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On the analytic properties of intertwining operators I: global normalizing factors

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ON THE ANALYTIC PROPERTIES OF INTERTWINING OPERATORS I: GLOBAL NORMALIZING FACTORS

T. FINIS AND E. LAPID*

To Freydoon Shahidi, for his 70th birthday

ABSTRACT. We provide a uniform estimate for the L^1-norm (over any interval of bounded length) of the logarithmic derivatives of global normalizing factors associated to intertwining operators for the following reductive groups over number fields: inner forms of $\mathrm{GL}(n)$; quasi-split classical groups and their similitude groups; the exceptional group G_2. This estimate is a key ingredient in the analysis of the spectral side of Arthur's trace formula. In particular, it is applicable to the limit multiplicity problem studied by the authors in earlier papers.
Keywords: Intertwining operators, L-functions, automorphic forms.
MSC(2010): Primary: 11F70; Secondary: 11F66.

1. Introduction

In this paper we study the analytic properties of the global intertwining operators associated to parabolic subgroups of reductive groups G over number fields F. In the previous papers [35] (joint with Werner Müller) and [34], we defined certain properties (TWN) and (BD) pertaining to these intertwining operators, and showed that these two properties together imply the solution of the limit multiplicity problem for congruence subgroups of lattices contained in $G(F)$. Property (TWN) is a global property concerning the scalar-valued normalizing factors, while (BD) is essentially a local property. In [35], these properties were verified for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$.

This paper is devoted to establishing property (TWN) in a number of other cases, namely for inner forms of the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, for quasi-split classical groups and their similitude groups, and for the exceptional group G_2. In fact, we prove without any more effort a finer estimate (already mentioned in [35]), which we call property (TWN+). In addition to the application to the limit multiplicity problem, this property may be useful in the study of other

asymptotic questions, where it is important to control the dependence on the archimedean parameters. The main examples of such problems are Weyl's law with remainder term and the asymptotics of Hecke operators in general families [84, 86, 87]. In any case, the results of this paper are sufficient to extend the limit multiplicity results of [35, 34] to inner forms of $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. This will be discussed in a future paper, where we will also study property (BD) for general groups G, and prove a weaker variant. This, together with the results of the current paper, will allow for an extension of the limit multiplicity results to quasi-split classical groups and their similitude groups and G_2 (at least under a technical restriction on the congruence subgroups in question).

We sketch the definition of property (TWN+), which is explained in more detail in Section 3 below. As usual, fix a minimal Levi subgroup M_0 of G defined over F, and consider a proper Levi subgroup M of G containing M_0 and a root $\alpha \in \Sigma_M$. Let U_α be the unipotent subgroup of G corresponding to α, M_α the group generated by M and $U_{\pm\alpha}$, and $\hat{M}_\alpha$ its F-simple normal subgroup generated by $U_{\pm\alpha}$. The group $\tilde{M}_\alpha = \hat{M}_\alpha \cap M$ is a maximal Levi subgroup of $\hat{M}_\alpha$. For $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ let $n_\alpha(\pi, s)$ be the normalizing factor for the global intertwining operators associated to pairs of parabolic subgroups of G adjacent along α. These factors are meromorphic functions of finite order of the complex variable s and satisfy the functional equation $|n_\alpha(\pi, it)| = 1$ for all $t \in \mathbb{R}$.

We say that G satisfies property (TWN+) (tempered winding numbers, strong version) if we have the estimate

$$\int_T^{T+1} |n'_\alpha(\pi, it)| \ dt \ll \log(|T| + \Lambda(\pi_\infty; p^{\mathrm{sc}}) + \mathrm{level}(\pi; p^{\mathrm{sc}}))$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and all real numbers T where the implied constant depends only on G. Here, $\mathrm{level}(\pi; p^{\mathrm{sc}})$ is essentially the level of π restricted to $\tilde{M}_\alpha$, and $\Lambda(\pi_\infty; p^{\mathrm{sc}})$ measures the size of the infinitesimal character of the restriction of π_∞ to $\tilde{M}_\alpha$. (See Section 2.3 and Section 3 for the precise definitions.) We note that property (TWN+) implies property (TWN) introduced in [35, Definition 5.2], and that property (TWN+) has been shown (as a consequence of the theory of Rankin-Selberg L-functions) for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ in [35, Proposition 5.5].

Our method of proving (TWN+) for the groups listed above, is to use functoriality to transfer the problem to a well-understood problem for $\mathrm{GL}(n)$. We start by an axiomatic treatment of automorphic L-functions in Section 2, which we then apply to the global normalizing factors in Section 3. In Section 4 we will show (TWN+) for inner forms of $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ using the Jacquet-Langlands correspondence (obtained in general in [7, 8]), which allows us to reduce the problem to known properties of Rankin-Selberg L-functions.

In Section [5], we will consider quasi-split classical groups. We first consider the twisted exterior and symmetric square L-functions for $\mathrm{GL}(n)$, as well as the Asai L-function for $\mathrm{Res}_{E/F}\mathrm{GL}(n)$, using known results obtained by the Langlands-Shahidi method and by the study of integral representations. Using Arthur's work on functoriality from the classical groups to $\mathrm{GL}(n)$ [3], extended by Mok to unitary groups [91], we will be able once again to reduce the remaining L-functions to Rankin-Selberg L-functions for $\mathrm{GL}(n)$.

Our approach is based on Arthur's work, which requires the full force of the stable twisted trace formula. Among the prerequisites of Arthur's results are [77, 105, 99, 33, 79, 111, 110, 90], to mention a few. One may contemplate whether there is a different approach to the problem which avoids functoriality (and is perhaps applicable to other groups). Unfortunately, at the moment we cannot say anything in this direction.

For the exceptional group G_2, which we treat in Section [6], we need to consider the (twisted) symmetric cube L-function for $\mathrm{GL}(2)$ which was studied by Kim–Shahidi.

1.1. It is a pleasure to dedicate this paper to Freydoon Shahidi for his upcoming 70th birthday. Shahidi's influence on the field of automorphic forms cannot be overestimated. Needless to say, the current paper also owes a lot to his work. On a personal level it has always been a pleasure to interact with Freydoon and we wish him the very best.

2. **Estimates for logarithmic derivatives of L-functions**

We begin with an axiomatic treatment of automorphic L-functions which isolates the precise properties needed for the main estimate (see Proposition [2.6] below).

Let us first recall some generalities about L-functions. Let G be a reductive group over a number field F and let $\mathbb{A}$ be the ring of adeles of F. Let $\mathbb{A}_{\mathrm{fin}}$ be the ring of finite adeles of F and $F_\infty = F \otimes \mathbb{R}$. Let $|\cdot|_{\mathbb{A}^\times}$ be the idele norm on $\mathbb{A}^\times$. Let T_G be the $\mathbb{Q}$-split part of the (Zariski) connected component of the center of $\mathrm{Res}_{F/\mathbb{Q}}G$ (restriction of scalars) and let $A_G = T_G(\mathbb{R})^\circ$ (topological connected component), viewed as a subgroup of $T_G(\mathbb{A}_\mathbb{Q})$ (and hence of $G(\mathbb{A})$). Let $G(\mathbb{A})^1 \subset G(\mathbb{A})$ be the intersection of the kernels $\ker |\chi|_{\mathbb{A}^\times}$ as χ ranges over the F-rational characters of G. Then $G(\mathbb{A})$ is the direct product of $G(\mathbb{A})^1$ and A_G. Let $\mathfrak{a}_G$ be the Lie algebra of A_G, a real vector space, and $\mathfrak{a}_G^*$ its dual space. We set $\mathfrak{a}_{G,\mathbb{CC}}^* = \mathfrak{a}_G^* \otimes \mathbb{CC}$.

We write $\Pi_{\mathrm{disc}}(G(\mathbb{A}))$ for the set of equivalence classes of automorphic representations of $G(\mathbb{A})$ which occur in the discrete spectrum of $L^2(A_G G(F)\backslash G(\mathbb{A}))$. We will also write $\Pi_{\mathrm{cusp}}(G(\mathbb{A}))$ for the subset of cuspidal representations.

For any $\pi = \otimes_v \pi_v \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ let $S(\pi)$ be the finite set of places of F such that at least one of the following conditions holds:

(1) v is archimedean.

(2) $F/\mathbb{Q}$ is ramified at v.

(3) G is ramified at v, i.e., G is not quasi-split over F_v or G does not split over an unramified extension of F_v.

(4) For every hyperspecial maximal compact subgroup K_v of $G(F_v)$, π_v does not have a nonzero vector invariant under K_v.

(The exclusion of the finite places which are ramified for F is inessential and is only made for convenience.) We write $S(\pi) = S_\infty \cup S_f(\pi)$, where S_∞ denotes the set of archimedean places of F and $S_f(\pi)$ denotes the non-archimedean places in $S(\pi)$. Let $S_{\mathbb{Q},f}(\pi)$ (or simply $S_{\mathbb{Q},f}$, if π is clear from the context) be the set of rational primes which lie below the primes in $S_f(\pi)$. Also set $S_\mathbb{Q}(\pi) = S_{\mathbb{Q},f}(\pi) \cup \{\infty\}$.

Let W_F be the Weil group of F and let $^L G$ be the L-group of G (cf. [11]). Let $r : {}^L G \to \mathrm{GL}(N, \mathbb{C})$ be a continuous and W_F-semisimple N-dimensional representation of $^L G$. For any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ and any place v of F outside of $S(\pi)$ we have the Hecke-Frobenius parameter $t_{\pi_v} \in {}^L G$. To each such v we associate the polynomial $P_v(X) = \det(1 - Xr(t_{\pi_v}))$ of degree N and the local L-factor $L_v(s, \pi, r) = P_v(q_v^{-s})^{-1}$ where q_v is the cardinality of the residue field of F_v. Since π is unitary, the absolute values of the eigenvalues of $r(t_{\pi_v})$ are bounded by q_v^α, where α depends only on G and r ([11, 80]). Therefore, for $S \supset S(\pi)$ the partial L-function

$$L^S(s, \pi, r) = \prod_{v \notin S} L_v(s, \pi, r)$$

is well-defined as an absolutely convergent product and holomorphic for $\mathrm{Re}\, s > \alpha + 1$. Because of the unitarity of π, we also have

$$L^S(s, \pi, r^\vee) = \overline{L^S(\bar{s}, \pi, r)},$$

where $r^\vee$ is the contragredient of r. A cornerstone of the Langlands program is the (largely conjectural) meromorphic continuation of these L-functions with finitely many poles in $\mathbb{C}$.

As usual, we write $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\mathbf{\Gamma}(s/2)$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\mathbf{\Gamma}(s) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$, where $\mathbf{\Gamma}(s)$ is the standard Gamma function.

Definition 2.1. We say that a pair (G, r) has property (FE), if for any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ there exist rational functions R_p, $p \in S_{\mathbb{Q},f}(\pi)$, an integer $m \geq 1$, a constant $C_\infty \in \mathbb{C}^*$ and parameters $\alpha_1, \ldots, \alpha_m, \alpha_1^\vee, \ldots, \alpha_m^\vee \in \mathbb{C}$, such that the partial L-function $L^S(s, \pi, r)$ admits a meromorphic continuation to $\mathbb{C}$ with a functional equation of the form

$$(2.1) \qquad L^{S(\pi)}(s, \pi, r) = \Big(\prod_{p \in S_\mathbb{Q}(\pi)} \gamma_p(s, \pi, r) \Big) L^{S(\pi)}(1 - s, \pi, r^\vee),$$

or equivalently,

$$L^{S(\pi)}(s, \pi, r) = \big(\prod_{p \in S_{\mathbb{Q}}(\pi)} \gamma_p(s, \pi, r) \big) \overline{L^{S(\pi)}(1 - \bar{s}, \pi, r)},$$

where for each $p \in S_{\mathbb{Q},f}(\pi)$, $\gamma_p(s, \pi, r) = R_p(p^{-s})$, and

$$(2.2) \qquad \gamma_\infty(s, \pi, r) = C_\infty \prod_{i=1}^{m} \frac{\Gamma_{\mathbb{R}}(1 - s + \alpha_i^\vee)}{\Gamma_{\mathbb{R}}(s + \alpha_i)}.$$

In general, this property is wide open except for a number of important special cases, some of which will be considered below.

2.1.   We first address the uniqueness of the terms in (2.1). This is of course standard. For completeness we give the details.

Lemma 2.2. *Assume that* (2.1) *is satisfied for some fixed* r *and* π. *Then*

(1) *The factors* $\gamma_p(s, \pi, r)$, $p \in S_{\mathbb{Q}}(\pi)$ *(and hence the rational functions* R_p, $p \in S_{\mathbb{Q},f}(\pi)$), *are uniquely determined up to non-zero constant multiples.*
(2) *The integer* m *is uniquely determined. (We call it the degree of* γ_∞.*)*
(3) *We have*

$$(2.3) \qquad \gamma_p(s, \pi, r) \overline{\gamma_p(1 - \bar{s}, \pi, r)} = const.$$

for all $p \in S_{\mathbb{Q}}(\pi)$.

(4) *The parameters* $\alpha_1^\vee, \ldots, \alpha_m^\vee$ *are determined by* $\alpha_1, \ldots, \alpha_m$ *(up to permutation).*
(5) *The expression*

$$\big(\prod_{i=1}^{m} (1 + |\alpha_i - 1/2|)(1 + |\alpha_i^\vee - 1/2|) \big)^{\frac{1}{2}}$$

is uniquely determined. It is called the archimedean conductor and denoted by $\mathfrak{c}_\infty(\pi, r)$ (*cf.* [56]).

(6) *We say that* $\alpha_1, \ldots, \alpha_m$ *are* reduced, *if* $\alpha_i + \alpha_j^\vee$ *is not a negative odd integer for any* $1 \le i, j \le m$. *We may choose the parameters* $\alpha_1, \ldots, \alpha_m$ *to be reduced. Moreover, in this case*

(a) *The zeros (resp., poles) of* $\gamma_\infty(s, \pi, r)$, *taken with multiplicities, coincide with those of* $\prod_{i=1}^{m} \Gamma_{\mathbb{R}}(s + \alpha_i)^{-1}$ (*resp.,* $\prod_{i=1}^{m} \Gamma_{\mathbb{R}}(1 - s + \alpha_i^\vee)$).
(b) *The multisets* $\{\alpha_1, \ldots, \alpha_m\}$ *and* $\{\alpha_1^\vee, \ldots, \alpha_m^\vee\}$ *are uniquely determined.*
(c) *As a multiset we have*

$$(2.4) \qquad \{\alpha_1^\vee, \ldots, \alpha_m^\vee\} = \{\overline{\alpha_1}, \ldots, \overline{\alpha_m}\}.$$

Proof. Suppose that we are given a function of the form

$$\phi(s) = CA^s \prod_{i=1}^{m} \frac{\Gamma_{\mathbb{R}}(1 - s + \alpha_i^\vee)}{\Gamma_{\mathbb{R}}(s + \alpha_i)} \prod_{p \in S_{\mathbb{Q},f}(\pi)} \tilde{R}_p(p^{-s}),$$

where C is a non-zero constant, A is a positive real number and for all $p \in S_{\mathbb{Q},f}(\pi)$, $\tilde{R}_p$ is a rational function with $\tilde{R}_p(0) = 1$. Up to a finite multiset, the zeros of the first product are given (with multiplicities) by $-\alpha_i - 2\mathbb{Z}^{\geq 0}$, $i = 1, \ldots, m$, while those of $\tilde{R}_p(p^{-s})$ are given by $s = (-\log x_{p,i} + 2\pi \mathrm{i}\mathbb{Z})/\log p$, where $x_{p,1}, \ldots, x_{p,d_p}$ are the zeros of $\tilde{R}_p$. Considering only the zeros with $\operatorname{Im} s$ sufficiently large, we see that the zeros of each rational function $\tilde{R}_p$ are determined by ϕ. Arguing similarly with the poles, we conclude that each $\tilde{R}_p$ is determined by ϕ.

Furthermore, if we have an equality

$$(2.5) \qquad \prod_{i=1}^{m} \frac{\Gamma_{\mathbb{R}}(1 - s + \alpha_i^{\vee})}{\Gamma_{\mathbb{R}}(s + \alpha_i)} = CA^s \prod_{i=1}^{m'} \frac{\Gamma_{\mathbb{R}}(1 - s + \beta_i^{\vee})}{\Gamma_{\mathbb{R}}(s + \beta_i)}$$

for some constants $A > 0$ and C, then an examination of the zeros and poles shows that $m' = m$, and that after possibly reindexing $\alpha_1, \ldots, \alpha_m$ and $\alpha_1^{\vee}, \ldots, \alpha_m^{\vee}$ we may assume that $\alpha_i - \beta_i, \alpha_i^{\vee} - \beta_i^{\vee} \in 2\mathbb{Z}$ for all i, in which case $\Gamma_{\mathbb{R}}(s + \alpha_i)/\Gamma_{\mathbb{R}}(s + \beta_i)$ and $\Gamma_{\mathbb{R}}(1 - s + \alpha_i^{\vee})/\Gamma_{\mathbb{R}}(1 - s + \beta_i^{\vee})$ are rational functions in s. Thus $A = 1$.

