2n+1}, Sp_{2n}, or SO_{2n}. Let $\omega_1, \omega_2, \cdots, \omega_n$ be the fundamental weights of $G^\vee(\mathbb{C})$, following the notation of N. Bourbaki ([7]). We denote by ρ_{ω_i} the corresponding irreducible fundamental representation of $G^\vee(\mathbb{C})$ with the highest weight ω_i. We define

$$\rho_2 = \begin{cases} \rho_{\omega_2}, & \text{if } G_n = \mathrm{SO}_{2n+1}; \\ \rho_{2\cdot\omega_1}, & \text{if } G_n = \mathrm{Sp}_{2n} \text{ or } \mathrm{SO}_{2n}. \end{cases} \tag{3.5}$$

We formulate the following conjecture.

Conjecture 3.4. *Assume that $\pi \in \mathcal{A}_{\mathrm{cusp}}^{\widetilde{\Phi}_2}(G_n)$ has a generic global Arthur parameter*

$$\phi = (\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_r, 1),$$

and ρ_2 is defined as in (3.5). The following hold.

(1) **Order of Pole:** *$r-1$ is equal to the order of the pole at $s=1$ of the automorphic L-function $L^S(s, \pi, \rho_2)$.*

(2) **Multiplicity:** *Define $\mathcal{H}_\pi^{\rho_2} = H_1^\vee \times \cdots \times H_r^\vee$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}_\pi}(\rho_2, 1) = r - 1 = m_\pi(\rho_2),$$

where $m_\pi(\rho_2)$ is the order of the pole at $s=1$ of the automorphic L-function $L^S(s, \pi, \rho_2)$, and H_i is the twisted endoscopic group determined by the simple global Arthur parameter $\phi_i = (\tau_i, 1)$.

Conjecture 3.4 has been proved in [27] for $G_n = \mathrm{SO}_{2n+1}$. The proof for Sp_{2n} and SO_{2n} is similar, and we will omit it here. In fact, the results in [27] are stronger and more precise. First, we prove the Langlands conjecture for analytic properties of automorphic L-functions for $L(s, \pi, \rho_2)$.

Theorem 3.5 ([27, Theorem 2.1])**.** *For $\pi \in \mathcal{A}_{\mathrm{cusp}}^{\widetilde{\Phi}_2}(\mathrm{SO}_{2n+1})$ and for ρ_2 as defined in (3.5), the second fundamental automorphic L-function $L(s, \sigma, \rho_2)$ enjoys following properties.*

(1) *There exists an irreducible admissible automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that*

$$L(s, \pi, \rho_2) = \frac{L(s, \tau, \wedge^2)}{\zeta_F(s)}$$

holds for the real part of s large.

(2) *The eulerian product defining the L-function $L(s, \sigma, \rho_2)$ converges absolutely for the real part of s greater than one, has meromorphic continuation to the whole complex plane, and satisfies the functional equation*

$$L(s, \sigma, \rho_2) = \epsilon(s, \sigma, \rho_2) L(1 - s, \sigma^\vee, \rho_2^\vee)$$

with $\epsilon(s, \sigma, \rho_2) = \epsilon(s, \pi, \wedge^2)$*, the* ϵ*-factor for the exterior square L-function* $L(s, \pi, \wedge^2)$*.*

(3) *The L-function* $L(s, \sigma, \rho_2)$ *has possible poles at* $s = 0, 1$*, besides other possible poles in the open interval* $(0, 1)$*.*

We note that [27, Theorem 2.1] was proved for π generic. It is clear that the proof in [27] can be extended to all π in $\mathcal{A}_{\text{cusp}}^{\widetilde{\Phi}_2}(\text{SO}_{2n+1})$ based on the endoscopic classification of Arthur in [2], since each global Arthur packet with a generic global Arthur parameter contains a generic member by the automorphic descent of Ginzburg-Rallis-Soudry in [23]. The same argument applies to other results stated here.

