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SOME BOUNDS ON UNITARY DUALS OF CLASSICAL GROUPS - NON-ARCHIMEDEN CASE

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Dedicated to Prof. Freydoon Shahidi for his 70^{th} *birthday*

Abstract. We first give bounds for domains where the unitarizabile subquotients can show up in the parabolically induced representations of classical groups over a p-adic field of characteristic 0. Roughly, they can show up only if the central character of the inducing irreducible cuspidal representation is dominated by the square root of the modular character of the minimal parabolic subgroup. For unitarizable subquotients supported by a fixed parabolic subgroup, or in a specific Bernstein component, a more precise bound is given.

For the reductive groups of rank at least two, the trivial representation is always isolated in the unitary dual (D. Kazhdan). Still, we may ask if the level of isolation is higher in the case of the automorphic duals, as it is a case in the rank one. We show that the answer is negative to this question for symplectic p-adic groups.
Keywords: Irreducible unitary representations, parabolic induction, unitary dual, automorphic dual.
MSC(2010): Primary: 22E50; Secondary: 11F70.

1. **Introduction**

Bounds on various parts of unitary duals of reductive groups over local fields can be very important, in particular in the number theory. Let ρ be an irreducible cuspidal representation of a Levi factor M of a parabolic subgroup P of a connected reductive group over a non-archimedean local field F of characteristic 0. In [28] we have proved that the set of all unramified characters χ of M such that $\mathrm{Ind}_P^G(\chi)$ contains an irreducible unitarizable subquotient is a compact subset of the set of all unramified characters of M[1] (observe that this fact does not hold for archimedean fields, already for $SL(2,\mathbb{R})$). In other words, this implies that the set of such characters where irreducible unitarizable

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[1] This result is very easy to prove using Bernstein center (see [30]). Further, one can drop the condition of cuspidality.

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subquotients can show up is bounded (for each fixed M and ρ). Two questions were left unanswered there. The first question is to find good bounds for the region where the unitarizability can show up in a fixed Bernstein component[2], i.e., how far from the unitary axis we can have unitary subquotients (for fixed ρ). The second question is if the set of all these regions where unitarizability can show up is bounded when one fixes M, and let ρ to run over all the irreducible cuspidal representations of M (i.e., if there exists a set of bounds which is bounded), and if it exists, to find as precise common bound as possible.

A result in the direction of the first question was obtained by L. Clozel and E. Ullmo in [7] for $G = Sp(2n, F)$, $M = A$ the maximal torus of G and $\rho = \mathbf{1}_A$, the trivial representation of A, and for the subset of automorphic (unitarizable) representations in this (unramified) Bernstein component. We shall briefly recall of their result. First we introduce some notation.

On $\mathbb{R}^q$ we shall consider two orderings. Let $x = (x_1, \ldots, x_q), y = (y_1, \ldots, y_q) \in \mathbb{R}^q$. Then we write

$$x \leq_w y \iff \textstyle\sum_{j=1}^{i} x_j \leq \sum_{j=1}^{i} y_j, \quad \forall i \in \{1, \ldots, q\},$$

$$x \leq_s y \iff x_i \leq y_i, \quad \forall i \in \{1, \ldots, q\}.$$

Denote by $\mathcal{O}_F$ the maximal compact subring of F, and by q_F the cardinality of its residual field. Fix an element ϖ_F which generates the maximal ideal in $\mathcal{O}_F$. Denote by $|\ |_F$ the normalized absolute value on F (it is determined by the condition $|\varpi_F^{-1}|_F = q_F$). For $i = 1, \ldots, q$, denote by $b_i = (1, \ldots, 1, \varpi_F^{-1}, 1, \ldots, 1)$, where ϖ_F^{-1} is placed at the i-th place.

We consider a maximal torus A in $Sp(2q, F)$ consisting of the diagonal matrices in the group (see the second section for more details regarding notation). Using an isomorphism $(x_1, \ldots, x_q) \mapsto (x_1, \ldots, x_q, x_q^{-1}, \ldots, x_1^{-1})$, we identify $(F^\times)^q$ with the maximal torus. Denote by P_{min} a minimal parabolic subgroup consisting of the diagonal matrices in $Sp(2q, F)$.