The first two parts follow. The third and fourth parts immediately follow from the first part.

Moreover, by an easy argument (e.g., [101]) the equality (2.5) holds if and only if $A = 1$, $m' = m$ and after reindexing $\alpha_1, \ldots, \alpha_m$ and $\alpha_1^{\vee}, \ldots, \alpha_m^{\vee}$ if necessary, there exists $0 \leq k \leq m$ such that

(1) $\alpha_i + \beta_i^{\vee} = \alpha_i^{\vee} + \beta_i = 1$ and $\alpha_i - \beta_i \in 2\mathbb{Z}$ for all $i = 1, \ldots, k$,
(2) $\alpha_i = \beta_i$ and $\alpha_i^{\vee} = \beta_i^{\vee}$ for $i > k$, and,
(3) $C = (-1)^{(\alpha_1 - \beta_1 + \cdots + \alpha_k - \beta_k)/2}$.

Part 5 follows.

Suppose that $\alpha_1, \ldots, \alpha_m$ are not reduced, so that there exist indices i and j such that $\alpha_i + \alpha_j^{\vee} = 1 - 2k$ for some positive integer k. We may then replace α_i and $\alpha_j^{\vee}$ by $1 - \alpha_j^{\vee}$ and $1 - \alpha_i$, respectively, and multiply C_{∞} by $(-1)^k$. We may repeat this process until $\alpha_1, \ldots, \alpha_m$ become reduced. The process must terminate after finitely many steps since $\sum_i (\alpha_i + \alpha_i^{\vee})$ increases by $4k$ in each step.

Once $\alpha_1, \ldots, \alpha_m$ are reduced, the poles of $\prod_{i=1}^{m} \Gamma_{\mathbb{R}}(1 - s + \alpha_i^{\vee})$ are disjoint from those of $\prod_{i=1}^{m} \Gamma_{\mathbb{R}}(s + \alpha_i)$. Therefore, the zeros of $\prod_{i=1}^{m} \Gamma_{\mathbb{R}}(s + \alpha_i)^{-1}$ are precisely those of $\gamma_{\infty}(s, \pi, r)$ (including multiplicities). In particular, $\{\alpha_1, \ldots, \alpha_m\}$ is determined as a multiset, and hence (2.4) follows from (2.3). $\qquad \square$

Assume that (2.1) is satisfied for some fixed r and π and take $\alpha_1, \ldots, \alpha_m$ to be reduced (hence uniquely determined). We set

$$L_{\infty}^{\mathrm{red}}(s, \pi, r) = \prod_{i=1}^{m} \Gamma_{\mathbb{R}}(s + \alpha_i),$$

so that

$$\gamma_\infty(s, \pi, r) = c_\infty \frac{\overline{L_\infty(1 - \bar{s}, \pi, r)}}{L_\infty(s, \pi, r)}$$

with $c_\infty = \pm C_\infty$. In this case the archimedean conductor simplifies to

$$(2.6) \qquad\qquad \mathfrak{c}_\infty(\pi, r) = \prod_{i=1}^m (1 + |\alpha_i - 1/2|).$$

For $p \in S_{\mathbb{Q}, f}(\pi)$ it follows from (2.3) that we can write in a unique fashion

$$(2.7) \qquad\qquad \gamma_p(s, \pi, r) = c_p p^{(\frac{1}{2} - s)\mathfrak{e}_p(\pi, r)} P_p(p^{-s}) / \bar{P}_p(p^{s-1}),$$

where $c_p \in \mathbb{CC}^*$, $\mathfrak{e}_p(\pi, r) \in \mathbb{Z}$, and P_p is a polynomial with $P_p(0) = 1$ such that no zeros α and β of P_p satisfy $\alpha\bar{\beta} = p^{-1}$. Here $\bar{P}$ is the polynomial obtained from P by taking complex conjugates of the coefficients. The degree of P_p is the number of zeros (or, equivalently, poles) of R_p in $\mathbb{CC}^\times$ (counted with multiplicities), and the integer $\mathfrak{e}_p(\pi, r)$ is the difference between the order of R_p at $X = 0$ and the degree of P_p. We have seen that $\mathfrak{e}_p(\pi, r)$ and P_p are uniquely determined by π and r. Analogously to the case $p = \infty$, the zeros of $P_p(p^{-s})$ are precisely the zeros of $\gamma_p(s, \pi, r)$. Although we expect that $\mathfrak{e}_p(\pi, r) \geq 0$, we do not impose this condition at the outset. We set

$$L_p^{\mathrm{red}}(s, \pi, r) = P_p(p^{-s})^{-1}, \quad p \in S_{\mathbb{Q}, f}(\pi),$$

and define the *reduced completed L-function*

$$L^{\mathrm{red}}(s, \pi, r) = \Big(\prod_{p \in S_{\mathbb{Q}}(\pi)} L_p^{\mathrm{red}}(s, \pi, r) \Big) L^{S(\pi)}(s, \pi, r),$$

and the reduced epsilon factor

$$\epsilon^{\mathrm{red}}(s, \pi, r) = c_\infty \prod_{p \in S_{\mathbb{Q}, f}(\pi)} c_p p^{(\frac{1}{2} - s)\mathfrak{e}_p(\pi, r)} = \mathfrak{n}(\pi, r)^{\frac{1}{2} - s} \prod_{p \in S_{\mathbb{Q}}(\pi)} c_p,$$

where

$$(2.8) \qquad\qquad \mathfrak{n}(\pi, r) = \prod_{p \in S_{\mathbb{Q}, f}(\pi)} p^{\mathfrak{e}_p(\pi, r)} \in \mathbb{Q}_{>0}$$

is the arithmetic, or finite, conductor. Thus, we can rewrite (2.1) as

$$L^{\mathrm{red}}(s, \pi, r) = \epsilon^{\mathrm{red}}(s, \pi, r) L^{\mathrm{red}}(1 - s, \pi, r^\vee) = \epsilon^{\mathrm{red}}(s, \pi, r) \overline{L^{\mathrm{red}}(1 - \bar{s}, \pi, r)}.$$

We denote by

$$L^{\mathrm{red}, f}(s, \pi, r) = \Big(\prod_{p \in S_{\mathbb{Q}, f}(\pi)} L_p^{\mathrm{red}}(s, \pi, r) \Big) L^{S(\pi)}(s, \pi, r)$$

the "finite" part of L^{red}.

Remark 2.3. In many cases there is an alternative procedure to define a completed L-function

$$(2.9) \qquad L(s) = \big(\prod_{p \in S_{\mathbb{Q}}(\pi)} \tilde{L}_p(s) \big) L^{S(\pi)}(s, \pi, r),$$

which may differ from $L^{\mathrm{red}}(s)$. Here $\tilde{L}_\infty(s) = \prod_{i=1}^{\tilde{m}} \Gamma_{\mathbb{R}}(s + \tilde{\alpha}_i)$ with some complex numbers $\tilde{\alpha}_i$, and $\tilde{L}_p(s) = \tilde{P}_p(p^{-s})^{-1}$ for $p \in S_{\mathbb{Q},f}(\pi)$, where $\tilde{P}_p$ are some polynomials satisfying $\tilde{P}_p(0) = 1$. (For brevity, we say that factors $\tilde{L}_p(s)$, $p \in S_{\mathbb{Q}}(\pi)$, of this shape are Euler factors.) The advantage in working with $L^{\mathrm{red}}(s)$ is that it is uniquely determined by the partial L-function $L^{S(\pi)}$ (and hence by π and r). Of course, it is only defined if property (FE) is known a priori.

In any case, as a consequence of the minimality of the local factors $L_p^{\mathrm{red}}(s)$, $p \in S_{\mathbb{Q}}(\pi)$, the reduced L-function $L^{\mathrm{red}}(s)$ satisfies the following minimality property. Suppose that a function $L(s)$ as in (2.9) satisfies a functional equation of the form

$$L(s) = cR^{\frac{1}{2}-s} L^\vee(1-s)$$

for some $c \in \mathbb{CC}^*$ and $R > 0$, where

$$L^\vee(s) = \big(\prod_{p \in S_{\mathbb{Q}}(\pi)} \tilde{L}_p^\vee(s) \big) L^{S(\pi)}(s, \pi, r^\vee),$$

for some Euler factors $\tilde{L}_p^\vee(s)$, $p \in S_{\mathbb{Q}}(\pi)$. Then for each $p \in S_{\mathbb{Q},f}(\pi)$, the polynomial $\tilde{P}_p$ is divisible by P_p and $\tilde{L}_\infty(s)/L_\infty^{\mathrm{red}}(s, \pi, r)$ is a polynomial. In particular, the quotient $L^{\mathrm{red}}(s)/L(s)$ is an entire function. Moreover,

$$\frac{\mathfrak{n}(\pi, r)}{R} = \prod_{p \in S_{\mathbb{Q},f}(\pi)} p^{\deg \tilde{P}_p - \deg P_p}$$

is a positive integer.

Consider for example the case of $G = \mathrm{GL}(n)$ with the standard representation $r = \mathrm{St}_n$. If $\pi \in \Pi_{\mathrm{cusp}}(G(\mathbb{A}))$ then the completed L-function $L^{\mathrm{GJ}}(s, \pi, \mathrm{St}_n)$ was defined and studied by Godement–Jacquet [49]. The Jacquet-Shalika bounds on the local parameters of generic representations imply that $L^{\mathrm{GJ}}(s, \pi, \mathrm{St}_n)$ is reduced. On the other hand, for any $m > 1$ the L-function

$$\prod_{j=1}^{m} L^{\mathrm{GJ}}(s + \frac{m+1}{2} - j, \pi, \mathrm{St}_n),$$

which is the completed L-function of a residual representation of $\mathrm{GL}(nm, \mathbb{A})$, is not reduced.

Our next task is to apply the standard argument of Riemann and von Mangoldt for estimating the number of zeros of an L-function (cf., e.g., [55, Section

5.3]). For our purposes we will need a weak form of this argument which works under the following supplementary conditions.

Definition 2.4. We say that (G, r) satisfies property (FE+), if it satisfies property (FE) and if in addition the following holds.

(1) There exists a polynomial $P(s)$, whose degree is bounded in terms of (G, r) only, such that $P(s)L^{S(\pi)}(s, \pi, r)$ is an entire function of finite order.

(2) The degrees of the polynomials P_p, $p \in S_{\mathbb{Q}, f}$, in (2.7), as well as the degree of the factor γ_∞ are bounded in terms of (G, r) only.

(3) There exists a real number β depending only on (G, r), such that the zeros of P_p, $p \in S_{\mathbb{Q}, f}$, have absolute value $\geq p^{-\beta}$ and such that $\gamma_\infty(s, \pi, r)$ has no zeros in the half-plane $\operatorname{Re} s > \beta$ (i.e., the reduced parameters α_i satisfy $\operatorname{Re} \alpha_i \geq -\beta$).

We say that (G, r) has virtually property (FE+), if there exist representations r_1 and r_2 for which (G, r_i) has property (FE+) and $r \oplus r_1 = r_2$.

In practice, we will have $\deg \gamma_\infty = N[F : \mathbb{Q}] \geq \deg P_p$ for all $p \in S_{\mathbb{Q}, f}(\pi)$. However, we do not demand it at the outset. The most elusive condition in property (FE+) is in fact the first one — the boundedness of the number of poles of $L^S(s, \pi, r)$ independently of π.

Remark 2.5. It is clear that (G, r) satisfies (FE+) if and only if the same is true for $(G, r^\vee)$. An analogous remark is applicable for all subsequent properties defined below.

We can now state the desired estimate. Recall the archimedean and arithmetic conductors defined in (2.6) and (2.8), respectively. Here and throughout we write

$$A \ll B$$

to mean that A is bounded by a constant multiple of B. The implied constant is allowed to depend only on the pair (G, r).

Proposition 2.6. *Suppose that (G, r) satisfies property* (FE+). *Then for any* $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ *the meromorphic function*

$$m(s, \pi, r) = \epsilon^{\mathrm{red}}(1, \pi, r)L^{\mathrm{red}}(s, \pi, r)/L^{\mathrm{red}}(s + 1, \pi, r)$$

satisfies $|m(\mathrm{it}, \pi, r)| = 1$ *for* $t \in \mathbb{R}$ *and*

$$\int_T^{T+1} |m'(\mathrm{it}, \pi, r)| \, dt \ll \log(|T| + 2) + \log \mathfrak{c}_\infty(\pi, r) + \log(\mathfrak{n}(\pi, r) \prod_{p \in S_{\mathbb{Q}, f}(\pi)} p)$$

for all $T \in \mathbb{R}$.

Proof. Although the argument is familiar, we will provide the details, since our assumptions are somewhat weaker than usual. Let

$$\Lambda(s,\pi,r) = \mathfrak{n}(\pi,r)^{s/2} L^{\mathrm{red}}(s,\pi,r).$$

Then

(2.10) $$\Lambda(s,\pi,r) = \epsilon^{\mathrm{red}}(\tfrac{1}{2},\pi,r)\overline{\Lambda(1-\overline{s},\pi,r)}.$$

Note that $m(s,\pi,r) = \Lambda(s,\pi,r)/\overline{\Lambda(-\overline{s},\pi,r)}$ and hence $m(s,\pi,r)$ is holomorphic on the imaginary axis and has absolute value one there. Moreover,

$$\frac{m'(\mathrm{i}t,\pi,r)}{m(\mathrm{i}t,\pi,r)} = 2\operatorname{Re}\frac{\Lambda'(\mathrm{i}t,\pi,r)}{\Lambda(\mathrm{i}t,\pi,r)}, \quad t\in\mathbb{R}.$$

The function $\Lambda(s) = \Lambda(s,\pi,r)$ is a quotient of holomorphic functions of order one, and hence a meromorphic function of order one. For instance, the argument of [39, Proposition 1] shows that there exist positive constants c_1, c_2, c_3, depending also on π, such that

$$\left|P(s)L^{S(\pi)}(s,\pi,r)\right| \leq c_1(1+|s|)^{\max(c_2,-c_3\operatorname{Re}(s))}, \quad s\in\mathbb{CC}.$$

Therefore $\Lambda(s)$ admits a Hadamard factorization

$$\Lambda(s) = e^{a+bs}s^{n(0)}\prod_{\rho\neq 0}[(1-s/\rho)e^{s/\rho}]^{n(\rho)},$$

where $a,b\in\mathbb{CC}$, the product ranges over the zeros and poles of $\Lambda(s)$ other than 0, and $n(\rho)$ is the order of the function $\Lambda(s)$ at $s=\rho$ (possibly negative). Also,

$$\sum_{\rho\neq 0}\frac{|n(\rho)|}{|\rho|^{1+\epsilon}} < \infty$$

for all $\epsilon > 0$. Thus,

$$\frac{\Lambda'(s)}{\Lambda(s)} = b + \frac{n(0)}{s} + \sum_{\rho\neq 0}n(\rho)\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right),$$

and hence

$$\operatorname{Re}\frac{\Lambda'(s)}{\Lambda(s)} = \sum_{\rho}\operatorname{Re}\frac{n(\rho)}{s-\rho} + \operatorname{Re}b + \sum_{\rho\neq 0}\operatorname{Re}\frac{n(\rho)}{\rho},$$

where the series of ρ are absolutely convergent because of

$$\sum_{\rho\neq 0}\left|\operatorname{Re}\frac{n(\rho)}{\rho}\right| = \sum_{\rho\neq 0}\frac{n(\rho)|\operatorname{Re}\rho|}{|\rho|^2} \ll \sum_{\rho\neq 0}\frac{n(\rho)}{|\rho|^2} < \infty.$$

Taking the logarithmic derivative of the functional equation (2.10), we also get

$$-\operatorname{Re}\frac{\Lambda'(s)}{\Lambda(s)} = \sum_{\rho}\operatorname{Re}\frac{n(\rho)}{1-s-\overline{\rho}} + \operatorname{Re}b + \sum_{\rho\neq 0}\operatorname{Re}\frac{n(\rho)}{\rho}.$$

Since $n(\rho) = n(1 - \bar{\rho})$, we conclude that $\operatorname{Re} b + \sum_{\rho \neq 0} \operatorname{Re} \frac{n(\rho)}{\rho} = 0$, and therefore

$$\operatorname{Re} \frac{\Lambda'(s)}{\Lambda(s)} = \sum_{\rho} n(\rho) \operatorname{Re} \frac{1}{s - \rho} = \sum_{\rho} n(\rho) \frac{\operatorname{Re}(s - \rho)}{|s - \rho|^2}.$$

By the second and third condition of Definition 2.4, there exist an integer $m \geq 1$ and a constant $A \geq 2$, depending only on G and r, such that for all primes p the Euler factor at p of $L^{\mathrm{red}}(s, \pi, r)$ is a product of at most m factors $(1 - \alpha_{p,i} p^{-s})^{-1}$ with $|\alpha_{p,i}| \leq p^{A-2}$ for all p and i, and such that the poles of the factor at infinity L_∞ lie in the half-plane $\operatorname{Re} s \leq A - 2$. (In fact, we may take $A = \max(\alpha, \beta) + 2$.) Therefore, the Euler product $L^{\mathrm{red},f}(s, \pi, r)$ is absolutely convergent for $\operatorname{Re} s > A - 1$, and because of the condition on L_∞ and the functional equation (2.10), all zeros and poles ρ of $\Lambda(s)$ lie in the strip $2 - A \leq \operatorname{Re} s \leq A - 1$.