Then we characterize the stability of cuspidal automorphic representations in the set $\mathcal{A}_{\text{cusp}}^{\widetilde{\Phi}_2}(\text{SO}_{2n+1})$ by means of the order of the pole at $s = 1$ of the L-function $L(s, \sigma, \rho_2)$. We recall from [2] that an irreducible cuspidal representation π of classical group $G_n(\mathbb{A})$ is called *stable* if it is not a non-trivial endoscopic transfer, or equivalently, it has a simple global Arthur parameter.

Theorem 3.6 ([27, Theorem 3.2])**.** *For* $\pi \in \mathcal{A}_{\text{cusp}}^{\widetilde{\Phi}_2}(\text{SO}_{2n+1})$ *and for* ρ_2 *as defined in* (3.5)*, the following hold.*

(1) **Stability:** *The partial L-function* $L^S(s, \pi, \rho_2)$ *has a pole of order* $r - 1$ *at* $s = 1$ *if and only if there exists a partition* $n = \sum_{j=1}^{r} n_j$ *with* $n_j > 0$ *such that* π *is an endoscopic transfer from an irreducible, stable, cuspidal automorphic representation* $\pi_1 \otimes \cdots \otimes \pi_r$ *of*

$$\text{SO}_{2n_1+1}(\mathbb{A}) \times \cdots \times \text{SO}_{2n_r+1}(\mathbb{A}).$$

In particular, π *is* **stable** *if and only if the full L-function* $L(s, \pi, \rho_2)$ *or equivalently, the partial L-function* $L^S(s, \pi, \rho_2)$ *is holomorphic at* $s = 1$*.*

(2) **Multiplicity:** *Let* $m_\pi(\rho_2)$ *be the order of the pole at* $s = 1$ *of* $L(s, \pi, \rho_2)$ *and let* $m_{\mathcal{H}_{[n_1 \cdots n_r]}}(\rho_2)$ *be the dimension of*

$$\text{Hom}_{\mathcal{H}_{[n_1 \cdots n_r]}(\mathbb{C})}(\rho_2, 1)$$

where $\mathcal{H}_{[n_1 \cdots n_r]} = \text{Sp}_{2n_1} \times \cdots \times \text{Sp}_{2n_r}$*. Then*

$$m_\pi(\rho_2) = m_{\mathcal{H}_{[n_1 \cdots n_r]}}(\rho_2).$$

We note that the order of the pole at $s = 1$ of the second fundamental L-function $L(s, \pi, \rho_2)$ only detects the number r that is the number of the simple global Arthur parameters in the global Arthur parameter ϕ of π. In order to determine precisely the size of each simple global Arthur parameters in ϕ, we need information from more L-functions, i.e. more information from the dimension data associated to ϕ or π. As proved in [27, Theorem 4.10], we have the following conjecture.

Conjecture 3.7. *For $\pi \in \mathcal{A}_{\mathrm{cusp}}^{\widetilde{\Phi}_2}(\mathrm{SO}_{2n+1})$, assume that the automorphic L-functions $L(s,\pi,\rho)$ attached to the fundamental representations*

$$\rho = \rho_3, \rho_4, \cdots, \rho_n$$

are analytic for the real part of S greater than one. The structure of the algebraic subgroup $\mathcal{H}_{[n_1,\cdots,n_r]}$ or the partition $n = \sum_{i=1}^r n_i$ is completely determined by the order of the pole at $s = 1$ of the L-function $L(s,\sigma,\rho)$ for $\rho \in \{\rho_2, \rho_4, \cdots, \rho_{2[\frac{n}{2}]}\}$ of the complex dual group $\mathrm{Sp}_{2n}(\mathbb{C})$. Moreover, the L-function $L(s,\sigma,\rho)$ is holomorphic at $s = 1$ for $\rho \in \{\rho_1, \rho_3, \cdots, \rho_{2[\frac{n}{2}]+1}\}$.