Fix an irreducible representation π of $Sp(2q, F)$. Then we can find a standard parabolic subgroup $P = MN$ of G such that the Levi factor M contains P_{min}, an irreducible unitarizable cuspidal representation ρ of M and a positive valued (unramified) character χ of M such that π is a subquotient of $\mathrm{Ind}_P^{Sp(2q,F)}(\chi\rho)$, and that holds

$$\log_{q_F}(|\chi(b_1)|_F) \geq \cdots \geq \log_{q_F}(|\chi(b_q)|_F) \geq 0.$$

Then the element $(\log_{q_F}(|\chi(b_1)|_F, \ldots, \log_{q_F}(|\chi(b_q)|_F)) \in (\mathbb{R}_{\geq 0})^q$ is uniquely determined by π, and it will be denoted by

$$||\pi||.$$

[2]Bernstein component of a non-unitary dual determined by ρ is the set of all equivalence classes of irreducible subquotients of representations $\mathrm{Ind}_P^G(\chi\rho)$ when χ runs over the set of all unramified characters of M.

Recall that for the trivial representation $\mathbf{1}_{Sp(2q,F)}$ we have

$$||\mathbf{1}_{Sp(2q,F)}|| = (q, q-1, \dots, 2, 1).$$

Now we shall recall of [7, Theorem 6.3] (we shall use logarithmic interpretation of it):

Theorem 1.1 (L. Clozel, E. Ullmo)**.** *Let k be a number field and v a place of k. Denote by F the completion of k at v. Let π be a non-trivial irreducible unramified representation of $Sp(2q,F)$ which shows up as a tensor factor of an irreducible representation of the adelic group $Sp(2q,\mathbb{A}_k)$ which is in the support of the representation $Sp(2q,\mathbb{A}_k)$ in the space of the square integrable automorphic forms $L^2(Sp(2q,k)\backslash Sp(2q,\mathbb{A}_k))$. Then*

$$||\pi|| \leq_w (q-1+\theta, q-2+\theta, \dots, 1+\theta, \theta),$$

where θ is the Ramanujan constant for $SL(2,k)$[3].

The proof of this theorem is based on a result proved by L. Clozel and E. Ullmo in [7], claiming that restriction preserves automorphicity. Later in [6], using [24], L. Clozel avoids use of the Ramanujan constant if $q \geq 2$, showing that for $q \geq 2$, θ can be taken to be 0 in the above estimate of $||\pi||$.

In particular, if we drop in the above theorem the assumption that π is non-trivial, we get obviously the estimate $(q, q-1, \dots, 2, 1)$ for (unitarizable) automorphic unramified irreducible representations, i.e.,

$$||\pi|| \leq_w (q, q-1, \dots, 2, 1)^{4}.$$

In other words, for automorphic representation in the unramified component the upper bound is $(q, q-1, \dots, 2, 1)$. In this paper we prove that for the above estimate we can avoid assumption of automorphicity, and moreover, that this estimate holds for any irreducible unitarizable representation (not only unramified). Actually, we prove a slightly stronger result then this, since the estimate is for $\leq_s$:

Theorem 1.2. *Let F be a local non-archimedean field of characteristic 0 and let π be an irreducible unitarizable representation of $G := Sp(2q,F)$. Then*

$$||\pi|| \leq_s ||\mathbf{1}_G||.$$

The equality holds if and only π is the trivial or the Steinberg representation.

The proof of the above theorem is a relatively easy consequence of [33] and a recent work of J. Arthur and C. Mœglin, which we use to get estimate for the cuspidal reducibility[5]. The above theorem holds also for other classical

[3] See [7] for more details.

[4] This estimate follows also easily from [13].

[5] It is possible that the above simple estimate is already known, but we do not know for it.

groups (with slight modification; see the third section). It would be interesting to know if such an estimate holds more generally (for $\leq_w$).