By definition, we have

$$\frac{\Lambda'(s)}{\Lambda(s)} = \frac{1}{2} \log \mathfrak{n}(\pi, r) + \frac{(L^{\mathrm{red},f})'(s, \pi, r)}{L^{\mathrm{red},f}(s, \pi, r)} + \frac{L'_\infty(s, \pi, r)}{L_\infty(s, \pi, r)}.$$

The absolute convergence of the Euler product for $\operatorname{Re} s > A - 1$ implies that

$$\left| \frac{(L^{\mathrm{red},f})'(s, \pi, r)}{L^{\mathrm{red},f}(s, \pi, r)} \right| \leq -m \frac{\zeta'(s - A + 2)}{\zeta(s - A + 2)}$$

in this half-plane. Using the fact that $\mathbf{\Gamma}'(s)/\mathbf{\Gamma}(s) = \log s + O(1/|s|)$ for $\operatorname{Re} s \geq 1$, we get that

$$\left| \operatorname{Re} \frac{\Lambda'(s)}{\Lambda(s)} \right| \leq \left| \frac{\Lambda'(s)}{\Lambda(s)} \right| \leq \frac{1}{2} \log \mathfrak{n}(\pi, r) + \frac{1}{2} \log \mathfrak{c}_\infty(\pi, r) + \frac{m}{2} \log |s| + c$$

for $\operatorname{Re} s \geq A$, where c depends only on G and r. Taking $\operatorname{Re} s = A$, we conclude that

$$\sum_{\rho} n(\rho) \frac{A - \operatorname{Re} \rho}{|A + \mathrm{i}T - \rho|^2} \leq \frac{1}{2} \log \mathfrak{n}(\pi, r) + \frac{1}{2} \log \mathfrak{c}_\infty(\pi, r) + \frac{m}{2} \log(1 + |T|) + c.$$

On the other hand, because of the first condition of Definition 2.4, up to finitely many exceptions the poles of $\Lambda(s)$ can only arise from the poles of the Euler factors $L_p(s, \pi, r) = P_p(p^{-s})$, $p \in S_{\mathbb{Q},f}(\pi)$. Therefore, using the identity $\sum_{n \in \mathbb{Z}} (1 + (n/x)^2)^{-1} = \pi x \coth(\pi x)$, we have

$$\sum_{\rho : n(\rho) < 0} |n(\rho)| \frac{A - \operatorname{Re} \rho}{|A + \mathrm{i}T - \rho|^2} \leq c + \frac{m}{2} \sum_{p \in S_{\mathbb{Q},f}(\pi)} \frac{1 + p^{-1}}{1 - p^{-1}} \log p,$$

where c again depends only on r. Hence

(2.11) $$\sum_\rho \frac{|n(\rho)|}{1 + (T - \operatorname{Im}\rho)^2} \ll$$

$$\log \mathfrak{n}(\pi, r) + \sum_{p \in S_{\mathbb{Q}, f}(\pi)} \log p + \log \mathfrak{c}_\infty(\pi, r) + \log(1 + |T|) + 1,$$

and in particular
(2.12)
$$\sum_{\rho : |\operatorname{Im}\rho - T| < 2} |n(\rho)| \ll \log \mathfrak{n}(\pi, r) + \sum_{p \in S_{\mathbb{Q}, f}(\pi)} \log p + \log \mathfrak{c}_\infty(\pi, r) + \log(1 + |T|) + 1.$$

We can now estimate $\int_T^{T+1} |m'(\mathrm{i}t, \pi, r)| \; dt = 2 \int_T^{T+1} \left| \operatorname{Re} \frac{\Lambda'(\mathrm{i}t)}{\Lambda(\mathrm{i}t)} \right| \; dt$ by writing

$$\operatorname{Re} \frac{\Lambda'(\mathrm{i}t)}{\Lambda(\mathrm{i}t)} = \sum_\rho n(\rho) \frac{\operatorname{Re}\rho}{|\mathrm{i}t - \rho|^2},$$

and splitting the sum into two parts according to whether $|\operatorname{Im}\rho - T| \geq 2$ or $|\operatorname{Im}\rho - T| < 2$. For the first sum we have

$$\left| \sum_{\rho : |\operatorname{Im}\rho - T| \geq 2} n(\rho) \frac{\operatorname{Re}\rho}{|\mathrm{i}t - \rho|^2} \right| \leq 2A \sum_{\rho : |\operatorname{Im}\rho - T| \geq 2} \frac{|n(\rho)|}{1 + (T - \operatorname{Im}\rho)^2}$$

for any $t \in [T, T+1]$ and we use (2.11). For the second sum we use the fact that for $\rho = \beta + \mathrm{i}\gamma$ with $\beta \neq 0$ we have

$$\int_T^{T+1} \frac{|\operatorname{Re}\rho|}{|\mathrm{i}t - \rho|^2} dt = \int_T^{T+1} \frac{|\beta|}{\beta^2 + (t - \gamma)^2} \; dt = \int_{(T-\gamma)/|\beta|}^{(T+1-\gamma)/|\beta|} \frac{1}{1 + t^2} \; dt < \pi.$$

Thus,

$$\int_T^{T+1} \left| \sum_{\rho : |\operatorname{Im}\rho - T| < 2} n(\rho) \frac{\operatorname{Re}\rho}{|\mathrm{i}t - \rho|^2} \right| \; dt \leq \pi \sum_{\rho : |\operatorname{Im}\rho - T| < 2} |n(\rho)| \, ,$$

which is bounded by (2.12). All in all we get the required estimate. $\qquad \square$

Remark 2.7. It is clear that in Proposition 2.6 it is in fact sufficient to require that (G, r) has virtually property (FE+). Similarly, it suffices to know that the number of poles of $L^{S(\pi)}(s, \pi, r)$ is $\ll \log \mathfrak{c}_\infty(\pi, r) + \log(\mathfrak{n}(\pi, r) \prod_{p \in S_{\mathbb{Q}, f}(\pi)} p)$.

In order to make the connection to asymptotic problems, we need to control $\mathfrak{n}(\pi, r)$ (resp., $\mathfrak{c}_\infty(\pi, r)$) in terms of the level of π (resp., the size of the infinitesimal character of π_∞). Fix once and for all a faithful F-rational representation $\rho : G \to \mathrm{GL}(V)$ and an $\mathfrak{o}_F$-lattice Λ in the representation space V. The stabilizer of $\hat{\Lambda} = \hat{\mathfrak{o}}_F \otimes \Lambda \subset \mathbb{A}_{\mathrm{fin}} \otimes V$ in $G(\mathbb{A}_{\mathrm{fin}})$ is an open compact subgroup $\mathbf{K}_{\mathrm{fin}}$,

and any maximal compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$ can be realized this way. For any non-zero ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ let

$$\mathbf{K}(\mathfrak{n}) = \mathbf{K}_G(\mathfrak{n}) = \{g \in G(\mathbb{A}_{\mathrm{fin}}) : \rho(g)v \equiv v \pmod{\mathfrak{n}\hat{\Lambda}}, \quad v \in \hat{\Lambda}\}$$

be the principal congruence subgroup of level $\mathfrak{n}$, a factorizable normal open subgroup of $\mathbf{K}_{\mathrm{fin}}$. The groups $\mathbf{K}(\mathfrak{n})$ form a neighborhood base of the identity element in $G(\mathbb{A}_{\mathrm{fin}})$. We denote by $N(\mathfrak{n}) = [\mathfrak{o}_F : \mathfrak{n}]$ the ideal norm of $\mathfrak{n}$. We define the level of an admissible representation π of $G(\mathbb{A})$ by $\mathrm{level}(\pi) = N(\mathfrak{n})$, where $\mathfrak{n}$ is the largest ideal such that $\pi^{\mathbf{K}(\mathfrak{n})} \neq 0$. Analogously, for any finite place v of F we define the level $\mathrm{level}_v(\pi_v)$ of a smooth representation π_v of $G(F_v)$. Thus, $\mathrm{level}(\pi) = \prod_v \mathrm{level}_v(\pi_v)$ where v ranges over the finite places of F and almost all of the factors are 1. Note that there exists an integer n, depending only on G, such that for any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ we have:

(2.13) p divides $\mathrm{level}(\pi)$ for any rational prime $p \in S_{\mathbb{Q},f}(\pi)$ coprime to n.

We fix a maximal compact subgroup $\mathbf{K}_\infty$ of $G(F_\infty)$ and set $\mathbf{K} = \mathbf{K}_\infty \mathbf{K}_{\mathrm{fin}}$.

Fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{CC}}$ of the complexified Lie algebra $(\mathfrak{g}_\infty)_{\mathbb{CC}}$ of $G(F_\infty)$ and a Euclidean norm $\|\cdot\|_{\mathbb{CC}}$ on $\mathfrak{h}_{\mathbb{CC}}^*$ which is invariant under the Weyl group $W((\mathfrak{g}_\infty)_{\mathbb{CC}}, \mathfrak{h}_{\mathbb{CC}})$. For any (smooth) non-zero representation π of $G(F_\infty)$, which has an infinitesimal character χ_π, we define

$$\Lambda(\pi) = 1 + \|\chi_\pi\|^2,$$

where we view χ_π as an orbit in $\mathfrak{h}_{\mathbb{CC}}^*$ under $W((\mathfrak{g}_\infty)_{\mathbb{CC}}, \mathfrak{h}_{\mathbb{CC}})$.

Remark 2.8. In [35, Section 5.1] the slightly different invariant

$$\Lambda'(\pi) = 1 + \lambda_\pi^2 + \min\|\tau\|^2$$

is used (and denoted there by $\Lambda(\pi)$), where λ_π is the eigenvalue of the Casimir operator of $G(F_\infty)$, τ ranges over the $\mathbf{K}_\infty$-types of π and the norm $\|\tau\|$ is the one defined by Vogan (cf. [28, Section 2.2]). By a standard argument (cf. [109, Sections 6.5 and 6.6]), there exist constants $c_1, c_2 > 0$, depending only on G, such that

$$c_1 \Lambda'(\pi) < \Lambda(\pi)^2 < c_2 \Lambda'(\pi).$$

Therefore, for all practical purposes there is no difference between $\Lambda(\pi)$ and $\Lambda'(\pi)$.

We thus make the following definition.

Definition 2.9. Suppose that the pair (G, r) satisfies property (FE+). We say that the conductor condition (CC) is satisfied, if there exists $c > 0$ (depending only on G and r), such that

(2.14) $\mathfrak{n}(\pi, r) \ll \mathrm{level}(\pi)^c$

and

(2.15) $\mathfrak{c}_\infty(\pi, r) \ll \Lambda(\pi_\infty)^c$

for all $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$.

Condition (CC) is certainly expected to hold in general (cf. [9]), although it is not clear whether, strictly speaking, its non-archimedean part (2.14) formally follows from Langlands's principle of functoriality or the local Langlands conjecture for G. For the archimedean part (2.15) see Remark 2.10 below.

2.2. The standard paradigm for proving (FE) (e.g., through the Langlands-Shahidi method or an integral representation, cf. [41]) goes by defining local factors $\gamma_v^\bullet(s, \pi, r)$, that satisfy certain properties, most importantly

(FE') $$L^S(s, \pi, r) = \big(\prod_{v \in S} \gamma_v^\bullet(s, \pi, r) \big) L^S(1 - s, \pi, r^\vee)$$

for any finite set $S \supset S(\pi)$. (In each particular case the superscript $\bullet$ will be replaced by an appropriate acronym.) The local factors $\gamma_v^\bullet(s, \pi, r)$ normally depend on a choice of a character ψ_v of F_v. We will suppress this choice by taking $\psi_F = \psi_\mathbb{Q} \circ \mathrm{Tr}_{F/\mathbb{Q}}$ where $\psi_\mathbb{Q} = \prod_p \psi_{\mathbb{Q}_p}$ is the standard character of $\mathbb{Q} \backslash \mathbb{A}_\mathbb{Q}$ (characterized by $\psi_\mathbb{R}(\cdot) = e^{2\pi \mathrm{i}(\cdot)}$) and writing $\psi_F = \otimes \psi_v$, i.e., $\psi_v = \psi_{F_v} = \psi_{\mathbb{Q}_p} \circ \mathrm{Tr}_{F_v/\mathbb{Q}_p}$, if v lies above p. If we want to emphasize that $\gamma_v^\bullet(s, \pi, r)$ depends only on π_v (as it is in the usual paradigm) we will write it as $\gamma_v^\bullet(s, \pi_v, r)$. However, we will not make it a part of our requirements. This flexibility will be useful in sections 4 and 5, where we will use functoriality to study analytic properties of L-functions.

In addition to (FE') we will impose the following conditions on $\gamma_v^\bullet(s, \pi, r)$ (which for simplicity we denote by (ΓF)). Here, n is a positive integer and β a real number which depend only on (G, r). (In practice $n = N = \deg r$.)

(1) For every finite v, the function $\gamma_v^\bullet(s, \pi, r)$ is a rational function in q_v^{-s}, and

(2.16) $$\gamma_v^\bullet(s, \pi, r) = \epsilon_v^\bullet(s, \pi, r) \frac{L_v^\bullet(1 - s, \pi, r^\vee)}{L_v^\bullet(s, \pi, r)},$$

where, denoting by Δ_v the ideal norm of the different of F_v (i.e., the conductor of ψ_v as above),

$$\epsilon_v^\bullet(s, \pi, r) = c_v q_v^{\mathfrak{c}_v^\bullet(\pi, r)(\frac{1}{2} - s)} \Delta_v^{N(\frac{1}{2} - s)},$$
$$L_v^\bullet(s, \pi, r) = P_v(q_v^{-s})^{-1},$$
$$L_v^\bullet(s, \pi, r^\vee) = Q_v(q_v^{-s})^{-1},$$

for some $\mathfrak{c}_v^\bullet(\pi, r) \in \mathbb{Z}$ and polynomials P_v and Q_v of degree $\leq n$ satisfying $P_v(0) = Q_v(0) = 1$.

(2) For $v \in S_\infty$ we have

(2.17) $$\gamma_v^\bullet(s, \pi, r) = c_v \prod_{i=1}^{m_v} \frac{\Gamma_{F_v}(1 - s + \alpha_{v,i}^\vee)}{\Gamma_{F_v}(s + \alpha_{v,i})}$$

for some $c_v \in \mathbb{CC}^*$, $\alpha_{v,1}, \ldots, \alpha_{v,m_v}, \alpha_{v,1}^\vee, \ldots, \alpha_{v,m_v}^\vee \in \mathbb{CC}$, where $m_v \leq n$.

   (3) For all $v \in S(\pi)$, the function $\gamma_v^\bullet(s, \pi, r)$ has no zeros for $\operatorname{Re} s > \beta$.

Let us make a few comments about these conditions.

First, as before, we do not impose that $\mathfrak{e}_v^\bullet(\pi, r) \geq 0$ for $v \in S_f(\pi)$ (although in practice this will always be the case). Also, for v finite we allow $P_v(X)$ and $Q_v(q_v^{-1} X^{-1})$ to have common zeros. Thus, $\mathfrak{e}_v^\bullet(\pi, r)$ does not depend only on the function $\gamma_v^\bullet(s, \pi, r)$ itself. However, this ambiguity is immaterial for our purpose since different presentations (2.16) give rise to values of $\mathfrak{e}_v^\bullet(\pi, r)$ which differ by an integer of absolute value $\leq n$.

Similarly, for $v \in S_\infty$ we allow the numerator and denominator in the expression (2.17) to have common poles. We set

$$\mathfrak{e}_v^\bullet(\pi, r) = \Big(\prod_{i=1}^{m_v} (1 + |\alpha_{v,i} - 1/2|)(1 + |\alpha_{v,i}^\vee - 1/2|) \Big)^{\frac{1}{2}}.$$

As in the proof of Lemma 2.2, $\mathfrak{e}_v^\bullet(\pi, r)$ depends only on the factor $\gamma_v^\bullet(s, \pi, r)$ and not on the choice of $\alpha_{v,1}, \ldots, \alpha_{v,m_v}, \alpha_{v,1}^\vee, \ldots, \alpha_{v,m_v}^\vee$. (In practice we always have $m_v = N$.)