This conjecture was proved in [27, Theorem 4.10], assuming that the Langlands conjecture holds for all fundamental L-functions $L(s,\sigma,\rho)$ with $\rho \in \{\rho_3, \rho_4, \cdots, \rho_n\}$ plus a mild assumption on invariants. Hence it suggests that the dimension datum $\mathfrak{D}_{\mathcal{H}_{[n_1,\cdots,n_r]}}$ is completely determined by the finite subset

$$\{\rho_2, \rho_4, \cdots, \rho_{2[\frac{n}{2}]}\}$$

in $R(\mathrm{Sp}_{2n}(\mathbb{C}))$, and therefore, the dimension datum $\mathfrak{D}_{\mathcal{H}_{[n_1,\cdots,n_r]}}$ is an isolated point in the image $\Im(\mathfrak{D})$ in $\mathbb{Z}^{R(\mathrm{Sp}_{2n}(\mathbb{C}))}$. Also we mention that the characterization of the stability or the endoscopic structure of cuspidal automorphic representations with generic global Arthur parameters can be given in terms of family of periods as introduced and discussed by Ginzburg and the author in [16]. Finally, we expect that the similar results hold for Sp_{2n} and SO_{2n}.

4. On tensor product L-functions: poles

Let G_n be either Sp_{2n} or any special orthogonal group $\mathrm{SO}_{\mathfrak{n}}$ defined over F with $\mathfrak{n} = 2n$ or $2n+1$. We consider the family of tensor product L-functions:

$$L^S(s, \pi \times \tau) = \prod_{\nu} L_\nu(s, \pi_\nu \times \tau_\nu). \tag{4.1}$$

where $\pi = \otimes_\nu \pi_\nu \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and $\tau = \otimes_\nu \tau_\nu \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_m)$. At unramified local places, Denote by c_{π_ν} the Satake parameter associated to the unramified π_ν, which is a semi-simple conjugacy class in $G_n^\vee(\mathbb{C})$, and the same for c_{τ_ν}. The local L-factors at unramified local places are given by

$$L_\nu(s, \pi_\nu \times \tau_\nu) = (\det(I - (c_{\pi_\nu} \otimes c_{\tau_\nu}) q_\nu^{-s}))^{-1}.$$

Langlands proves that the (partial) L-functions $L^S(s, \pi \times \tau)$ converge absolutely for the real part of s large and have meromorphic continuation to the whole complex plane $\mathbb{C}$. It is a conjecture of Langlands that $L(s, \pi \times \tau)$ satisfies a functional equation relating s to $1-s$ and has finitely many poles at $s > \frac{1}{2}$. This conjecture has been verified by the Langlands-Shahidi method [58], and by the Rankin-Selberg method [21] for π to be generic. When $\pi \in \mathcal{A}_{\mathrm{cusp}}^{\widetilde{\Phi}_2}(G_n)$ and G_n is F-quasisplit, this conjecture has been proved by Arthur in [2] using the theory of endoscopic classification of the discrete spectrum of G_n. For general classical groups, this follows from [54] and [43]. When π has a non-generic global Arthur parameter, the situation becomes more complicated. We refer to the work of Mœglin [51], and also the work of [17, 20, 37].

We would like to discuss briefly the poles of the tensor product (partial) L-functions $L^S(s, \pi \times \tau)$ and the endoscopic structure of π. Let ψ be the global Arthur parameter of π. It can be written as

$$\psi = (\tau_1, b_1) \boxplus \cdots \boxplus (\tau_r, b_r).$$

If the partial L-function $L^S(s, \pi \times \tau)$ is holomorphic for the real part of s greater than s_0 and has a pole at $s = s_0$, then it is not hard to show that $\tau = \tau_{i_0}$ for some $i_0 \in \{1, 2, \cdots, r\}$ and $s_0 = \frac{b_{i_0}+1}{2}$. The converse is also true. Hence the right-most pole at $s = s_0$ of $L^S(s, \pi \times \tau)$ detects the occurrence of the simple global Arthur parameter $(\tau, 2s_0 - 1)$ in the global Arthur parameter ψ of π. It remains very interesting to consider the following problem

Problem 4.1. *For a given $\pi \in \mathcal{A}_{\text{cusp}}(G_n)$, determine the upper bound for the integer b in the simple global Arthur parameter (τ, b) that occurs in the global Arthur parameter ψ of π.*

In order to understand this problem, we formulate an approach, called the (τ, b)-*theory* in [28], which will be recalled briefly below.