For particular M, we can often get much better estimates. They are in the following

Theorem 1.3. *Suppose* $\mathrm{char}(F) = 0$ *and let* P *be a standard parabolic subgroup*[6] *of* $Sp(2n, F)$ *whose Levi factor* M *is isomorphic to*

$$GL(p_1, F)^{n_1} \times \dots GL(p_k, F)^{n_k} \times Sp(2q, F),$$

where $p_i \neq p_j$ *for* $i \neq j$*. Let* $|\det|_F^{e_1} \rho_1 \otimes \dots \otimes |\det|_F^{e_k} \rho_l \otimes \sigma$ *be an irreducible cuspidal representation of* M*, where* ρ_i *are irreducible unitarizable cuspidal representations of general linear groups,* $e_i \in \mathbb{R}$ *and* σ *is an irreducible cuspidal representations of* $Sp(2q, F)$*. Suppose that the parabolically induced representation*

$$\mathrm{Ind}_P^{Sp(2n,F)}(|\det|_F^{e_1} \rho_1 \otimes \dots \otimes |\det|_F^{e_k} \rho_k \otimes \sigma)$$

contains an irreducible unitarizable subquotient.

Denote by $e'_1, e'_2, \dots, e'_{n_{i_0}}$ *the subsequence of the sequence* $e_1, e_2, \dots, e_l$ *consisting of all* e_i *such that* ρ_i *is a representation of* $GL(p_{i_0}, F)$*. After renumeration, we can assume* $|e'_1| \geq |e'_2| \geq \dots \geq |e'_{n_{i_0}}|$*. Then*

$$(|e'_1|, |e'_2|, \dots, |e'_{n_{i_0}}|) \leq_s (r, r-1, \dots, c+1, c)$$

for appropriate r*, where*

$$c = \max\left\{t \in 1/2\mathbb{Z}; t \leq \sqrt{\tfrac{2q+1}{p_{i_0}}}\right\}.$$

For other classical groups the upper bound c is very similar (see the third section).

Consider the example where $k = 1$, $p_1 = 2$, $n_1 = 5$ and $q = 6$ in the above theorem. The above theorem gives the bound $(\frac{13}{2}, \frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{5}{2})$. In other words,

$$||\pi|| \leq_s (\tfrac{13}{2}, \tfrac{13}{2}, \tfrac{11}{2}, \tfrac{11}{2}, \tfrac{9}{2}, \tfrac{9}{2}, \tfrac{7}{2}, \tfrac{7}{2}, \tfrac{5}{2}, \tfrac{5}{2}, 0, 0, 0, 0, 0, 0). \tag{1.1}$$

This is much sharper estimate then the one given by Theorem 1.2, which is $(16, 15, \dots, 2, 1)$.

If we want upper bound for specific Bernstein component, we need to have a data parameterizing the cuspidal representation σ. We shall use Jordan blocks (see the third section). Fix an irreducible square integrable representation σ of some $Sp(2q, F)$ and fix a self dual irreducible cuspidal representation ρ of a general linear group. For $k \in \mathbb{Z}_{\geq 1}$, the square integrable representation attached by Bernstein-Zelevinsky to the segment $[|\det|_F^{-(k-1)/2}\rho, |\det|_F^{(k-1)/2}\rho]$ will be denoted by $\delta(\rho, k)$ (see the third section). Then for one parity of $k \in \mathbb{Z}_{\geq 1}$,

[6] Parabolic subgroups that we shall consider in the paper will always contain all the upper triangular matrices in the group, and they will be called standard

the representation $\mathrm{Ind}(\delta(\rho,k)\otimes\sigma)$ is always irreducible[7]. For the other parity we have reducibility, with finitely many exceptions. Denote by $\mathrm{Jord}(\sigma)$ the set of all such representations which are exceptions, when ρ runs over all the self dual irreducible cuspidal representations of general linear groups[8]. For a self dual irreducible cuspidal representation ρ of a general linear group, denote by $\mathrm{Jord}_\rho(\sigma)$ the set of all k such that there exist $\delta(\rho,k)\in\mathrm{Jord}(\sigma)$.

The following theorem gives upper bounds for individual Bernstein components.