Note that the functional equation (FE') implies that for all $v \notin S(\pi)$ we have

$$\gamma_v^\bullet(s, \pi, r) = \frac{L_v(1 - s, \pi_v, r^\vee)}{L_v(s, \pi_v, r)}.$$

Finally, it implies (FE) with

$$\gamma_p(s, \pi, r) = \prod_{v \in S(\pi) : v | p} \gamma_v^\bullet(s, \pi, r)$$

for each $p \in S_\mathbb{Q}(\pi)$, and the second and third conditions of property (FE+) are satisfied. Regarding condition (CC), by Remark 2.3 the quotient

$$\frac{\mathfrak{n}(\pi, r)}{\prod_{v \in S_f(\pi)} q_v^{\mathfrak{e}_v^\bullet(\pi, r)}}$$

is a positive integer which divides $\prod_{p \in S_{\mathbb{Q},f}(\pi)} p^{n[F:\mathbb{Q}]}$. Therefore, by (2.13) the non-archimedean part of condition (CC) is implied by the condition

(CC') $\mathfrak{e}_v^\bullet(\pi, r) \ll 1 + \log_{q_v} \operatorname{level}_v(\pi_v), \quad v \in S_f(\pi).$

Note that this property (in contrast to the value of $\mathfrak{e}_v^\bullet(\pi, r)$ itself) depends only on $\gamma_v^\bullet(s, \pi, r)$. The archimedean part of condition (CC) is implied by the condition

(AF) $\mathfrak{e}_v^\bullet(\pi, r) \ll \Lambda(\pi_v)^c, \quad v \in S_\infty,$

where c depends only on (G, r).

Remark 2.10. In many cases one knows that the archimedean factors are compatible with the Langlands classification in the following sense. For $v \in S_\infty$, let W_v be the Weil group of F_v and let $\phi_v : W_v \to {}^L G$ be the Langlands parameter attached to π_v. Then

(CL)
$$\gamma_v^\bullet(s, \pi, r) = \gamma_v(s, r \circ \phi_v, \psi_v) := \epsilon_v(s, r \circ \phi_v, \psi_v) L_v(1 - s, r^\vee \circ \phi_v) / L_v(s, r \circ \phi_v),$$

where the L- and ϵ-factors on the right-hand side are as in [108, §3]. It is easy to see that condition (CL) for all $v \in S_\infty$ implies (AF) (with $c = N/2$).

We sum up the discussion as follows.

Corollary 2.11. *For a given pair* (G, r), *suppose that for all* $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ *the following conditions are satisfied.*

(1) *There exists a polynomial* $P(s)$, *whose degree is bounded in terms of* (G, r) *only, such that* $P(s) L^{S(\pi)}(s, \pi, r)$ *extends to an entire function of finite order.*

(2) *There exist local factors* $\gamma_v^\bullet(s, \pi, r)$ *(for all places* v *of* F*) satisfying* (FE'), *(ΓF),* (AF) *and* (CC').

Then (G, r) *satisfies properties* (FE+) *and* (CC).

The following observation will be useful.

Lemma 2.12. *Suppose that for all triplets* (M, π, r'), *where* M *is a Levi subgroup of* G *defined over* F *(including* G *itself),* $\pi \in \Pi_{\mathrm{cusp}}(M(\mathbb{A}))$ *is a* cuspidal *representation of* $M(\mathbb{A})$, *and* r' *is an irreducible constituent of* $r\big|_{{}^L M}$, *conditions* (1) *and* (2) *of Corollary* 2.11 *are satisfied (with* M *instead of* G *and* r' *instead of* r*). Then* (G, r) *satisfies properties* (FE+) *and* (CC).

Proof. Let $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$. Then there exist $\sigma \in \Pi_{\mathrm{cusp}}(M(\mathbb{A}))$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ such that π is a subquotient of the representation $I(\sigma, \lambda)$ parabolically induced from the twist of σ by the character of $M(\mathbb{A})/M(\mathbb{A})^1$ determined by λ [12, Supplement]. We have $S(\sigma) \subset S(\pi)$, $\mathrm{level}(\sigma) \ll \mathrm{level}(\pi)$ and $\Lambda(\sigma_\infty) + \|\lambda\|^2 \ll \Lambda(\pi_\infty)$. Also, by the unitarity of π, $\|\mathrm{Re}\,\lambda\|$ is bounded in terms of G only. Decompose $r\big|_{{}^L M} = \oplus_{i=1}^k r_i$ according to the central character, and let $\beta_i^\vee$ be the element of $\mathfrak{a}_M$ corresponding to the central character of r_i. Then

$$L^{S(\pi)}(s, \pi, r) = \prod_{i=1}^k L^{S(\pi)}(s + \langle \lambda, \beta_i^\vee \rangle, \sigma, r_i).$$

We take

$$\gamma_v^\bullet(s, \pi, r) = \prod_{i=1}^k \gamma_v^\bullet(s + \langle \lambda, \beta_i^\vee \rangle, \sigma, r_i).$$

Then properties (1) and (2) for (G, π, r) immediately follow from the corresponding properties of (M, σ, r_i), $i = 1, \ldots, k$ (cf. Remark 2.3). $\square$

2.3. At this stage it will be useful to introduce a slight refinement of the notion of level. For the rest of this section let G' be a closed connected normal subgroup of G and $p : H \to G'$ a finite covering of G' (with H connected), both defined over F. Note that $p(H(F))$ is a normal subgroup of $G'(F)$ and that $G'(F)/p(H(F))$ is an abelian group of finite exponent (bounded by the size of the kernel of p), and similarly for $p(H(\mathbb{A})) \subset G'(\mathbb{A})$.

For any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ we write $\mathrm{level}(\pi; p) = \mathrm{N}(\mathfrak{n})$, where $\mathfrak{n}$ is the largest ideal such that $\pi^{\mathbf{K}(\mathfrak{n}) \cap p(H(\mathbb{A}))} \neq 0$. (Note that the same notion was considered in [35, §5.1], where the notation $\mathrm{level}(\pi; p(H(\mathbb{A})))$ was used.) Analogously, we define $\mathrm{level}_v(\pi_v; p)$ for a smooth representation π_v of $G(F_v)$. We also set

$$(2.18) \qquad\qquad \Lambda(\pi_\infty; p) = \Lambda^{G'}(\pi_\infty\big|_{G'(F_\infty)}),$$

where on the right-hand side Λ is taken with respect to G'. Alternatively, $\Lambda(\pi_\infty; p) = 1 + \|\chi_{\pi_\infty; p}\|^2$, where $\chi_{\pi_\infty; p}$ is the projection of χ_{π_∞} to $(\mathfrak{h}_{\mathbb{CC}} \cap \mathrm{Lie}(G'(F_\infty))_{\mathbb{CC}})^*$.

Lemma 2.13. *There exists an integer N_1, depending only on p and G, such that for any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ there exists $\sigma \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ and a character χ of $G'(\mathbb{A})$ trivial on $G'(F)p(H(\mathbb{A}))$ such that $\mathrm{level}(\sigma)$ divides $N_1 \mathrm{level}(\pi; p)$ and $\sigma\chi$ is a subrepresentation of $\pi\big|_{G'(\mathbb{A})}$. In particular, $\Lambda^{G'}(\sigma) = \Lambda(\pi_\infty; p)$.*

Proof. We first reduce the lemma to the case $G' = G$. Namely, we show that there exists an integer N_2, depending only on G and G', such that for any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ there exists a subrepresentation $\sigma \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ of $\pi|_{G'(\mathbb{A})}$ with $\mathrm{level}(\sigma; p)$ dividing $N_2 \mathrm{level}(\pi; p)$.

Let C be the centralizer of G' in G. Then $CG' = G$, and therefore $G'(F_\infty)C(F_\infty)$ has finite index in $G(F_\infty)$. Since the maximal compact subgroup $\mathbf{K}_\infty$ meets every connected component of $G(F_\infty)$, we have $G(F_\infty) = G'(F_\infty)C(F_\infty)\mathbf{K}_\infty$. Combining this fact with the well-known finiteness of the class number of G [96, Theorem 8.1], we see that the coset space

$$X = G(F)\backslash G(\mathbb{A})/G'(\mathbb{A})C(F_\infty)\mathbf{K}$$

is finite. Fix a set of representatives $\{g_1, \ldots, g_r\}$ for the classes of X.

Let $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$. Note that since $CG' = G$, $G(\mathbb{A})/C(\mathbb{A})G'(\mathbb{A})$ is compact, and therefore, the restriction $\pi\big|_{G'(\mathbb{A})}$ decomposes into a direct sum of irreducible representations. Let ϕ be an automorphic form on $G(\mathbb{A})$ in the isotypic space of π, which is right-invariant under the group $\mathbf{K}(\mathfrak{n}) \cap p(H(\mathbb{A}_{\mathrm{fin}}))$. Then there exist $i = 1, \ldots, r$, $k \in \mathbf{K}_M$ and $c \in C(F_\infty)$, such that the function $\tilde{\phi} := \phi(\cdot cg_ik)$ does not vanish on $G'(F)\backslash G'(\mathbb{A})$. The function $\tilde{\phi}$ is clearly invariant under $g_i\mathbf{K}(\mathfrak{n})g_i^{-1} \cap p(H(\mathbb{A}_{\mathrm{fin}}))$. Decomposing the span of $\tilde{\phi}\big|_{G'(\mathbb{A})}$ under $G'(\mathbb{A})$ into irreducibles, we obtain an irreducible subrepresentation $\sigma \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ of $\pi|_{G'(\mathbb{A})}$ such that $\mathrm{level}(\sigma; p)$ divides $N_2 \mathrm{level}(\pi; p)$, where the integer N_2 depends on G only, as required.

So assume from now on that $G' = G$. It remains to show that there exists an integer N_3, depending only on p, such that for any $\sigma \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ there exists a character $\chi : G(\mathbb{A})/G(F)p(H(\mathbb{A})) \to \mathbb{CC}^\times$ such that $\mathrm{level}(\sigma\chi^{-1})$ divides $N_3 \mathrm{level}(\sigma; p)$. Recall from [35, Lemma 5.1] that

$$\mathrm{level}_v(\sigma_v; p) = \min_{\chi_v} \mathrm{level}_v(\sigma_v\chi_v^{-1}),$$

where χ_v ranges over the characters of $G(F_v)/p(H(F_v))$. Since $G(F_v)/p(H(F_v))$ is finite for all v, it suffices to show the following assertion. There exists a finite set S_0 of finite places of F, depending on p only, such that for any finite set S of finite places of F outside S_0 and a family of characters $\tilde{\chi}_v$ of $G(F_v)/p(H(F_v))$ for $v \in S$, there exists a global character $\chi : G(\mathbb{A})/G(F)p(H(\mathbb{A})) \to \mathbb{CC}^\times$, unramified outside $S \cup S_0$, such that $\chi_v\tilde{\chi}_v^{-1}$ is unramified for all $v \in S$. (We use the fact that $\mathrm{level}_v(\pi_v\chi_v) \ll \mathrm{level}_v(\pi_v)$ for any character χ_v of $G(F_v)/p(H(F_v))$, $v \in S_0$.)

For convenience we write $\tilde{\mathbf{K}} = G(F_\infty)\mathbf{K}_{\mathrm{fin}}$. We first show that the group

$$(G(F) \cap p(H(\mathbb{A}))\tilde{\mathbf{K}})/p(H(F))$$

is finitely generated. Indeed, choose representatives $x_1, \ldots, x_k$ for the finite double coset space $H(F)\backslash H(\mathbb{A})/p^{-1}(\tilde{\mathbf{K}})$, and let $y_i = p(x_i)$. Then $G(F) \cap p(H(\mathbb{A}))\tilde{\mathbf{K}}$ is the union over $i = 1, \ldots, k$ of

$$G(F) \cap p(H(F))y_i\tilde{\mathbf{K}} = p(H(F))(G(F) \cap y_i\tilde{\mathbf{K}}).$$

If $\delta_i \in G(F) \cap y_i\tilde{\mathbf{K}}$, then $G(F) \cap y_i\tilde{\mathbf{K}} = (G(F) \cap \delta_i\tilde{\mathbf{K}}\delta_i^{-1})\delta_i$, which is a coset of a finitely generated group. Our claim follows.

Let Γ be the image of $G(F) \cap p(H(\mathbb{A}))\tilde{\mathbf{K}}$ in $G(\mathbb{A}_{\mathrm{fin}})/p(H(\mathbb{A}_{\mathrm{fin}}))$. Since the latter is abelian of finite exponent, Γ is necessarily finite, and therefore projects injectively into $G(F_{S_0})/p(H(F_{S_0}))$ for a suitable finite set S_0 depending only on p. It is therefore possible to extend the character $\prod_{v \in S} \tilde{\chi}_v$ of $\mathbf{K}_S$ to a character of $\tilde{\mathbf{K}}/(G(F)p(H(\mathbb{A})) \cap \tilde{\mathbf{K}})$, which is trivial on $\mathbf{K}_v$ for any $v \notin S \cup S_0$. Extending this character to $G(\mathbb{A})/G(F)p(H(\mathbb{A}))$ we obtain the desired χ.

The last assertion of the lemma is clear, since χ is trivial on the connected component of the identity of $G(F_\infty)$. $\qquad\square$

The following observation will also be useful.

Lemma 2.14. *Assume that G' contains the derived group of G and let $^L\iota : {}^LG \to {}^LG'$ be the corresponding homomorphism of L-groups. Let r' be a representation of $^LG'$ and let $r = r' \circ {}^L\iota$. Assume that (G, r) satisfies (FE+) and (CC). Then (G', r') satisfies (FE+) and (CC).*

We first need the following standard result.

Lemma 2.15. *Let T be a torus over F. Then there exists a compact subset C of the Pontryagin dual $T(F_\infty)^D = \mathrm{Hom}(T(F_\infty), \mathbb{CC}^1)$ of $T(F_\infty)$, such that*

for any character χ of $T(\mathbb{A})$ there exists a character $\tilde{\chi}$ of $T(F)\backslash T(\mathbb{A})$ such that $\tilde{\chi}\chi^{-1}$ is unramified at all finite places and $\tilde{\chi}_\infty \chi_\infty^{-1} \in C$.

Proof. Indeed, let $T(F_\infty)^1 = T(F_\infty) \cap T(\mathbb{A})^1$. Then

$$X := T(F)T(F_\infty)^1 \prod_{v \text{ finite}} T(\mathcal{O}_v)$$

is a closed subgroup of finite index of $T(\mathbb{A})^1$ and the group $\Gamma := T(F) \cap T(F_\infty)^1 \prod_{v \text{ finite}} T(\mathcal{O}_v)$ is a lattice in $T(F_\infty)^1$. Therefore, there exists a compact subset C of $(T(F_\infty)^1)^D$, such that its image under the restriction map

$$(T(F_\infty)^1)^D \to \Gamma^D$$

is onto. Thus, there exists a character $\tilde{\chi}$ of X, trivial on $T(F)$, whose restriction to $T(F_\infty)^1$ is in $\chi_\infty C$ and whose restriction to $\prod_{v \text{ finite}} T(\mathcal{O}_v)$ is χ. Extending $\tilde{\chi}$ arbitrarily to $T(\mathbb{A})^1$ and setting $\tilde{\chi}|_{A_T} = \chi|_{A_T}$, we obtain the assertion. $\square$

Proof of Lemma 2.14. We show that for any given $\pi' \in \Pi_{\text{disc}}(G'(\mathbb{A}))$ we can find $\pi \in \Pi_{\text{disc}}(G(\mathbb{A}))$ such that π' is a subrepresentation of $\pi|_{G'(\mathbb{A})}$, level$(\pi) \ll$ level(π') and $\Lambda(\pi_\infty) \ll \Lambda(\pi'_\infty)$. By [54, Theorem 4.13, Remark 4.23] there exists $\pi \in \Pi_{\text{disc}}(G(\mathbb{A}))$ such that π' occurs in $\pi|_{G'(\mathbb{A})}$. Let $T = G/G'$. As in the proof of [35, Lemma 5.1] we have level$_v(\pi'_v) \geq \min_{\omega_v}$ level$_v(\pi_v \otimes \omega_v)$, where ω_v ranges over the characters of $G(F_v)/G'(F_v)$, and similarly for $\Lambda(\pi'_\infty)$. Since the map $G(F_v)/G'(F_v) \to T(F_v)$ is injective, we can vary over the characters of $T(F_v)$ instead. Hence the claim follows from Lemma 2.15.