4.1. **On (τ, b)-theory: $\tau = \chi$.** It is a program of S. Rallis [57] in collaboration with I. Piatetski-Shapiro and S. Kudla to understand the poles of the partial L-functions $L^S(s, \pi \times \chi)$ with a quadratic character χ with connection to the structure of the theta lifting of π, via the doubling method [55] and the regularized Siegel-Weil formula [46]. We may reformulate the program of Rallis as the (χ, b)-theory, as a special case taking τ to be χ of the general (τ, b)-theory.

For example, take $G_n = \mathrm{Sp}_{2n}$. The doubling method of Piatetski-Shapiro and Rallis [55] gives a global zeta integral to represent the L-function $L^S(s, \pi \times \chi)$ with quadratic character χ and $\pi \in \mathcal{A}_{\text{cusp}}(\mathrm{Sp}_{2n})$. By means of the regularized Siegel-Weil formula for Sp_{4n}, Kudla and Rallis prove the following theorem in [46].

Theorem 4.2 (Kudla-Rallis, [46])**.** *Let $\pi \in \mathcal{A}_{\text{cusp}}(\mathrm{Sp}_{2n})$ and χ be a unitary character of $\mathbb{A}^\times / F^\times$.*

(1) *If $\chi^2 \neq 1$, then $L^S(s, \pi \otimes \chi)$ is holomorphic for the real part of s greater than or equal to $\frac{1}{2}$.*
(2) *If $\chi^2 = 1$, then the possible poles of $L^S(s, \pi \otimes \chi)$ for $s \geq \frac{1}{2}$ are at most simple and are located at the points*

$$s_0 \in \{1, 2, \cdots, [\frac{n}{2}] + 1\}.$$

With the endoscopic classification of Arthur citeA13, we have the following consequence.

Corollary 4.3. *Let χ be a quadratic character of $\mathbb{A}^\times/F^\times$. For any $\pi \in \mathcal{A}_{\text{cusp}}(\text{Sp}_{2n})$, if the simple global Arthur parameter (χ, b) with an integer b occurs in the global Arthur parameter ψ of π, then $b \leq 2[\frac{n}{2}] + 1$.*

It is worthwhile to remark that the integer b in the simple global Arthur parameter (τ, b) occurring in the global Arthur parameter ψ of π measures how far away of the cuspidal automorphic representation π is from the Ramanujan bound according to the generalized Ramanujan conjecture for the cuspidal spectrum of Sp_{2n}. In our recent paper joint with Baiying Liu [29], we consider the precise upper bound of Ramanujan type for the whole cuspidal spectrum of Sp_{2n} and relate this problem to the construction and the characterization of the so called *small* cuspidal automorhic representations of Sp_{2n}. For instance, the cuspidal automorphic representation π of Sp_{4n} as constructed by T. Ikeda in [25] has the global Arthur parameter $(\tau, 2n)$ with $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_2)$ of symplectic type is the worst cuspidal automorphic representation of non-unipotent type in the cuspidal spectrum in the sense of the generalized Ramanujan conjecture. Note that the construction of Ikeda needs to assume that the ground field F must be totally real. We show in [29] that if the ground field F is totally imaginary and $n \geq 5$, then the Ikeda construction does not exist for Sp_{2n}. Hence we can get a better Ramanujan type bound in such a circumstance for Sp_{2n}. We refer to [29] for the detailed discussion of this issue and other related problems.