Theorem 1.4. *Let* $\mathrm{char}(F)=0$. *Fix an irreducible cuspidal representation* σ *of* $Sp(2q,F)$ *and let* $\rho_1,\dots,\rho_k$ *be irreducible unitarizable cuspidal representations of general linear groups* $GL(p_1,F),\dots,GL(p_k,F)$, *respectively, such that* $\rho_i\not\cong|\det|_F^\beta\rho_j$ *for any* $i\neq j$ *and any* $\alpha\in\mathbb{C}$. *Let* $n_1,n_2,\dots,n_k$ *be positive integers and let* $\beta_{i,j}, 1\le i\le k, 1\le j\le n_i$ *be a set of complex numbers such that the representation of appropriate symplectic group, parabolically induced by*

$$|\det|_F^{\beta_{1,1}}\rho_1\otimes\ldots\otimes|\det|_F^{\beta_{1,n_1}}\rho_1\otimes\ldots\ldots\otimes|\det|_F^{\beta_{k,1}}\rho_1\otimes\ldots\otimes|\det|_F^{\beta_{k,n_k}}\rho_1\otimes\sigma$$

contains an irreducible unitarizable subquotient. Fix some index i. *After renumeration we can assume that for real parts of complex exponents hold* $|\Re(\beta_{i,1})|\ge\cdots\ge|\Re(\beta_{i,n_i})|$. *Denote by* X_i *the set of all unramified characters* χ *of* $GL(p_i,F)$ *such that* $\chi\rho_i$ *is a self dual representation. Then* X_i *is a finite set. If* $X_i\neq\emptyset$, *denote*

$$c_i=\tfrac{1+\max\{\mathrm{card}(\mathrm{Jord}_{\chi\rho_i}(\sigma));\chi\in X_i\}}{2}.$$

Then

$$(|\Re(\beta_{i,1})|,\dots,|\Re(\beta_{i,n_i})|)\le_s(c_i+n_i-1,\dots,c_i+1,c_i)$$

if $X_i\neq\emptyset$, *and*

$$(|\Re(\beta_{i,1})|,\dots,|\Re(\beta_{i,n_i})|)\le_s(\tfrac{n_i}{2},\tfrac{n_i-1}{2},\dots,1,\tfrac12)$$

if $X_i=\emptyset$.

The second topic of this paper is also devoted to bounding some parts of the unitary dual, but in a different way. We shall explain this below. It is again related to the result of L. Clozel and E. Ullmo which we have mentioned above.

Let us recall that D. Kazhdan introduced in [10] property (T) for a locally compact group G. This property means that the trivial representation is isolated in the unitary dual $\hat{G}$ of G, i.e., in the set of all equivalence classes of the irreducible unitary representations of G, supplied with a natural topology.

[7]It is even if $\mathrm{Ind}(|\det|_F^{1/2}\rho\otimes\mathbf{1}_{Sp(0,F)})$ is irreducible. Otherwise it is odd.

[8]C. Mœglin has shown that using the local Langlands correspondence for general linear groups, $\mathrm{Jord}_\rho(\sigma)$ transfers to the admissible homomorphism of the Weyl-Deligne group attached by J. Arthur to σ in [1] (see [35] for a little bit more precise explanation, but still avoiding too much technical details).

He proved there that a simple group $G(F)$ over a locally compact non-discrete field F of split rank different from one has property (T), and he obtained some very interesting arithmetic consequences from that. Property (T) has shown to be related to a number of interesting facts. There is a vast literature on this, which we shall not discuss here. We shall mention only one result in that direction related to the exception of rank one in the Kazhdan result.

L. Clozel proved in [5] τ-conjecture, which is a weaker condition than property (T), but holds for any rank. This weaker property tells that the trivial representation is isolated in the automorphic dual of $G(F)$. More precisely, if F is a completion of a number field k at a place v, then the automorphic dual of $G(k)$ at v is the support of the representation $G(k_v)$ on the space $L^2(G(k)\backslash G(\mathbb{A}_k))$ of the square integrable automorphic forms.

Now that one knows that the trivial representation is isolated in the automorphic dual, further important question may be: how far is it from the other irreducible automorphic representations (in other words, what is the quantitative level of isolation of the trivial representation). Let us recall that such type of questions in rank one case are important in the number theory. One may ask such question also in the higher ranks. Recall that the trivial representation is not isolated in the unitary dual in rank one case, except in several archimedean completions. In the higher ranks, the trivial representation is isolated in both unitary and automorphic duals. Considering the rank one case, one may ask if the level of isolation is higher in the automorphic dual than in the unitary dual in higher ranks. Theorem 1.1 of L. Clozel and E. Ullmo for symplectic groups gives an estimate in the automorphic case (for $q \geq 2$, taking $\theta = 0$). Moreover, this estimate cannot be improved in this case (for $\theta = 0$), since G. Muić has proved in [22] that there exists an irreducible unitarizable automorphic representation for which we have equality in their theorem (this representation is the Aubert-Schneider-Stuhler involution of an irreducible square integrable representation which is the "closest" to the Steinberg representation[9]).