Finally, since π' occurs in $\pi|_{G'(A)}$ we have $S(\pi') \subset S(\pi)$ and $L^{S(\pi)}(s, \pi, r) = L^{S(\pi)}(s, \pi', r')$. It is now easy to conclude the assertion of the lemma from Remark 2.3. $\square$

3. **Global normalizing factors and L-functions**

In this section we consider the global normalizing factors associated to intertwining operators. We switch therefore to a slightly different setting. Let G be an isotropic reductive group defined over F, and M a proper Levi subgroup of G containing a fixed maximal F-split torus T_0. As usual, we let $\mathcal{P}(M)$ be the set of all parabolic subgroups of G, defined over F, with Levi subgroup M. For any $P \in \mathcal{P}(M)$ with unipotent radical N_P let $\mathcal{A}^2(P)$ be the space of automorphic forms φ on $N_P(\mathbb{A})M(F)\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\varphi(\cdot k) \in L^2_{\text{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ for all $k \in \mathbf{K}$, where δ_P denotes the modulus function of $P(\mathbb{A})$. For each pair $P, Q \in \mathcal{P}(M)$ there is a global intertwining operator

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q),$$

which has meromorphic continuation in $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ [1, Section 1].

Let $\Sigma_M \subset \mathfrak{a}^*_M$ be the set of reduced roots of T_M on the Lie algebra of G, and for any $P \in \mathcal{P}(M)$ let $\Sigma_P \subset \Sigma_M$ be the set of reduced roots of T_M on

the Lie algebra $\mathfrak{n}_P$ of N_P. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are adjacent along $\alpha \in \Sigma_M$, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. The study of the operators $M_{Q|P}(\lambda)$ reduces to this case. Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$. We are interested in the corresponding normalizing factors $n_\alpha(\pi, s)$ introduced by Langlands and Arthur (see [1, Section 6], [2]). They are meromorphic functions of a complex variable s, and are closely connected to certain automorphic L-functions studied by Langlands and Shahidi. (As the notation suggests, n_α depends only on α and not on the choice of parabolic subgroups $P|^\alpha Q$. The precise definition of n_α and its relation to the intertwining operators $M_{Q|P}(\lambda)$ is described in [1]. We will not need it here.)

To describe the relevant representations of the L-group $^L M$, let U_α be the unipotent subgroup of G corresponding to α (so that the eigenvalues of T_M on the Lie algebra of U_α are positive integer multiples of α). Let M_α be the group generated by M and $U_{\pm\alpha}$. It is a Levi subgroup of G defined over F containing M as a co-rank one Levi subgroup. Let $\hat{M}_\alpha$ be the subgroup of $M_\alpha \cap G^{\mathrm{der}}$ generated by $U_{\pm\alpha}$. By [13, Proposition 4.11] $\hat{M}_\alpha$ is a connected normal subgroup of M_α defined over F. Hence $\tilde{M}_\alpha := M \cap \hat{M}_\alpha$ is a normal subgroup of M. Moreover, since M has co-rank one in M_α, precisely one simple root β of M_α restricts to α, which implies that the root system of $\hat{M}_\alpha$ is the irreducible component of the root system of M_α containing β. The group $\hat{M}_\alpha$ is therefore F-simple, and $\tilde{M}_\alpha$ is a Levi subgroup of co-rank one. (In particular, $\tilde{M}_\alpha$ is connected.) Let $\hat{M}_\alpha^{\mathrm{sc}}$ be the simply connected cover of $\hat{M}_\alpha$, and $\widetilde{p^{\mathrm{sc}}} : \hat{M}_\alpha^{\mathrm{sc}} \to \hat{M}_\alpha$ the natural projection. By abuse of notation we write $\tilde{M}_\alpha^{\mathrm{sc}} = (p^{\mathrm{sc}})^{-1}(\tilde{M}_\alpha)$. (Of course $\tilde{M}_\alpha^{\mathrm{sc}}$ is not semisimple.) We denote by $p^{\mathrm{sc}} : \tilde{M}_\alpha^{\mathrm{sc}} \to \tilde{M}_\alpha$ the restriction of $\widetilde{p^{\mathrm{sc}}}$ to $\tilde{M}_\alpha^{\mathrm{sc}}$. We note that $p^{\mathrm{sc}}(\tilde{M}_\alpha^{\mathrm{sc}}(\mathbb{A})) = \widetilde{p^{\mathrm{sc}}}(\hat{M}_\alpha^{\mathrm{sc}}(\mathbb{A})) \cap \tilde{M}_\alpha(\mathbb{A})$.

The L-group $^L M_\alpha$ of M_α admits a parabolic subgroup $^L P_\alpha = {}^L M \ltimes {}^L U_\alpha$ corresponding to α whose Levi part $^L M$ is the L-group of M. The representation of $^L M$ relevant for the theory of intertwining operators is (the contragredient of) the adjoint representation of $^L M$ on $\mathrm{Lie}(^L U_\alpha)$. We have a sequence of homomorphisms of reductive groups

$$\hat{M}_\alpha^{\mathrm{sc}} \longrightarrow\!\!\!\!\!\rightarrow \hat{M}_\alpha \lhook\joinrel\longrightarrow M_\alpha,$$

which restricts to

$$\tilde{M}_\alpha^{\mathrm{sc}} \longrightarrow\!\!\!\!\!\rightarrow \tilde{M}_\alpha \lhook\joinrel\longrightarrow M.$$

The dual sequences of homomorphisms of L-groups are

$$^L M_\alpha \longrightarrow {}^L \hat{M}_\alpha \longrightarrow {}^L(\hat{M}_\alpha^{\mathrm{sc}}) = (^L \hat{M}_\alpha)^{\mathrm{ad}}$$

and

$$^L M \longrightarrow {}^L \tilde{M}_\alpha \longrightarrow {}^L(\tilde{M}_\alpha^{\mathrm{sc}}) = {}^L \tilde{M}_\alpha / Z(^L \hat{M}_\alpha),$$

respectively. The adjoint action of ^LM on $\mathrm{Lie}(^LU_\alpha)$ clearly factors through the composed homomorphism. We decompose the contragredient of the adjoint representation of $^L\tilde{M}_\alpha$ on $\mathrm{Lie}(^LU_\alpha)$ as $\oplus_{j=1}^l r_j$, where the (irreducible) subrepresentation r_j is the eigenspace of weight $-j$ of the fundamental weight corresponding to α, considered as a cocharacter of the center of $^L\tilde{M}_\alpha/Z(^L\hat{M}_\alpha)$.

The normalizing factor $n_\alpha(\pi, s)$ is closely related to the L-functions $L^S(js, \pi, r_j)$, $j = 1, \dots, l$, which emerged in the famous computation by Langlands of the constant term of the corresponding Eisenstein series [81]. Langlands used this to show the meromorphic continuation of these L-functions to the entire complex plane (at least in the cuspidal case, but the general case follows from the cuspidal case as in the argument of Lemma 2.12). These L-functions are also known to have finite order as meromorphic functions. In the cuspidal case this is [39, Theorem 2], which is based on the results of Müller [92, 93]. The general case follows once again from the argument of Lemma 2.12. However, the finer analytic properties of $L^S(s, \pi, r_j)$, such as properties (FE+) and (CC) considered in this paper, are more elusive (cf. Remark 3.7 below).

We now summarize the pertinent properties of the normalizing factors $n_\alpha(\pi, s)$. The first one is the functional equation

$$n_\alpha(\pi, s)\overline{n_\alpha(\pi, -\bar{s})} = 1,$$

which is equivalent to

$$|n_\alpha(\pi, \mathrm{i}t)| = 1, \qquad t \in \mathbb{R}.$$

The second is a factorization

$$n_\alpha(\pi, s) = \prod_v n_{\alpha,v}(\pi, s)$$

as an absolutely convergent product for $\mathrm{Re}\, s$ sufficiently large. The local factors $n_{\alpha,v}(\pi, s)$ are assumed to satisfy the following properties.

(1) For all finite v, $n_{\alpha,v}(\pi, s)$ is a rational function in $X = q_v^{-s}$, whose degree is bounded in terms of G only and which is regular and non-zero at $X = 0$.

(2) If $v \in S_\infty$, then

$$n_{\alpha,v}(\pi, s) = c_v \prod_{i=1}^{N_v} \frac{\Gamma_{\mathbb{R}}(j_i s + \alpha_i)}{\Gamma_{\mathbb{R}}(j_i s + \alpha_i + 1)},$$

where $c_v \neq 0$, $\alpha_1, \dots, \alpha_{N_v} \in \mathbb{CC}$ and the integers $N_v \geq 1$ and $j_i \geq 1$, $i = 1, \dots, N_v$, are bounded in terms of G only.

(3) Write $\pi = \otimes \pi_v$. If v is finite, $\tilde{M}_\alpha$ is unramified at v and the pull-back $\pi_v \circ p^{\mathrm{sc}}$ of π_v to $\tilde{M}_\alpha^{\mathrm{sc}}(F_v)$ contains an unramified subrepresentation σ_v,

then

(3.1)
$$n_{\alpha,v}(\pi, s) = \prod_{j=1}^{l} \frac{L_v(js, \sigma_v, r_j)}{L_v(js + 1, \sigma_v, r_j)}.$$

Remark 3.1. The first and second property are clear from [2] (and moreover $n_{\alpha,v}(\pi, s)$ depends only on π_v, or in fact on $\pi_v \circ p^{\text{sc}}$ – cf. [27, Theorem 3.3.4]). For $v \in S_\infty$, by [2, §3] we have in fact

$$n_{\alpha,v}(\pi_v, s) = \prod_{j=1}^{l} \frac{L_v(js, r_j \circ \phi_v)}{L_v(js + 1, r_j \circ \phi_v)},$$

where $\phi_v : W_v \to {}^L M$ is the Langlands parameter associated to π_v. However, we will only need the qualitative property stated above. The third property is implicit in [2]. We omit the details.

Remark 3.2. We can rewrite the assumption on π_v in (3) in an equivalent way by saying that $\pi_v|_{\tilde{M}_\alpha(F_v)}$ contains an irreducible subrepresentation of the form $\sigma_v \chi_v$, where σ_v is an unramified representation of $\tilde{M}_\alpha(F_v)$ and χ_v is a character of $\tilde{M}_\alpha(F_v)/p^{\text{sc}}(\tilde{M}_\alpha^{\text{sc}}(F_v))$.

We now introduce the property (TWN+), which is the main object of this paper.

Definition 3.3. The group G satisfies property (TWN+) (tempered winding numbers, strong version) if, for any proper Levi subgroup M of G defined over F, and any root $\alpha \in \Sigma_M$ we have the estimate

$$\int_T^{T+1} |n_\alpha'(\pi, \mathrm{i}t)| \ dt \ll \log(|T| + \Lambda(\pi_\infty; p^{\text{sc}}) + \text{level}(\pi; p^{\text{sc}}))$$

for all $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ and all real numbers T.

Recall that the invariants $\Lambda(\pi_\infty; p^{\text{sc}})$ and $\text{level}(\pi; p^{\text{sc}})$ are defined in Section 2.3. (Of course, the ambient group is M in our case.) As explained in [35, Remark 5.3, part 2], this property implies property (TWN) for G formulated in [35, Definition 5.2], which is relevant to the limit multiplicity problem.

In view of the description of the unramified factors of $n_\alpha(\pi, s)$, it is no wonder that the property (TWN+) is intimately related to analytic properties of the automorphic L-functions $L^S(s, \pi, r_j)$. In the special cases $G = \text{GL}(n)$ and $G = \text{SL}(n)$ this was spelled out in [35, Proposition 5.5]. Here, we will analyze it in the general case.

Definition 3.4. We say that G satisfies property (L), if for any standard Levi subgroup M, any $\alpha \in \Sigma_M$, and any irreducible constituent $r = r_j$ of the representation $(\text{Ad}_{L_{U_\alpha}})^\vee$ of the group ${}^L \tilde{M}_\alpha$, the pair $(\tilde{M}_\alpha, r)$ satisfies properties (FE+) and (CC).

Remark 3.5. It is clear from the definition that property (L) depends only on the derived group of G.

Remark 3.6. We expect that every group satisfies property (L). Using [42, Proposition 3.1], it is possible to reduce this question to the constituent r_1.

Remark 3.7. Suppose that G is quasi-split and that π is a *generic* cuspidal representation of $M(\mathbb{A})$. (If M is isogenous to a product of general linear groups, then this condition is satisfied for all cuspidal representations π.) By the results of Shahidi, the meromorphic function $L^{S(\pi)}(s, \pi, r)$ satisfies then a functional equation of the form (2.1). Moreover, local factors $\gamma_v^{\mathrm{Sh}}(s, \pi_v, r)$ are defined and satisfy (FE'), (ΓF) and (CL) [103]. Shahidi's work also gives that under some mild assumptions which are known in almost all cases (see [74] and the references therein), these L-functions admit finitely many poles and are of order one (see [42], with some complements in [39]). However, the poles are controlled by those of the corresponding Eisenstein series in the right-half plane, and in general it is not clear how to bound the number of the latter in terms of G only. (This is known in several cases and is expected to hold in general.) Therefore, even granted the reduction to the generic case, without additional input we cannot conclude from Shahidi's work by itself that r has property (FE+).

In addition, it is not clear how to approach property (CC') using the Langlands-Shahidi method (although it is certainly not excluded that this is possible).

In the cases at hand we will supplement the information from Shahidi's work by using integral representations of Rankin–Selberg type, which give better control of the poles of (at least) the partial L-functions as well as of the local γ-factors.

We now have the following implication.

Proposition 3.8. *Suppose that G satisfies property (L). Then it satisfies property (TWN+).*

This result essentially follows from Proposition 2.6. We first need a simple lemma to account for the ramified local factors.

Lemma 3.9. *Let P be a polynomial of degree d and let $f(s) = P(p^{-s})$. Then*

$$\int_T^{T+1} \left| \mathrm{Re}\, \frac{f'(\mathrm{i}t)}{f(\mathrm{i}t)} \right| \, dt \leq d(\pi + \log p)$$

for all $T \in \mathbb{R}$. Similarly let $f(s) = \frac{\Gamma_\mathbb{R}(s+\alpha)}{\Gamma_\mathbb{R}(s+1+\alpha)}$ for some $\alpha \in \mathbb{CC}$. Then

$$\int_T^{T+1} \left| \mathrm{Re}\, \frac{f'(\mathrm{i}t)}{f(\mathrm{i}t)} \right| \, dt \ll 1$$

for all $T \in \mathbb{R}$.

Proof. For the first part, it is enough to consider the case $P(z) = 1 - \alpha z$ with $\alpha \in \mathbb{C}$. (The left-hand side vanishes for $P(z) = z$.) Absorbing the argument of α into T, we may also assume without loss of generality that $\alpha \in \mathbb{R}$. Note that then

$$\log p \int \left| \operatorname{Re} \frac{1}{1 - \alpha p^{-it}} - \frac{1}{2} \right| dt = \frac{1}{2} \log p \int \frac{\left| 1 - \alpha^2 \right|}{1 + \alpha^2 - 2\alpha \cos(t \log p)} dt$$

$$= \arctan\left(\left| \frac{1 + \alpha}{1 - \alpha} \right| \tan\left(\frac{t \log p}{2} \right) \right) + C,$$

and hence, since the integrand is periodic with period $2\pi / \log p$,

$$\log p \int_T^{T+1} \left| \operatorname{Re} \frac{1}{1 - \alpha p^{-it}} - \frac{1}{2} \right| dt \leq \pi + \frac{1}{2} \log p.$$

Thus,

$$\int_T^{T+1} \left| \operatorname{Re} \frac{f'(it)}{f(it)} \right| dt = \log p \int_T^{T+1} \left| \operatorname{Re} \frac{\alpha p^{-it}}{1 - \alpha p^{-it}} \right| dt \leq \pi + \log p.$$

For the second part we can assume once again that $\alpha \in \mathbb{R}$. Note that

$$\frac{\Gamma'_{\mathbb{R}}(z)}{\Gamma_{\mathbb{R}}(z)} - \frac{\Gamma'_{\mathbb{R}}(z+1)}{\Gamma_{\mathbb{R}}(z+1)} = \sum_{k=0}^{\infty} \left(\frac{1}{z + 2k + 1} - \frac{1}{z + 2k} \right) = \sum_{k=0}^{\infty} \frac{1}{(z + 2k)(z + 2k + 1)}.$$

Thus,

$$\frac{\Gamma'_{\mathbb{R}}(\alpha + it)}{\Gamma_{\mathbb{R}}(\alpha + it)} - \frac{\Gamma'_{\mathbb{R}}(\alpha + it + 1)}{\Gamma_{\mathbb{R}}(\alpha + it + 1)} = A + B,$$

where upon writing $\alpha = n + \delta$ with $n \in \mathbb{Z}$ and $-\frac{1}{2} < \delta \leq \frac{1}{2}$, we have

$$A = \begin{cases} \frac{(-1)^{n+1}}{\delta + it} & \text{if } \alpha < \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$|B| \leq \frac{1}{|\delta + it + (-1)^n|} + \sum_{k \geq 0,\, k \neq \lfloor \frac{-n}{2} \rfloor} \frac{1}{|\alpha + it + 2k|\,|\alpha + it + 2k + 1|}$$

$$\leq \frac{1}{|\delta + (-1)^n|} + \sum_{k \neq \lfloor \frac{-n}{2} \rfloor} \frac{1}{|\alpha + 2k|\,|\alpha + 2k + 1|} \ll 1.$$