We also remark that the characterization of the poles of the partial L-functions $L^S(s, \pi \otimes \chi)$ can be given in terms of certain periods of π. Such a characterization has important applications to arithmetic and geometric problems as in the work of Bergeron, Millson and Mœglin [4] and [5], and also [3]. The whole theory works for all classical groups as indicated in [18,35] and [36].

4.2. **On (τ, b)-theory: general τ.** In order to extend the (χ, b)-theory to the general (τ, b)-theory for irreducible cuspidal automorphic representations $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_a)$, the theory of global zeta integrals for the tensor product L-functions $L(s, \pi \times \tau)$ for general irreducible cuspidal automorphic representations $\pi \in \mathcal{A}_{\text{cusp}}(G)$ with G being any classical group is necessary. Currently, certain types of global zeta integrals are known to give the tensor product L-functions $L(s, \pi \times \tau)$, see [17, 20, 37, 38, 40, 59–61] and. However, it is not hard to discover that the global zeta integrals in such circumstances are different from the doubling integrals developed by Piatetski-Shapiro and Rallis [55]. Although the author has explained in [28] what should be expected for a general (τ, b)-theory, and also there are conjectures that may be relevant made in [15] and [19], it is still not clear about how to develop such a theory. The key point is to bound the integer b for the simple global Arthur parameter (τ, b) to occur in the global Arthur parameter of π. As explained in [38] and [40], this issue may have a close relation to the global Gan-Gross-Prasad conjecture [14], and hence should be deeply involved in the arithmetic of automorphic representations. We will leave a further discussion of this issue in the next section and will be considered in the future.

5. On the tensor product L-functions: central values

Let G be a classical group defined over a number field F. For $\pi \in \mathcal{A}_{\mathrm{cusp}}(G)$ and for $\tau \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_a)$, we consider the functorial meaning of the cental value $L(\frac{1}{2}, \tau \times \pi)$. In Section 4, we take π as the main object to study and use τ to test if the global Arthur parameter ψ of π has a simple summand (τ, b) for some b. In this section, we want to turn around the roles of τ and π.

Let G^* be an F-quasisplit classical group and G be one of its pure inner form. Assume that $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ is an irreducible generic isobaric automorphic representation of $\mathrm{GL}_N(\mathbb{A})$ with $N = N_{G^*}$ as before, which determines a generic global Arthur parameter ϕ of G^*. Then the endoscopic classification provides the following diagram:

$$\begin{array}{ccc} & \widetilde{\Phi}_2(G^*) & \\ & \phi & \\ & \swarrow \quad\quad \searrow & \\ \mathcal{A}^{\widetilde{\Phi}_2}_{\mathrm{cusp}}(G) \cap \widetilde{\Pi}_\phi(G) & \Longleftrightarrow & \widetilde{\Pi}_\phi(G^*) \cap \mathcal{A}^{\widetilde{\Phi}_2}_{\mathrm{cusp}}(G^*). \end{array} \tag{5.1}$$

Here is a basic problem.

Problem 5.1. *Determine when the generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G^*)$ is G-relevant.*

It is clear that if $\phi \in \widetilde{\Phi}_2(G^*)$ is G-relevant, then both global Arthur packets $\widetilde{\Pi}_\phi(G)$ and $\widetilde{\Pi}_\phi(G^*)$ are non-empty and the transfer from $\widetilde{\Pi}_\phi(G)$ to $\widetilde{\Pi}_\phi(G^*)$ is the generalized Jacquet-Langlands correspondence between G^* and its pure inner form G. The theory of the *twisted automorphic descents* as developed in [38] and [40], and also [31], is to give an approach to resolve Problem 5.1. The idea is to bring the idea of the global Gan-Gross-Prasad conjecture [14] into the game of the explicit construction of the endoscopic transfers and the generalized Jacquet-Langlands correspondence as illustrated in diagram (5.1).