In this paper we prove the following result:

Theorem 1.5. *Let F be a local non-archimedean field of characteristic* 0 *and let π be a non-trivial irreducible unramified representation of $Sp(2q, F)$, $q \geq 2$. Then*

$$||\pi|| \leq_w (q-1, q-2, \ldots, 1, 0),$$

and

$$||\pi|| \leq_s (q-1, q-2, \ldots, 2, 1, \tfrac{1}{2}).$$

The first inequality of the theorem tells that the trivial representation is not more isolated in the automorphic dual than it is in the unitary dual (here we consider the ranks at least two).

[9] G. Savin has also told us of such an example.

The proof of the above theorem is an elementary application of the classification of the unramified unitary duals in [23] (this also reproves the result of L. Clozel and E. Ullmo, and adds a new estimate there).

We are thankful to G. Muić for discussions during writing of this paper, to L. Clozel for some comments and to the referee for corrections.

The contents of this paper is as follows. After introduction, in the second section we introduce notation that shall be used in the paper. The third section is devoted to the bounds where the unitarizability can show up. We recall of the unramified unitary dual of p-adic symplectic groups in the fourth section, while the fifth section brings bounds for the trivial representation from the rest of the unramified irreducible unitary representations.

2. **Notation**

We fix a local non-archimedean field F of characteristic 0. The normalized absolute value on F is denoted by $|\ \ |_F$ and the character $g \mapsto |\det|_F$, $GL(n,F) \to \mathbb{R}^\times$ is denoted by ν. For each irreducible essentially square integrable representation δ of $GL(n,F)$ there exist a unique $e(\delta) \in \mathbb{R}$ and a unique up to an equivalence (unitarizable) irreducible square integrable representation δ^u of $GL(n,F)$ such that

$$\delta \cong \nu^{e(\delta)}\delta^u.$$

For smooth representations π_i of $GL(n_i,F)$ by $i=1,2$, $\pi_1 \times \pi_2$ denotes the smooth representation of $GL(n_1+n_2,F)$ parabolically induced by

$$\pi_1 \otimes \pi_2$$

from appropriate maximal parabolic subgroup, which is standard with respect to the subgroup of the upper triangular matrices (parabolic induction that we consider is normalized).

Fix an irreducible square integrable representation τ of a general linear group and a positive integer k. Then the representation

$$\nu^{(k-1)/2}\tau \times \nu^{(k-1)/2-1}\tau \times \ldots \times \nu^{-(k-1)/2}\tau$$

has a unique irreducible quotient, which is denoted by

$$u(\tau,k).$$

For an irreducible square integrable representation τ of a general linear group there exists an irreducible unitarizable cuspidal representation ρ of a general linear group and a positive integer ℓ such that τ is isomorphic to the unique irreducible subrepresentation of

$$\nu^{(\ell-1)/2}\rho \times \nu^{(\ell-1)/2-1}\rho \times \ldots \times \nu^{-(\ell-1)/2}\rho.$$

Then we write

$$\tau = \delta([\nu^{-(\ell-1)/2}\rho, \nu^{(\ell-1)/2}\rho]),$$

or

$$\tau = \delta(\rho, \ell).$$

For an irreducible representation π of $GL(n, F)$ there exist irreducible cuspidal representations $\rho_1, \dots, \rho_k$ of general linear groups such that π is isomorphic to a subquotient of $\rho_1 \times \cdots \times \rho_k$. The multiset of equivalence classes $(\rho_1, \dots, \rho_k)$ is called the cuspidal support of π, and it is denoted by $\mathrm{supp}(\pi)$.