It remains to note that

$$\int_T^{T+1} \left| \operatorname{Re} \frac{1}{\delta + it} \right| dt = |\delta| \int_T^{T+1} \frac{dt}{\delta^2 + t^2} < \pi. \qquad \square$$

Proof of Proposition 3.8. Assume that G satisfies property (L), and let M, α, and $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ be given. We apply Lemma 2.13 with respect to $p^{\text{sc}} : \tilde{M}_\alpha^{\text{sc}} \to \tilde{M}_\alpha \lhd M$. Let $\sigma \in \Pi_{\text{disc}}(\tilde{M}_\alpha(\mathbb{A}))$ and χ be as in that lemma. Then by Remark 3.2 we have

$$(3.2) \qquad \prod_{v \notin S(\sigma)} n_{\alpha,v}(\pi, s) = \prod_{j=1}^{l} \frac{L^{S(\sigma)}(js, \sigma, r_j)}{L^{S(\sigma)}(js+1, \sigma, r_j)}.$$

As in §2.1 define

$$(3.3) \qquad m(\sigma, s) = \prod_{j=1}^{l} \frac{\epsilon^{\text{red}}(1, \sigma, r_j) L^{\text{red}}(js, \sigma, r_j)}{L^{\text{red}}(js+1, \sigma, r_j)}.$$

By Proposition 2.6 we have $|m(\sigma, \mathrm{i}t)| = 1$ for all $t \in \mathbb{R}$, and

$$\int_T^{T+1} |m'(\sigma, \mathrm{i}t)| \; dt \ll$$

$$\log(|T| + 2) + \sum_j [\log \mathfrak{n}(\sigma, r_j) + \log \mathfrak{c}_\infty(\sigma, r_j)] + \log \text{level}(\sigma).$$

By condition (CC) for $(\tilde{M}_\alpha, r_j)$ and Lemma 2.13, for all j we have here

$$\log \mathfrak{n}(\sigma, r_j) \ll \log \text{level}(\sigma) + 1 \ll \log \text{level}(\pi; p^{\text{sc}}) + 1,$$

while

$$\log \mathfrak{c}_\infty(\sigma, r_j) \ll \log \Lambda^{\tilde{M}_\alpha}(\sigma_\infty) + 1 = \log \Lambda(\pi_\infty; p^{\text{sc}}) + 1.$$

It remains to compare $m(\sigma, s)$ and $n_\alpha(\pi, s)$. Consider the quotient $\phi(s) = m(\sigma, s)/n_\alpha(\pi, s)$. We have $|\phi(\mathrm{i}t)| = 1$, and therefore $\phi'(\mathrm{i}t)/\phi(\mathrm{i}t) \in \mathbb{R}$ for $t \in \mathbb{R}$. On the other hand, from (3.2) and (3.3) we get that

$$\phi(s) = c \prod_{j=1}^{l} \prod_{p \in S_{\mathbb{Q}}(\sigma)} \frac{L_p^{\text{red}}(js, \sigma, r_j)}{L_p^{\text{red}}(js+1, \sigma, r_j)} \prod_{v \in S(\sigma)} n_{\alpha,v}(\pi, s)^{-1}$$

for some non-zero constant c. By the nature of the local factors $L_p^{\text{red}}(js, \sigma, r_j)$ and $n_{\alpha,v}(\pi, s)$, we may conclude from Lemma 3.9 and (2.13) that

$$\int_T^{T+1} |\phi'(\mathrm{i}t)| \; dt = \int_T^{T+1} \left| \text{Re} \, \frac{\phi'(\mathrm{i}t)}{\phi(\mathrm{i}t)} \right| \; dt \ll_G [F : \mathbb{Q}] + \sum_{v \in S_f(\sigma)} \log q_v$$

$$\ll \log \text{level}(\pi; p^{\text{sc}}) + 1.$$

This concludes the proof of the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.10. For Proposition 3.8 to hold, we may replace the condition (FE+) in the definition of property (L) by the weaker condition that (FE+) holds virtually (cf. Remark 2.7). It would be interesting to know whether one can further weaken the assumptions in Proposition 3.8.

In the next sections we will prove:

Theorem 3.11. *The following groups satisfy property (L), and hence also property (TWN+).*

(1) $\mathrm{GL}(n)$ *and its inner forms.*
(2) *Quasi-split classical groups.*
(3) *The exceptional group G_2.*

Thus, by Remark 3.5, the same holds for any group whose derived group coincides with the derived group of any one of the groups above.

The proof is based on a case-by-case analysis of the L-functions appearing in the definition of property (L). The quasi-splitness assumption in part 3 comes from the fact that we use functoriality to $\mathrm{GL}(n)$, which at the moment is only available in this case. (See Remark 5.7 below.) In each case we will use Corollary 2.11 for appropriately defined local factors. We will also often use Lemma 2.12 to reduce to the cuspidal case.

4. **Inner forms of** $\mathrm{GL}(n)$

We first consider the groups $\mathrm{GL}(n)$ and their inner forms. In order to prove Theorem 3.11 in this case, in view of Lemma 2.14 it suffices to show the following

Theorem 4.1. *Let $G = G_1 \times G_2$, where G_i is an inner form over F of $\mathrm{GL}(n_i)$, $i = 1, 2$, and $r = \mathrm{St}_{n_1} \otimes \mathrm{St}_{n_2}$ where St_n is the standard n-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$. Then the pair (G, r) satisfies properties (FE+) and (CC).*

This theorem will be proved in the rest of this section.

4.1.   We start with the case $n_2 = 1$. Let G be an inner form of $\mathrm{GL}(n)$ and let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(n, \mathbb{A}))$. For any Hecke character χ of F we have

$$L^S(s, \pi \times \chi, \mathrm{St}_n \times \mathrm{St}_1) = L^S(s, \pi \otimes \chi, \mathrm{St}),$$

where $\pi \otimes \chi$ denotes the twist of π by χ. Therefore, it suffices to show that (G, St_n) satisfies (FE+) and (CC). The standard L-function for π was studied by Godement–Jacquet [49]. In particular, St_n has property (FE+) and the local factors

$$\gamma^{\mathrm{GJ}}(s, \pi_v, \mathrm{St}_n) = \epsilon^{\mathrm{GJ}}(s, \pi_v, \mathrm{St}_n) \frac{L^{\mathrm{GJ}}(1 - s, \pi_v^\vee, \mathrm{St}_n)}{L^{\mathrm{GJ}}(s, \pi_v, \mathrm{St}_n)}$$

defined by Godement–Jacquet (with respect to the fixed character ψ_v as in Section 2.2 satisfy (FE'), (ΓF) and (CL) (and in particular (AF)). (Of course, in the case $n = 1$ these are Tate's local factors.) For brevity we write for v finite

$$\mathfrak{c}(\pi_v) := \mathfrak{c}_v^{\mathrm{GJ}}(\pi_v, \mathrm{St}_n)$$

(see (2.16)). This is the usual conductor of π_v.

Property (CC') for (G, St_n) follows from the following standard result.[1]

Lemma 4.2. *Let F be a non-archimedean field with residue field $\mathbb{F}_q$, D a central division algebra over F of degree d and $G = \mathrm{GL}(m, D)$. Let $\mathcal{O}_D$ be the maximal order of D and let ϖ_D be a prime element of D. Let π be an irreducible representation π of G and let*

$$l = \min\{i \geq 0 : \pi^{K_i} \neq 0\},$$

where $K_0 = \mathrm{GL}(m, \mathcal{O}_D)$ and $K_i = 1 + \varpi_D^i M_m(\mathcal{O}_D)$, $i > 0$. Then $\mathfrak{c}(\pi) \leq n + m(l-1)$ where $n = md$.

Proof. The result follows from the relation between conductor and depth proved recently in [9]. (See also [20, 18, 19, 82, 22] and the appendix of [6].) For convenience we provide an easy, self-contained argument.

If $l = 0$, i.e. if $\pi^{K_0} \neq 0$, then $\mathfrak{c}(\pi) = m(d-1) = n - m$ [49, Section I.6], and the lemma is clear. Assume therefore that $l \geq 1$. Let Nrd (resp., Trd) be the reduced norm (resp., trace) on $M_m(D)$. For $\Phi \in \mathcal{S}(M_m(D))$ and a matrix coefficient f of π let

$$Z(s, f, \Phi) = \int_G f(g)\Phi(g)\,|\mathrm{Nrd}(g)|^{s + \frac{n-1}{2}}\ dg.$$

Consider the functional equation

(4.1) $$\frac{Z(1 - s, f^\vee, \hat{\Phi})}{L^{\mathrm{GJ}}(1 - s, \pi^\vee)} = (-1)^{n-m} \epsilon^{\mathrm{GJ}}(s, \pi, \psi)\frac{Z(s, f, \Phi)}{L^{\mathrm{GJ}}(s, \pi)},$$

where $\hat{\Phi}$ is the Fourier transform of Φ given by

$$\hat{\Phi}(x) = \int_{M_m(D)} \Phi(y)\psi(\mathrm{Trd}(xy))\ dy$$

for a suitable Haar measure on $M_m(D)$, a non-trivial additive character ψ of F, and $f^\vee(g) = f(g^{-1})$ (a matrix coefficient of $\pi^\vee$). Take f to be a bi-K_l-invariant matrix coefficient of π such that $f(1) = 1$ and $\Phi = 1_{K_l}$. Then $Z(s, f, \Phi) \equiv \mathrm{vol}(K_l)$ so that $\gamma^{\mathrm{GJ}}(s, \pi)$ is a scalar multiple of $Z(1-s, f^\vee, \hat{\Phi})$. On the other hand, if ψ has conductor $\mathcal{O}_F$ then $\hat{\Phi}$ is supported in $\varpi_D^{1-d-l} M_m(\mathcal{O}_D)$ [113, Chapter X, Section 2, Proposition 5]. Thus, $Z(1 - s, f^\vee, \hat{\Phi})$ is a Laurent series $\sum a_i x^i$ in $x = q^{-s}$ such that $a_i = 0$ for all $i > m(d + l - 1)$. Hence, the left-hand side of (4.1) is a Laurent polynomial $\sum b_i x^i$ with $b_i = 0$ for

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[1]In the case where $D = F$ and π_v is generic, it follows from the well-known description of $\mathfrak{c}(\pi_v)$ due to Jacquet–Piatetski-Shapiro–Shalika [63], that $\mathfrak{c}(\pi_v) \geq \log_q \mathrm{level}_v(\pi_v)$. In [35, Lemma 5.6] it was incorrectly asserted that we always have $\mathfrak{c}(\pi_v) \leq \log_q \mathrm{level}_v(\pi_v)$ for $D = F$. Fortunately, replacing this erroneous statement by the correct upper bound $\mathfrak{c}(\pi_v) \leq n \log_q \mathrm{level}_v(\pi_v)$ of the current Lemma 4.2, the rest of the argument of [35] goes through without further change.

all $i > m(d + l - 1)$. Since $\epsilon^{\mathrm{GJ}}(s, \pi, \psi) = \epsilon^{\mathrm{GJ}}(0, \pi, \psi) x^{\mathfrak{c}(\pi)}$, we conclude that $\mathfrak{c}(\pi) \leq m(d + l - 1) = n + m(l - 1)$, as required. $\square$

We also remark that for $v \in S_\infty$ we have

$$(4.2) \qquad \Lambda(\pi_v) \ll \mathfrak{c}_\infty(\pi, r)^2 \ll \Lambda(\pi_v)^n.$$

4.2.   Next consider $G = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ with the tensor product representation $r = \mathrm{St}_{n_1} \otimes \mathrm{St}_{n_2}$. This L-function was studied independently by Rankin and Selberg for $n_1 = n_2 = 2$ ([98, 100]) and in the general case by Jacquet, Piatetski-Shapiro and Shalika ([57, 60, 61, 62, 64, 66, 67, 65, 70], see also [29, 30] and the references therein). In particular, they showed that for π, σ cuspidal the function

$$s(s - 1) L^{S(\pi) \cup S(\sigma)}(s, \pi \times \sigma, r)$$

is an entire function of order one. Alternatively, this L-function can be also studied using the Langlands–Shahidi method. The local factors arising from either method coincide. We denote them by

$$\gamma_v^{\mathrm{JPSS, Sh}}(s, \pi_v \times \sigma_v, r) = \epsilon_v^{\mathrm{JPSS, Sh}}(s, \pi_v \times \sigma_v, r) \frac{L^{\mathrm{JPSS, Sh}}(1 - s, \pi_v \times \sigma_v, r^\vee)}{L^{\mathrm{JPSS, Sh}}(s, \pi_v \times \sigma_v, r)},$$

where again implicitly ψ_v are as in Section 2.2. These γ-factors satisfy (FE'), (ΓF) and (CL). Thus r has property (FE+). In the non-archimedean case we set for brevity $\mathfrak{c}(\pi_v \times \sigma_v) = \mathfrak{c}_v^{\mathrm{JPSS, Sh}}(\pi_v \times \sigma_v, r)$ (see (2.16)). By a result of Bushnell–Henniart [21] we have

$$(4.3) \qquad \mathfrak{c}(\pi_v \times \sigma_v) \leq n_2 \mathfrak{c}(\pi_v) + n_1 \mathfrak{c}(\sigma_v).$$

(It is worthwhile to mention that this result does not depend on the Bushnell–Kutzko classification of supercuspidal representations [24], unlike the lower bound for $\mathfrak{c}(\pi_v \times \sigma_v)$ proved in [23].) By Lemma 4.2 we conclude that property (CC') holds for $\gamma_v^{\mathrm{JPSS, Sh}}(s, \pi_v \times \sigma_v, r)$.

4.3.   To finish the proof of Theorem 4.1 in the general case, we first recall the Jacquet–Langlands correspondence [32, 59], proved in this generality by Badulescu–Grbac [7], using results of Arthur–Clozel [4], with some complements in [8]. Let $G' = \mathrm{GL}(n)$ and let G be an inner form of G'. Then for any $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))$ there exists a unique $\pi' \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ such that $\pi'_v = \pi_v$ for all v where G' splits. The representation π' is called the Jacquet–Langlands transfer of π and will be denoted by $\mathrm{JL}(\pi)$. For any place v we have

$$(4.4) \qquad \gamma_v^{\mathrm{GJ}}(s, \pi_v, \psi_v) = \gamma_v^{\mathrm{GJ}}(s, \mathrm{JL}(\pi)_v, \psi_v).$$

Note that it is not true in general that $\mathfrak{c}(\pi_v) = \mathfrak{c}(\mathrm{JL}(\pi)_v)$. This is related to the fact that $\mathrm{JL}(\pi)_v$ does not depend only on π_v. (It is true however that π_v is determined by $\mathrm{JL}(\pi)_v$.) For instance, if π is the identity representation of G, then $\mathrm{JL}(\pi)$ is the identity representation of G, so that $\mathfrak{c}(\mathrm{JL}(\pi)_v) = 0$ for all finite v. On the other hand, if G does not split at v, then $\mathfrak{c}(\pi_v) > 0$. However,

as was pointed out after (CC'), this is immaterial for our purposes, since (4.4) implies that in any case

$$(4.5) \qquad |\mathfrak{c}(\pi_v) - \mathfrak{c}(\mathrm{JL}(\pi)_v)| \leq n.$$

Now let $G = G_1 \times G_2$ where G_i is an inner form of $G'_i = \mathrm{GL}(n_i)$, $i = 1, 2$.

Let $\pi \in \Pi_{\mathrm{disc}}(G_1)$, $\sigma \in \Pi_{\mathrm{disc}}(G_2)$ and let $\mathrm{JL}(\pi)$ and $\mathrm{JL}(\sigma)$ be their Jacquet–Langlands transfers to G'_1 and G'_2, respectively. We have

$$(4.6)$$
$$L^{S(\pi) \cup S(\sigma)}(\pi \times \sigma, \mathrm{St}_{n_1} \times \mathrm{St}_{n_2}) = L^{S(\pi') \cup S(\sigma')}(\mathrm{JL}(\pi) \times \mathrm{JL}(\sigma), \mathrm{St}_{n_1} \times \mathrm{St}_{n_2}).$$

Since we do not have at our disposal an independent theory of Rankin–Selberg convolutions for $G_1 \times G_2$, we will resort to the one on $G'_1 \times G'_2$. That is, we simply define

$$\gamma_v^{\mathrm{JPSS,Sh} \circ \mathrm{JL}}(s, \pi \times \sigma, \mathrm{St}_{n_1} \times \mathrm{St}_{n_2}) = \gamma_v^{\mathrm{JPSS,Sh}}(s, \mathrm{JL}(\pi)_v \times \mathrm{JL}(\sigma)_v, \mathrm{St}_{n_1} \times \mathrm{St}_{n_2}).$$

(In fact, these γ-factors depend only on π_v and σ_v, but we do not need to use this fact.) These factors satisfy properties (FE') and (ΓF) by (4.6) and the fact that $\gamma_v^{\mathrm{JPSS,Sh}}$ satisfy (FE') and (ΓF) for $G'_1 \times G'_2$. Property (AF) for $\gamma_v^{\mathrm{JPSS,Sh} \circ \mathrm{JL}}$ follows from the corresponding result for $\gamma_v^{\mathrm{JPSS,Sh}}$ and the fact that (under a suitable identification) $\mathrm{JL}(\pi)_v$ has the same infinitesimal character as π_v [8]. Combining (4.3), (4.5) and Lemma 4.2 (for both π_v and σ_v), we conclude property (CC') for the factors $\gamma_v^{\mathrm{JPSS,Sh} \circ \mathrm{JL}}$. Thus, we can apply Corollary 2.11 to conclude the proof of Theorem 4.1.