In order to explain the idea, we take G_n^* to be the F-split SO_{2n+1} and G_n be a pure inner form of G_n^* over F. Take H_m^* to be an F-quasisplit SO_{2m} and H_m be a pure inner form of H_m^* over F. We refer to [14,41,62] [2, Chapter 9] and [42] for further discussion of the inner forms in general. Assume that $G_n^* \times H_m^*$ is a relevant pair in the framework of the global Gan-Gross-Prasad conjecture [14, Section 2.1], and $G_n \times H_m$ is a *relevant pure inner form* of $G_n^* \times H_m^*$ over F. For any $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$, the non-vanishing of the central value $L(\frac{1}{2}, \tau \times \sigma)$ should detect whether the representation τ or the generic global Arthur parameter ϕ determined by τ parameterizes cuspidal automorphic representations of $G_n(\mathbb{A})$. We remark that here the L-function $L(s, \tau \times \sigma)$ is viewed as an L-function of τ twisted by σ. In other words, τ is given and σ and the group H_m are changing. This is the reason that in [31, 38] and [40], we call the theory of automorphic descents based on the non-vanishing condition:

$$L(\frac{1}{2}, \tau \times \sigma) \neq 0$$

the *theory of twisted automorphic descents*, which extends the theory of the automorphic descents of Ginzburg, Rallis and Soudry [23] to the great generality.

We are going to use the following two conditions to build a source representation for the twisted automorphic descent:

(1) $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ is an irreducible generic isobaric automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, which determines a generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n^*)$ for the F-split G_n^* of type SO_{2n+1}.

(2) the central value $L(\frac{1}{2}, \tau \times \sigma_0) \neq 0$ for some $\sigma_0 \in \mathcal{A}_{\mathrm{cusp}}^{\widetilde{\Phi}_2}(H_m)$ for some pure inner form H_m of an F-quasisplit H_m^* of type SO_{2m}.

Let ϕ_0 be the generic global Arthur parameter of σ_0 and define

$$\phi^{(2)} \boxplus \phi_0 := (\tau_1, 2) \boxplus \cdots \boxplus (\tau_r, 2) \boxplus \phi_0. \tag{5.2}$$

It is clear that $\phi^{(2)} \boxplus \phi_0$ is a global Arthur parameter for H_{2n+m}, which is a pure inner F-form of an F-quasisplit SO_{4n+2m}. The source representations for the twisted automorphic descent are the automoprhic members in the global Arthur packet $\widetilde{\Pi}_{\phi^{(2)} \boxplus \phi_0}(H_{2n+m})$.

We first want to construct some automorphic members in the global Arthur packet $\widetilde{\Pi}_{\phi^{(2)} \boxplus \phi_0}(H_{2n+m})$. For the given generic global Arthur parameter ϕ_0 of H_m^*, if ϕ_0 is H_m-relevant for a pure inner F-form H_m of H_m^*, we define as in [2, Chapter 9] the global Arthur packet $\widetilde{\Pi}_{\phi_0}(H_m)$, and if ϕ_0 is not H_m-relevant, we define the global Arthur packet $\widetilde{\Pi}_{\phi_0}(H_m)$ to be empty. The global Vogan packet attached to the global Arthur parameter ϕ_0 is defined by

$$\widetilde{\Pi}_{\phi_0}[H_m^*] := \cup_{H_m} \widetilde{\Pi}_{\phi_0}(H_m), \tag{5.3}$$