While in the case of general linear groups we follow mainly notation of [36], in the case of the classical p-adic groups, we follow the notation of [32]. The $n \times n$ matrix having 1's on the second diagonal and all other entries 0 is denoted by J_n. The identity matrix is denoted by I_n. For a $2n \times 2n$ matrix S, denote

$$^{\times}S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^{t}S \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix},$$

where ${}^{t}S$ is the transposed matrix of S. Then $Sp(n, F)$ is the group of all $2n \times 2n$ matrices over F which satisfy ${}^{\times}S\, S = I_{2n}$ (we take $Sp(0, F)$ to be the trivial group).

By $O(m, F)$ we denote the group of all $m \times m$ matrices X with entries in F which satisfy ${}^{\tau}X\, X = I_m$, where ${}^{\tau}X$ denotes the transposed matrix of X with respect to the second diagonal. Denote $SO(m, F) = O(m, F) \cap SL(m, F)$.

In the case of groups that we consider in this paper, we always fix the minimal parabolic subgroup consisting of all upper triangular matrices in the group (denoted by $P_{\min}$).

In the sequel, we denote by S_n either the group $Sp(n, F)$ or $SO(2n+1, F)$. Parabolic subgroups which contain the minimal parabolic subgroup which we have fixed will be called standard parabolic subgroups.

Most of the results of Section 3 hold also for other classical groups considered in [19], and also for unitary groups (also considered in [19]). When some statement of this paper is specific for symplectic or split odd-orthogonal groups, it will be specified in the statement.

In the case of unitary groups, a separable quadratic extension F' of F is fixed. We denote by θ the non-trivial element of the Galois group of F' over F. Whenever in the non-unitary case appears a representations of $GL(n, F)$, we need to replace it with a representations of $GL(n, F')$, and the contragredient representations $\tilde{\pi}$ of a representation π of $GL(n, F)$, with the representations $g \mapsto \pi(\theta(g))$ of $GL(n, F)$[10].

The Jacquet module of a representation π of S_n for the standard maximal parabolic subgroup whose Levi factor is a direct product of $GL(k, F)$ and a classical group S_{n-k}, is denoted by

$$s_{(k)}(\pi).$$

[10] We can handle the case of unitary groups uniformly, as we did in [19].

Each irreducible representation τ of some classical group S_ℓ is a subquotient of a representation of the form

$$\rho_1 \times \dots \cdots \times \rho_k \rtimes \sigma,$$

where $\rho_1, \dots, \rho_k$ are irreducible cuspidal representations of general linear groups and σ is an irreducible cuspidal representation of some S_m. The representation σ is called the partial cuspidal support of τ and it is denoted by

$$\tau_{cusp}.$$

If $\rho_1 \times \dots \cdots \times \rho_k$ is a representation of $GL(p, F)$, then the Jacquet module $s_{(p)}(\tau)$ will be denoted by

$$s_{GL}(\tau).$$

An irreducible cuspidal representation ρ of a general linear group is called a factor of τ if there exists an irreducible subquotient $\pi \otimes \tau_{cusp}$ of $s_{GL}(\tau)$ such that ρ is in the cuspidal support of π. Then the set of all factors of τ is contained in

$$\{\rho_1, \tilde{\rho}_1, \rho_2, \tilde{\rho}_2, \dots, \rho_k, \tilde{\rho}_k\}$$

(recall that our τ is a subquotient of $\rho_1 \times \dots \cdots \times \rho_k \rtimes \sigma$). Further, for each $1 \leq i \leq k$, at least one representation from $\{\rho_i, \tilde{\rho}_i\}$ is a factor of τ.

Let ρ and σ be unitarizable irreducible cuspidal representations of a general linear group and of S_n, respectively. Then if $\nu^\alpha \rho \rtimes \sigma$ reduces for some $\alpha \in \mathbb{R}$, then $\rho \cong \tilde{\rho}$. Further, if $\rho \cong \tilde{\rho}$, then we have always reduction for unique $\alpha \geq 0$ (see [27]). This reducibility point will be denoted by

$$\alpha_{\rho,\sigma}.$$

A very non-trivial fact which follows from the recent work of J. Arthur and C. Mœglin is that always $\alpha_{\rho,\sigma} \in (\frac{1}{2})\mathbb{Z}$.

3. Bounding unitarizability

The following proposition from [33] (Proposition 3.2 there[11]) will be used several times in this paper to get some bounds where the unitarizability can show up in the parabolically induced representations (it was also used in [33] for similar purpose).