5. **Classical groups**

We now consider the case of quasi-split classical groups, i.e. the second part of Theorem 3.11.

By a classical group we mean either a symplectic group (which is automatically split), a special orthogonal group, or a unitary group. In the latter case we denote by E the quadratic extension of F pertaining to the hermitian form defining the unitary group (i.e., the quadratic extension over which the group splits). In all other cases let $E = F$. The L-group of a classical group is equipped with a natural embedding

$$\mathrm{Can}_G : {}^L G \to {}^L \mathrm{Res}_{E/F} \mathrm{GL}(m),$$

where, if r is the $\bar{F}$-rank of G, then

$$m = \begin{cases} 2r + 1, & \text{in the symplectic case,} \\ 2r, & \text{in the orthogonal case,} \\ r, & \text{in the unitary case.} \end{cases}$$

Here, in the unitary case we have

$${}^L \mathrm{Res}_{E/F} \mathrm{GL}(m) \simeq (\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})) \rtimes W_F,$$

where W_F acts on $\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})$ via $\mathrm{Gal}(E/F)$ by permuting the factors.

Let $p_n : {}^L \mathrm{Res}_{E/F} \mathrm{GL}(n) \to \mathrm{GL}([E : F]n, \mathbb{C})$ be the representation given by the projection ${}^L \mathrm{GL}(n) \to \mathrm{GL}(n, \mathbb{C})$ if $E = F$, and, in the case $E \neq F$, by $p_n((x,y), e) = \mathrm{diag}(x, y)$, $x, y \in \mathrm{GL}(n, \mathbb{C})$, and

$$p_n(I_n, I_n, \sigma) = \begin{cases} I_{2n}, & \text{if the image of } \sigma \text{ in } \mathrm{Gal}(E/F) \text{ is the identity,} \\ \left(\begin{smallmatrix} & I_n \\ I_n & \end{smallmatrix} \right), & \text{otherwise,} \end{cases}$$

using the description of ${}^L \mathrm{Res}_{E/F} \mathrm{GL}(n)$ above. We also consider the representation

$$T_{m,n} : {}^L (\mathrm{Res}_{E/F} \mathrm{GL}(m) \times \mathrm{Res}_{E/F} \mathrm{GL}(n)) \to \mathrm{GL}([E : F]mn, \mathbb{C})$$

obtained by the composition of p_{nm} with the "tensor product" homomorphism

$${}^L (\mathrm{Res}_{E/F} \mathrm{GL}(m) \times \mathrm{Res}_{E/F} \mathrm{GL}(n)) \to {}^L \mathrm{Res}_{E/F} \mathrm{GL}(mn).$$

In the case $E = F$, $T_{m,n} = \mathrm{St}_m \times \mathrm{St}_n$.

Theorem 5.1. *The following pairs (G, r) satisfy properties* (FE+) *and* (CC):

(1) $(E = F)$, $G = \mathrm{GL}(n) \times \mathrm{GL}(1)$, $r = \wedge^2 \otimes \mathrm{St}_1$, *where $\wedge^2$ is the exterior square representation.*

(2) $(E = F)$, $G = \mathrm{GL}(n) \times \mathrm{GL}(1)$, $r = \mathrm{Sym}^2 \otimes \mathrm{St}_1$, *where Sym^2 is the symmetric square representation.*

(3) $(E \neq F)$, $G = \mathrm{Res}_{E/F} \mathrm{GL}(n)$, *and $r = \mathrm{As}^{\pm}$ is either the Asai or the twisted Asai representation* (*cf.* [31]).

(4) $G = G' \times \mathrm{Res}_{E/F} \mathrm{GL}(n)$, *where G' is a quasi-split classical group and $r = T_{m,n} \circ (\mathrm{Can}_{G'} \otimes \mathrm{id}_n)$.*

This theorem implies Theorem 3.11 for quasi-split classical groups by the explicit description of the representations appearing in the definition of property (L) (e.g., [104, Appendix C]). Actually, for Theorem 3.11, in the first two parts of Theorem 5.1 it is enough to consider the exterior square and the symmetric square representations themselves (without the twist), but we include the slightly more general statement since it does not incur additional difficulty and the twisted representations are relevant for the GSpin groups (cf. [5]). (For the Asai L-function there is no need to consider $\mathrm{As}^{\pm} \times \mathrm{St}_1$, since we can incorporate the twist into the representation.)

5.1.   In this subsection we consider the first three cases of Theorem 5.1. By Lemma 2.12 and Theorem 4.1 it is enough to consider the cuspidal case. Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(n, \mathbb{A}_E))$. By Remark 3.7, the partial L-function has finitely many poles and of order one, and local factors $\gamma_v^{\mathrm{Sh}}(s, \pi \times \chi, r)$ are defined and satisfy (FE'), (ΓF) and (CL).

There are several methods to study the poles of L-functions. The results of [50, 51] address the analytic properties of the completed L-functions. They are

rather delicate and rely on Arthur's endoscopic classification (extended by Mok to unitary groups). For our purposes we only care about the partial L-functions and this can be analyzed using the Rankin–Selberg method in a rather crude form. We summarize it in the following.

Lemma 5.2. *Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(n, \mathbb{A}_E))$ and χ a Hecke character of E. Then the following functions are entire and of finite order. (In the first three cases $E = F$.)*

(1) $s(1 - s)L^S(s, \pi \times \chi, \mathrm{Sym}^2 \times \mathrm{St}_1)$.
(2) $s(1 - s)L^S(s, \pi \times \chi, \wedge^2 \times \mathrm{St}_1)$, *(n even)*.
(3) $L^S(s, \pi \times \chi, \wedge^2 \times \mathrm{St}_1)$ *(n odd)*.
(4) $s(1 - s)L^S(s, \pi \times \chi, \mathrm{As}^\pm \times \mathrm{St}_1)$.

Proof. The most difficult case is the twisted symmetric square, since it involves Eisenstein series on a double cover of $\mathrm{GL}(n)$ (introduced in [71]). This case was worked out recently by Takeda [106, 107] who extended earlier work by Bump–Ginzburg [17].[2] The other cases are easier since they involve the well-understood mirabolic Eisenstein series on $\mathrm{GL}(n)$. The argument is completely standard. For completeness we include it for the Asai L-function of a representation $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(n, \mathbb{A}_E))$. (The twisted Asai L-function is obtained by twisting π.)

For any Schwartz-Bruhat function $\Phi \in \mathcal{S}(\mathbb{A}^n)$ let $\mathcal{E}_\Phi$ be the normalized Eisenstein series on $\mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A})$ given by

$$\mathcal{E}_\Phi(g, s) = \int_{\mathbb{R}_{>0}} \sum_{\xi \in F^n \backslash \{0\}} \Phi(\xi tg) \left| \det tg \right|^s \, d^*t.$$

Here we embed $\mathbb{R} \hookrightarrow F \otimes \mathbb{R} \hookrightarrow \mathbb{A}$ by $x \mapsto 1 \otimes x$. As for the Riemann zeta function, using the Poisson summation formula we have

$$\mathcal{E}_\Phi(g, s) = \int_1^\infty \sum_{\xi \in F^n \backslash \{0\}} \Phi(\xi tg) \left| \det tg \right|^s \, d^*t - \frac{\Phi(0)}{s} +$$

$$\int_1^\infty \sum_{\xi \in F^n \backslash \{0\}} \hat{\Phi}(\xi t\iota(g)) \left| \det tg \right|^{1-s} \, d^*t + \frac{\hat{\Phi}(0)}{s - 1}$$

where $\iota(g) = {}^t g^{-1}$. In particular, the function $s(s - 1)\mathcal{E}_\Phi(g, s)$ is entire.

The Rankin–Selberg integral

(5.1) $$\int_{\mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A}_F)} \varphi(g)\mathcal{E}_\Phi(g, s) \, dg$$

for a cuspidal automorphic form φ on $\mathrm{GL}(n, E) \backslash \mathrm{GL}(n, \mathbb{A}_E)$ is a twisted version of the usual integral for $\mathrm{GL}(n) \times \mathrm{GL}(n)$. It was considered in [36]. Let N_n be

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[2]For our purposes we could have avoided this case by using Remark 2.7.

the group of upper unitriangular matrices in $\mathrm{GL}(n)$. Fix a non-degenerate character ψ_N of $N_n(\mathbb{A}_E)$ which is trivial on $N_n(\mathbb{A})$. Let

$$W(g) = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi(ng)\psi_N(n) \ dn$$

be the Fourier coefficient of φ. Assume that φ corresponds to a factorizable vector in $\pi = \otimes\pi_v$ (as an abstract representation). Then we can write $W(g) = \prod_v W_v(g_v)$ where W_v is in the Whittaker model of π_v (with respect to ψ_{N_v}), and $W_v(e) = 1$ for all $v \notin S$. For $\Phi = \otimes\Phi_v$ and $\mathrm{Re}\, s$ large, (5.1) unfolds to

$$L^S(s, \pi, \mathrm{As}^+) \prod_{v \in S} I_v(s, W_v, \Phi_v)$$

where

$$I_v(s, W_v, \Phi_v) = \int_{N_n(F_v)\backslash \mathrm{GL}(n, F_v)} W_v(g)\Phi_v(e_n g) \left|\det g\right|^s \ dg,$$

e_n is the row vector $(0, \ldots, 0, 1) \in F^n$. Since $s(1-s)\mathcal{E}_\Phi(g, s)$ is entire, it remains to show that for any $s \in \mathbb{CC}$ we can choose data W_v, Φ_v such that $I_v(s, W_v, \Phi_v) \neq 0$. We can write

(5.2) $$I_v(s, W_v, \Phi_v) = \int_{F_v^n} J_v(W_v, \xi, s)\Phi_v(\xi) \ d\xi,$$

where

$$J_v(W_v, e_n g, s) = \int_{N_n(F_v)\backslash P_n(F_v)} W_v(pg) \left|\det pg\right|^{s-1} \ dp$$

$$= \left|\det g\right|^{s-1} \int_{K_{n-1,v}} \int_{T_{n-1}(F_v)} W_v(tkg) \left|\det t\right|^{s-1} \delta_{B_{n-1}}(t)^{-1} \ dt.$$

Here P_n is the stabilizer of e_n, T_{n-1} is the group of diagonal matrices with 1 in the right bottom corner, B_{n-1} is the group of upper triangular matrices of $\mathrm{GL}(n-1)$ embedded in $\mathrm{GL}(n)$ via $b \mapsto \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $K_{n-1,v}$ is the standard maximal compact subgroup of $\mathrm{GL}(n-1, F_v)$. Using the asymptotics of the Whittaker function (see [26, 83] for the p-adic case, [112, Chapter 15] for the archimedean case), the function $J_v(W_v, \xi, s)$ admits a meromorphic continuation in s. More precisely, in the non-archimedean case there exists a polynomial $P_v(x)$ such that for any fixed W_v, $P_v(q_v^{-s})J_v(W_v, \xi, s)$ is a polynomial in q_v^{-s} which is locally constant in $\xi \in F_v^n \setminus \{0\}$. In the archimedean case, for any real number A there exists a polynomial $P_v(s)$ such that $P_v(s)J_v(W_v, \xi, s)$ is continuous in $(\xi, s) \in F_v^n \setminus \{0\} \times \{\mathrm{Re}\, s > A\}$ and holomorphic in s for $\mathrm{Re}\, s > A$. Hence, $P_v(s)J_v(W_v, \xi, s), \xi \in F_v^n \setminus \{0\}$ is a normal family of analytic functions on $\mathrm{Re}\, s > A$. Taking $\Phi_v \in \mathcal{S}(F_v^n \setminus \{0\})$ we infer that in both the archimedean and non-archimedean cases (5.2) holds for all s. Assume on the contrary that

for some fixed $s \in \mathbb{CC}$, $I_v(s, W_v, \Phi_v) = 0$ for all $\Phi_v \in \mathcal{S}(F_v^n \setminus \{0\})$ and W_v. Varying Φ_v we infer that $J_v(W_v, \xi, s) = 0$ for all ξ. In particular,

$$\int_{N_n(F_v) \backslash P_n(F_v)} W_v(p) \left| \det p \right|^{s-1} \, dp = 0$$

for all W_v. Since we can take W_v such that its restriction to $P_v(E_v)$ is an arbitrary smooth left $(N_v(E_v), \psi_v)$-invariant function which is compactly supported modulo $N_v(E_v)$ [43, 58, 72] we get a contradiction.

For the twisted exterior square case one can argue in a similar way using the Jacquet–Shalika integral representation [69]. (In the even case, the pole will come from the Eisenstein series as above. In the odd case there is no Eisenstein series and the zeta integral is entire.) The integral J_v in this case involves an extra unipotent integration, but its meromorphic continuation (with the extra uniformity property mentioned above) follows from the argument of [69]. (See [10] for more details.) Alternatively, one can use the Bump-Friedberg integral [15] (which does not involve an extra unipotent integration, and hence the argument above applies with little change) to infer that $s_2(1 - s_2)L^S(s_1, \pi, \mathrm{St}_n)L^S(s_2, \pi \times \chi, \wedge^2 \times \mathrm{St}_1)$ (and hence also $s(1-s)L^S(s, \pi \times \chi, \wedge^2 \times \mathrm{St}_1)$) is entire. Strictly speaking, only the case $\chi = 1$ is considered, in [15] but it is a simple matter to incorporate a non-trivial χ into the integral (and the unramified calculation is essentially the same).

We also remark that in the case where n is odd, it easily follows from the theory of Eisenstein series that $L^S(s, \pi \otimes \chi, \wedge^2 \otimes \mathrm{St}_1)$ is holomorphic (see [73] for a more precise result). □

We turn to property (CC'). By standard properties of the Shahidi local factors we have

$$\epsilon_v^{\mathrm{Sh}}(s, \pi_v \times \chi_v, \mathrm{Sym}^2 \times \mathrm{St}_1)\epsilon_v^{\mathrm{Sh}}(s, \pi_v \times \chi_v, \wedge^2 \times \mathrm{St}_1) = \epsilon_v^{\mathrm{JPSS,Sh}}(s, \pi_v \times (\pi_v \otimes \chi_v)).$$

In particular,

$$\mathfrak{e}_v^{\mathrm{Sh}}(\pi_v \times \chi_v, \mathrm{Sym}^2 \times \mathrm{St}_1) + \mathfrak{e}_v^{\mathrm{Sh}}(\pi_v \times \chi_v, \wedge^2 \times \mathrm{St}_1) = \mathfrak{e}_v^{\mathrm{JPSS,Sh}}(\pi_v \times \pi_v \otimes \chi_v),$$

and by (4.3) we conclude that

$$\mathfrak{e}_v^{\mathrm{Sh}}(\pi_v \times \chi_v, \mathrm{Sym}^2 \times \mathrm{St}_1) + \mathfrak{e}_v^{\mathrm{Sh}}(\pi_v \times \chi_v, \wedge^2 \times \mathrm{St}_1) \le$$
$$n(\mathfrak{c}(\pi_v) + \mathfrak{c}(\pi_v \otimes \chi_v)) \le 2n\mathfrak{c}(\pi_v) + n^2\mathfrak{c}(\chi_v).$$

Both $\mathfrak{e}_v^{\mathrm{Sh}}(\pi_v \times \chi_v, \mathrm{Sym}^2 \times \mathrm{St}_1)$ and $\mathfrak{e}^{\mathrm{Sh}}(\pi_v \times \chi_v, \wedge^2 \times \mathrm{St}_1)$ are non-negative by [53] (which relies on the validity of the local Langlands conjecture for $\mathrm{GL}(n)$). Therefore, we infer (CC') for $\mathrm{Sym}^2 \times \mathrm{St}_1$ and $\wedge^2 \times \mathrm{St}_1$ from the corresponding relation for St_n.