where the union is taken over all the pure inner F-forms of the given F-quasisplit H_m^*. For any $\sigma \in \widetilde{\Pi}_{\phi_0}[H_m^*]$, take $\tau \otimes \sigma$ to be an automorphic representation of $\mathrm{GL}_{2n} \times H_m$, which is the Levi subgroup of a standard parabolic subgroup of H_{2n+m}. Hence we may form an Eisenstein series $E(\cdot, \Phi_{\tau\otimes\sigma}, s)$ following [48] and [53], and also [32]. By [38, Proposition 5.3], the Eisenstein series $E(\cdot, \Phi_{\tau\otimes\sigma}, s)$ has a pole at $s = \frac{1}{2}$ of order r if and only if the central value $L(\frac{1}{2}, \tau \times \sigma)$ is nonzero. In this case, following [32], the iterated residue of $E(\cdot, \Phi_{\tau\otimes\sigma}, s)$ at $s = \frac{1}{2}$, which is denoted by $\mathcal{E}_{\tau\otimes\sigma}$, is square-integrable and has the global Arthur parameter $\phi^{(2)} \boxplus \phi_0$. Following [62] and [14], we define the global Vogan packet by

$$\widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}[H^*_{2n+m}] := \cup_{H_{2n+m}} \widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}(H_{2n+m})$$

where H_{2n+m} runs over all pure inner F-forms of the F-quasisplit H^*_{2n+m} determined by the F-quasisplit H^*_m. Again if the global Arthur parameter $\phi^{(2)} \boxplus \phi_0$ is not H_{2n+m}-relevant, then we define the global Arthur packet $\widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}(H_{2n+m})$ to be empty. Hence when σ runs in the global Vogan packet $\widetilde{\Pi}_{\phi_0}[H^*_m]$ with the property that $L(\frac{1}{2}, \tau \times \sigma)$ is nonzero, then we construct nonzero automorphic members $\mathcal{E}_{\tau\otimes\sigma}$ in the global Vogan packet $\widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}[H^*_{2n+m}]$. According to the construction given in [30], in addition to the residual representations as constructed here, there should be cuspidal automorphic members in the global Vogan packet $\widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}[H^*_{2n+m}]$.

As explained before, the theory of twisted automorphic descents is to take τ that determines the global Arthur parameter $\phi \in \widetilde{\Phi}_2(G^*_n)$ as the main object for study, and to use ϕ_0, H_m and σ as tools to proceed the investigation or the construction. Hence we may let m run in $\{0, 1, 2, \cdots, n\}$, let H^*_m run over all F-quasisplit SO_{2m}, and let ϕ_0 run in $\widetilde{\Phi}_2(H^*_m)$. Therefore, the *source representations* for the construction of the twisted automorphic descents from global Arthur parameter $\phi \in \widetilde{\Phi}_2(G^*_n)$ to the automorphic members in the global Vogan packet $\widetilde{\Pi}_\phi[G^*_n]$ may be taken from the following set

$$\mathfrak{S}(\phi) := \cup_{m=0}^{n} \cup_{H^*_m} \cup_{\phi_0 \in \widetilde{\Phi}_2(H_m{}^*)} \widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}[H^*_{2n+m}]. \tag{5.4}$$

We give below the outline of the construction of the twisted automorphic descents.

For the given irreducible generic isobaric automorphic representation $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ of $\mathrm{GL}_{2n}(\mathbb{A})$, which determines a generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G^*_n)$, where G^*_n is the F-split SO_{2n+1}, we take an irreducible automorphic member $\Sigma \in \mathfrak{S}(\phi)$. We may assume that $\Sigma \in \widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}[H^*_{2n+m}]$ for a choice of data. Hence we must have that

$$\Sigma \in \mathcal{A}_2(H_{2n+m}) \cap \widetilde{\Pi}_{\phi^{(2)}\boxplus\phi_0}(H_{2n+m})$$

for some integer m, some generic global Arthur parameter ϕ_0, and some pure inner F-form of the F-quasisplit H^*_{2n+m}. The twisted automorphic descent of τ is defined to be the Fourier coefficient of Σ attached to the partition

$$\underline{p}_d := [(2n + 2m - 1)1^{2n+1}].$$

From the choice of the Σ, the partition $\underline{p}_d$ of $4n+2m$ is H_{2n+m}-relevant. Hence the partition $\underline{p}_d$ corresponds to the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_d}$ in the Lie algebra of H_{2n+m}. For each F-rational orbit $\mathcal{O}_d$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_d}(F)$, the Fourier coefficient attached to the $\mathcal{O}_d$ of the residual representation Σ is denoted by $\mathcal{D}^{\psi_{\mathcal{O}_d}}(\Sigma)$. We refer to [31] or [38] for the detailed account of the explicit construction of such Fourier coefficients. As proved in [31, Theorem 1.2], [38, Proposition 2.2], and [38, Theorem 6.4], we have the following theorem.