Similarly, if v is a place of F which is inert in E and w is the place of E above v then we may view π_v as a representation π_w of $\mathrm{GL}(n, E_w)$ and we have

$$\mathfrak{e}_v^{\mathrm{Sh}}(\pi_v, \mathrm{As}^+) + \mathfrak{e}_v^{\mathrm{Sh}}(\pi_v, \mathrm{As}^-) = \mathfrak{e}_w^{\mathrm{JPSS,Sh}}(\pi_w \times \pi_w^\tau, T_{n,n})$$

where τ is the Galois involution of E_w/F_v. Thus, by [53]

$$\mathfrak{e}_v^{\mathrm{Sh}}(\pi_v, \mathrm{As}^{\pm}) \le \mathfrak{e}_w^{\mathrm{JPSS,Sh}}(\pi_w \times \pi_w^{\tau}, T_{n,n}),$$

and therefore

$$\mathfrak{e}_v^{\mathrm{Sh}}(\pi_v, \mathrm{As}^{\pm}) \le n(\mathfrak{c}(\pi_w) + \mathfrak{c}(\pi_w^{\tau})) = 2n\mathfrak{c}(\pi_w).$$

In the split case, if w_1, w_2 are the places of E above v (with $E_{w_1}, E_{w_2} \simeq F_v$) and $\pi_v = \pi_{w_1} \otimes \pi_{w_2}$ then

$$\epsilon_v^{\mathrm{Sh}}(\pi_v, \mathrm{As}^{\pm}) = \epsilon_v^{\mathrm{Sh}}(\pi_{w_1} \times \pi_{w_2}, \mathrm{St}_n \times \mathrm{St}_n) \le n(\mathfrak{c}(\pi_{w_1}) + \mathfrak{c}(\pi_{w_2})).$$

In both cases the relation (CC') for $\mathrm{As}^{\pm}$ follows from the case of St_n.

5.2.   Next we consider the case where $G = G' \times \mathrm{Res}_{E/F} \mathrm{GL}(1)$ and $r = T_{m,1} \circ (\mathrm{Can}_{G'} \otimes \mathrm{id}_1)$, where G' is a classical group and m, $T_{m,1}$ and $\mathrm{Can}_{G'}$ are as in the beginning of the section. The L-functions in this case were studied by Piatetski-Shapiro and Rallis using the doubling method [40, 94, 95], see also [85] for some complements.[3] The analysis of poles was carried out in [40, 78] and completed by Yamana in [114]. In particular, r has property (FE+). Let $\gamma_v^{\mathrm{PSR}}(s, \pi \times \chi, r)$ be the local factors conceived by Piatetski-Shapiro–Rallis and explicated in [85]. They satisfy properties (FE'), (ΓF), and (CL).

We now turn our attention to condition (CC'). It is tempting to try to prove it using the definition of $\gamma_v^{\mathrm{PSR}}(s, \pi \times \chi, r)$, analogously to Lemma 4.2. However, at this stage we are unfortunately unable to carry this idea through. Instead, we will resort to a different approach using Arthur's work (adapted by Mok to the case of unitary groups) on functoriality for classical groups.

First, we recall the following stability result.

Proposition 5.3 ([14, 97]). *Let v be a non-archimedean place and π_1, π_2 irreducible representations of $G(F_v)$. In the unitary case assume that π_1 and π_2 have the same central character. Then there exists a constant $N = N(\pi_1, \pi_2)$ such that*

$$\gamma_v^{\mathrm{PSR}}(s, \pi_1 \times \chi, r) = \gamma_v^{\mathrm{PSR}}(s, \pi_2 \times \chi, r)$$

for any character χ of E_v^* such that $\mathfrak{c}(\chi) > N$.

We caution that unfortunately there is a mistake in [14] (in both the statement and the proof), but it is easy to fix the argument to obtain the correct statement above. We omit the details.

The argument of [97] explicates $N(\pi_1, \pi_2)$. Moreover, by taking π_2 to be a representation induced from a minimal Levi subgroup of G, in which case $\gamma_v^{\mathrm{PSR}}(s, \pi_2 \times \chi, r)$ is computed explicitly in [85, Theorem 4], we conclude

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[3]See [37, Section 7.2] for a small correction.

Corollary 5.4. *There exists a constant $C > 0$ such that for any non-archimedean place v, any irreducible representation π of $G(F_v)$ and any character χ of E_v^* such that $\mathfrak{c}(\chi) > C \operatorname{level}(\pi)$ we have*

$$\gamma_v^{\mathrm{PSR}}(s, \pi \times \chi, r) = \gamma_v^{\mathrm{Tate}}(s, \chi)^{m-1} \gamma_v^{\mathrm{Tate}}(s, \chi\chi_1).$$

Here χ_1 is trivial in the symplectic and odd orthogonal cases, while in the even orthogonal case χ_1 is the Hilbert symbol with $(-1)^{m/2}D$, where D is the discriminant of the quadratic form defining G (we denote this quadratic character by τ_D), and $\chi_1(x) = \omega_\pi(x/x^\tau)$ in the unitary case.

Now we turn to Arthur's work on classical groups. In order to state the result that we need, denote by $\Pi_{\mathrm{aut}}(\mathrm{GL}(n, \mathbb{A}))$ the set of irreducible representations of $\mathrm{GL}(n, \mathbb{A})$ which are weakly contained in $L^2(\mathrm{GL}(n, F) \backslash \mathrm{GL}(n, \mathbb{A}))$. These are the representations which are parabolically induced from some $\sigma \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ where M is a Levi subgroup of $\mathrm{GL}(n)$. As explained in [3, Section 1.3], by the Jacquet–Shalika classification theorem [66, 67] and the Mœglin–Waldspurger description of the discrete spectrum of $\mathrm{GL}(n)$ [89], for any finite set S of places of F, any $\pi \in \Pi_{\mathrm{aut}}(\mathrm{GL}(n, \mathbb{A}))$ is determined by the collection t_{π_v}, $v \notin S \cup S(\pi)$. Given $\pi \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ and $\sigma \in \Pi_{\mathrm{aut}}(\mathrm{GL}(m, \mathbb{A}_E))$, we say that σ is the transfer of π if for any $v \notin S(\pi)$, σ_v is unramified and $\mathrm{Can}_{G'}(t_{\pi_v}) = t_{\sigma_v}$. By the above, this condition determines σ and its central character is given by

$$(5.3) \qquad \omega_\sigma = \begin{cases} 1, & \text{in the symplectic or odd orthogonal case,} \\ \tau_D, & \text{in the even orthogonal case,} \\ \omega_\pi(x/x^\tau), & \text{in the unitary case.} \end{cases}$$

We will need the following consequence of Arthur's work (taking also [88] into account).[4]

Theorem 5.5 ([3], [91]). *Any $\pi \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ admits a transfer $\mathrm{Ar}(\pi) \in \Pi_{\mathrm{aut}}(\mathrm{GL}(m, \mathbb{A}_E))$. Consequently, for any $n \geq 1$ and any $\pi' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(n, \mathbb{A}_E))$ we have*

$$(5.4) \qquad L^{S(\sigma) \cup S(\pi')}(s, \pi \times \pi', r) = L^{S(\sigma) \cup S(\pi')}(s, \mathrm{Ar}(\pi) \times \pi', T_{m,n}).$$

Thanks to the work of Mœglin and others, a great deal is known about the local representations π_v in terms of $\mathrm{Ar}(\pi)$. For our purposes we will only need to know the preservation of γ-factors, namely that

$$(5.5) \qquad \gamma^{\mathrm{PSR}}(s, \pi_v \times \chi_v, r) = \gamma^{\mathrm{GJ}}(s, \mathrm{Ar}(\pi)_v \times \chi_v, T_{m,1})$$

for all v and characters χ_v of E_v^*. If π_v is unramified, this follows from the multiplicativity properties of the γ-factors on both sides ([49, 85]). If $\mathfrak{c}(\chi_v)$ is sufficiently large (with respect to π_v and $\mathrm{Ar}(\pi)_v$), then (5.5) follows from the stability properties of the γ-factors ([68] for the left-hand side and Corollary 5.4

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[4]We thank Colette Mœglin for helpful discussions on Arthur's local results in [3].

for the right-hand side) together with the description of the central character of $\mathrm{Ar}(\pi)$. The general case of (5.5) for finite v follows from the special cases above by the argument of [52, Theorem 4.1], namely, by using (5.4) (for $n = 1$) and comparing the functional equations

$$L^S(s, \pi \times \chi, r) = \big(\prod_{v \in S} \gamma_v^{\mathrm{PSR}}(s, \pi_v \times \chi_v, r) \big) L^S(1 - s, \pi \times \chi, r^\vee)$$

and

$$\begin{aligned} L^S(s, \mathrm{Ar}(\pi) \times \chi, T_{m,1}) \\ = \big(\prod_{v \in S} \gamma_v^{\mathrm{GJ}}(s, \mathrm{Ar}(\pi)_v \times \chi_v, T_{m,1}) \big) L^S(1 - s, \mathrm{Ar}(\pi) \times \chi, T_{m,1}^\vee) \end{aligned}$$

for suitable Hecke characters χ (with $S = S(\pi) \cup S(\chi)$). For archimedean v one can argue in a similar way using Lemma 2.15 (with T being the multiplicative group), replacing Corollary 5.4 by the fact that for any $s \in \mathbb{CC}$ we have

$$\lim_{|t| \to \infty} \gamma^{\mathrm{GJ}}(s, \mathrm{Ar}(\pi)_v \times |\cdot|^{\mathrm{it}}, T_{m,1}) / \gamma^{\mathrm{PSR}}(s, \pi_v \times |\cdot|^{\mathrm{it}}, T_{m,1}) = 1.$$

Indeed, this follows from Stirling's formula, the description of γ^{GJ} and γ^{PSR} in the archimedean case and (5.3).

Finally, we prove property (CC'). By (5.5) it is enough to show that

$$(5.6) \qquad \mathfrak{c}(\mathrm{Ar}(\pi)_v) \ll \log_q \mathrm{level}(\pi_v)$$

for every finite place v. We may assume that $\mathrm{level}(\pi_v) > 0$, i.e., that π_v is ramified, since otherwise $\mathrm{Ar}(\pi)_v$ is also unramified. We use Corollary 5.4 with any χ_v of conductor $C \, \mathrm{level}(\pi_v) + 1$. We obtain

$$\mathfrak{e}^{\mathrm{PSR}}(\pi_v \times \chi_v) \ll \log_q \mathrm{level}(\pi_v)$$

and hence by (5.5),

$$\mathfrak{c}(\mathrm{Ar}(\pi)_v \otimes \chi_v) \ll \log_q \mathrm{level}(\pi_v).$$

Since $\mathfrak{c}(\mathrm{Ar}(\pi)_v) \leq \mathfrak{c}(\mathrm{Ar}(\pi)_v \otimes \chi_v) + m\mathfrak{c}(\chi_v)$ we deduce that $\mathfrak{c}(\mathrm{Ar}(\pi)_v) \ll \log_q \mathrm{level}(\pi_v)$ as required.

We also note that by (4.2), and Remark 2.10 for $v \in S_\infty$ we have

$$(5.7) \qquad \Lambda(\mathrm{Ar}(\pi)_v) \ll \mathfrak{c}_v^{\mathrm{St}_m}(\mathrm{Ar}(\pi)_v)^2 = \mathfrak{c}_v^{\mathrm{PSR}}(\pi_v)^2 \ll \Lambda(\pi_v)^m.$$

5.3. Finally, consider $G = G' \times \mathrm{GL}(n)$ where G' is a quasi-split classical group and $r = T_{m,n} \circ (\mathrm{Can}_{G'} \otimes \mathrm{id}_n)$. We argue as in Section 4.3. Namely, for $\pi \in \Pi_{\mathrm{disc}}(G'(\mathbb{A}))$ and $\pi' \in \Pi_{\mathrm{disc}}(\mathrm{GL}(n, \mathbb{A}_E))$ we take

$$\gamma_v^{\mathrm{JPSS,Sh} \circ \mathrm{Ar}}(s, \pi \times \pi', r) := \gamma_v^{\mathrm{JPSS,Sh}}(s, \mathrm{Ar}(\pi)_v \times \pi'_v, T_{m,n}).$$

By using (5.6) and (5.7), properties (ΓF), (AF) and (CC') hold for $\gamma_v^{\mathrm{JPSS,Sh} \circ \mathrm{Ar}}$ since they hold for $\gamma_v^{\mathrm{JPSS,Sh}}$.

Remark 5.6. The same argument will give properties (FE+) and (CC) for $(G, r) = (G' \times G'', r = T_{m,n} \circ (\mathrm{Can}_{G'} \otimes \mathrm{id}_n))$, where G' is a quasi-split classical group and G'' is an inner form of $\mathrm{Res}_{E/F} \mathrm{GL}(n)$.

Remark 5.7. Once Arthur's work is extended to general classical groups and their inner forms, Theorem 3.11 will hold for them as well, with the same proof. (The stability argument of [97] should carry over to inner forms without too much trouble.) In fact, one can hope that the methods of Arthur carry over to the GSpin groups (as well as their inner forms). By the previous remark, once this is done, Theorem 3.11 and its proof will extend to this case as well.

Remark 5.8. One may contemplate whether Arthur's work (which invokes the full force of the stable twisted trace formula) is absolutely necessary for the question at hand. A possible different approach would be to use the Rankin–Selberg integrals that were studied in [46, 47]. In the case of generic representations, a great deal is known about these L-functions. In the general case, it seems that more input is necessary to address their finer analytic properties. We also mention the recent preprint [25], where a more uniform approach for these L-functions is laid out. It seems likely that some further local analysis of these integrals can shed light on the analytic issues at hand.

6. The exceptional group G_2

Finally, the last of Theorem 3.11 follows from Theorem 4.1 (for $n_1 = 2$, $n_2 = 1$) and the following

Theorem 6.1. *The pair*

$$(G = \mathrm{GL}(2), r = \mathrm{Sym}^3 \otimes (\wedge^2)^{-1} = \mathrm{Sym}^3 \otimes (\det)^{-1}),$$

where Sym^3 *is the (four-dimensional) symmetric cube representation of* $\mathrm{GL}(2)$, *satisfies properties* (FE+) *and* (CC):

Of course, it is enough to consider $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(2, \mathbb{A}))$. Once again by Remark 3.7 the local factors $\gamma_v^{\mathrm{Sh}}(s, \pi_v, r)$ are defined and satisfy (ΓF) and (CL). The poles of $L^{S(\pi)}(s, \pi, r)$ (and in fact, of the completed L-function) were analyzed by Kim-Shahidi in [75]. Alternatively, we can analyze the poles using the symmetric cube lift to $\mathrm{GL}(4)$, also due to Kim–Shahidi [76]. Even better, using Remark 3.10 and the fact that $r = \mathrm{St}_2^\vee \otimes \mathrm{Sym}^2 - \mathrm{St}_2$, we do not need any information about the poles of $L^{S(\pi)}(s, \pi, r)$ (but we need to know the existence of the symmetric square lift from $\mathrm{GL}(2)$ to $\mathrm{GL}(3)$ [38] to infer that $\mathrm{St}_2 \otimes \mathrm{Sym}^2$ satisfies (FE+)).

Finally, we prove (CC'). Combining [102, Proposition 2.2] and [103, Theorem 3.5] (applied to the exceptional group G_2), we have

$$\gamma^{\mathrm{Sh}}(s, \pi, r) = \gamma^{\mathrm{JPSS,Sh}}(s, \pi \times \mathrm{Sym}^2 \pi, \mathrm{St}_2^\vee \times \mathrm{St}_3)/\gamma^{\mathrm{GJ}}(s, \pi, \mathrm{St}_2^\vee),$$

where $\mathrm{Sym}^2\,\pi$ is the symmetric square lift of π to GL(3). Thus,

$$\mathfrak{e}^{\mathrm{Sh}}(\pi, r) \leq \mathfrak{c}(\pi^\vee \times \mathrm{Sym}^2\,\pi) \leq 3\mathfrak{c}(\pi) + 2\mathfrak{c}(\mathrm{Sym}^2\,\pi).$$

Also,

$$\gamma^{\mathrm{GJ}}(s, \mathrm{Sym}^2\,\pi) = \gamma^{\mathrm{JPSS,Sh}}(s, \pi \times \pi)/\gamma^{\mathrm{Tate}}(s, \omega_\pi),$$

and therefore

$$\mathfrak{c}(\mathrm{Sym}^2) \leq \mathfrak{c}(\pi \times \pi) \leq 4\mathfrak{c}(\pi).$$

We conclude that

$$\mathfrak{e}^{\mathrm{Sh}}(\pi, r) \leq 11\mathfrak{c}(\pi).$$

Remark 6.2. For higher rank exceptional groups, some of the L-functions on the list of [81] can be studied by Rankin–Selberg integrals, at least under a genericity assumption (e.g., [16, 44, 45, 48], to mention a few). One can also take into account Remark 3.10 to study further cases. However, more work has to be done in order to show that other exceptional groups satisfy properties (L) and (TWN+). We will not pursue this matter here any further.

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