Theorem 5.2. *The twisted automorphic descent $\mathcal{D}^{\psi_{\mathcal{O}_d}}(\Sigma)$ as defined above is a cuspidal automorphic representation of $G_n(\mathbb{A})$ for some pure inner F-form G_n of the F-split G_n^*. If it is nonzero, it may be written as*

$$\mathcal{D}^{\psi_{\mathcal{O}_d}}(\Sigma) = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k \oplus \cdots$$

with π_k's are all irreducible cuspidal automorphic representations of $G_n(\mathbb{A})$. In this case, the decomposition is of multiplicity free and all π_k's belong to the global Arthur packet $\widetilde{\Pi}_\phi(G_n)$.

As proved in [38], by the local Gan-Gross-Prasad conjecture at all local places, the twisted automorphic descent $\mathcal{D}^{\psi_{\mathcal{O}_d}}(\Sigma)$ is irreducible or zero if $\Sigma = \mathcal{E}_{\tau\otimes\sigma}$ for some cuspidal $\sigma \in \widetilde{\Pi}_{\phi_0}(H_m)$ with H_m determining H_{2n+m}. Hence we have:

Theorem 5.3. *For $\sigma \in \widetilde{\Pi}_{\phi_0}(H_m)$ with the property that $L(\frac{1}{2}, \tau\times\sigma)$ is nonzero, the twisted automorphic descent $\mathcal{D}^{\psi_{\mathcal{O}_d}}(\mathcal{E}_{\tau\otimes\sigma})$ is irreducible or zero.*

It follows that the twisted automorphic descent may construct many cuspidal members in the global Vogan packet $\widetilde{\Pi}_\phi[G_n^*]$. Here is the main conjecture in the theory of twisted automorphic descents.

Conjecture 5.4. *Let G_n^* be the F-split SO_{2n+1}. For an irreducible generic isobaric automorphic representation*

$$\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$$

of $\mathrm{GL}_{2n}(\mathbb{A})$, which determines a generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n^*)$, any cuspidal automorphic member π in the global Vogan packet

$$\widetilde{\Pi}_\phi[G_n^*] := \cup_{G_n}\widetilde{\Pi}_\phi(G_n),$$

where G_n runs over all pure inner F-forms of G_n^*, can be constructed as a twisted automorphic descent.

It is clear that there are two technical issues in the proof of Conjecture 5.4. One is the (global) non-vanishing of the twisted automorphic descent $\mathcal{D}^{\psi_{\mathcal{O}_d}}(\Sigma)$ and the other is the completeness of the construction of the twisted automorphic descents. These two issues can be resolved by extending the Arthur-Burger-Sarnak principle ([8, 9], and [10]) to certain cases of Fourier coefficients of automorphic forms associated to partitions or nilpotent orbits. This has been done in [38] for the Bessel-Foruier case and in [40] for the Fourier-Jacobi case. One finds relevant explanation in [39]. It is clear that the above theory can be formulated for all classical groups. In particular, in the spirit of the global Gan-Gross-Prasad conjecture [14], the approach given in [38] and [40], and also in [39] uses the property that the non-vanishing of the central value $L(\frac{1}{2}, \tau \times \sigma)$ is nonzero. Hence the central value of the L-function of τ twisted by σ has deep impact to the explicit construction of certain types of the Langlands functorial transfers.

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(Dihua Jiang)
E-mail address: dhj55119@yahoo.com