Proposition 3.1. *Let π be an irreducible representation of a classical group S_q.*

(i) *Let X be a set of irreducible cuspidal representations of general linear groups which satisfies:*

(1) $\nu^{\pm 1}\rho \notin \tilde{X}$[12], *for any $\rho \in X$.*

(2) $X \cap \tilde{X} = \emptyset$.

(3) *There is no element in $X \cup \tilde{X}$ which is a factor of π.*

[11] We only corrected the last sentence, which is not stated correctly in [33].

[12] Here $\tilde{X} = \{\tilde{\rho}; \rho \in X\}$.

(4) *$\rho \rtimes \pi_{cusp}$ is irreducible for any $\rho \in X$.*
(5) *$\rho \times \rho'$ and $\tilde{\rho} \times \rho'$ are irreducible for any $\rho \in X$ and any factor ρ' of π.*

Suppose that θ is an irreducible representation of a general linear group whose cuspidal support is contained in X. Then

$$\theta \rtimes \pi$$

is irreducible.

(ii) *Suppose that we can find sets X and Y of (equivalence classes of) irreducible cuspidal representations of general linear groups such that $X \cup \tilde{X} \cup Y \cup \tilde{Y}$ contains all the factors of π, $X \cap (Y \cup \tilde{Y}) = \emptyset$, and that hold conditions* (1), (2) *and* (4) *from* (i). *Further suppose that $\rho \times \rho'$ and $\tilde{\rho} \times \rho'$ are irreducible for all $\rho \in X \cup \tilde{X}$ and $\rho' \in Y$ (i.e., that holds condition* (5) *in* (i) *for all $\rho \in X \cup \tilde{X}$ and ρ' in Y).*

Then there exists an irreducible representation θ of a general linear group whose cuspidal support is contained in X (i.e., each representation of the support), and there exists an irreducible representation π' of a classical group whose all factors are contained in $Y \cup \tilde{Y}$, such that

$$\pi \cong \theta \rtimes \pi'.$$

The partial cuspidal support of π' is π_{cusp}. Further, π determines θ and π' as above up to equivalence.

If X' is the set of all $\rho \in X$ which are factors of π, then each representation from X' shows up in the cuspidal support of θ.

We shall use it also in this paper to get some additional bounds.

Proposition 3.2. *Let π be an irreducible unitarizable representation of a classical group S_q. Let ρ be a factor of π. Suppose that $\rho_1, \dots, \rho_n$ are all the factors τ of π such that $\tau^u \cong \rho^u$.*

(1) *Let $\rho^u \not\cong \widetilde{\rho^u}$. Renumerate $\rho_1, \dots, \rho_n$, $n \geq 1$, in a way that $|e(\rho_1)| \leq |e(\rho_2)| \leq \dots \leq |e(\rho_n)|$. Then*

$$|e(\rho_i)| \leq \tfrac{i}{2}, \quad 1 \leq i \leq n.$$

(2) *Suppose $\rho^u \cong \widetilde{\rho^u}$. Write the set of all $|e(\rho_i)| > \alpha_{\rho^u,\pi_{cusp}}$, $1 \leq i \leq n$, as*

$$\{\alpha_1, \dots, \alpha_\ell\},$$

where $\ell \geq 0$ and $\alpha_1 < a_2 < \dots < \alpha_\ell$. Then

$$\alpha_i - \alpha_{i-1} \leq 1 \quad \text{for each} \quad i = 2, 3, \dots, \ell,$$

if $\ell \geq 2$. Further

(i) *If $\alpha_{\rho^u,\pi_{cusp}} = 0$, then*

$$\alpha_i \leq i - \tfrac{1}{2}; \quad i = 1, \dots, \ell.$$

(ii) *If* $\alpha_{\rho^u,\pi_{cusp}} \geq \frac{1}{2}$, *then there exists index* i *such that*

$$|e(\rho_i)| \leq \alpha_{\rho^u,\pi_{cusp}}.$$

Denote

$$\alpha^{(\pi)}_{\rho^u,\pi_{cusp}} = \max\{|e(\rho_i)|; |e(\rho_i)| \leq \alpha_{\rho^u,\pi_{cusp}} \ \& \ 1 \leq i \leq n\}.$$

Then the following conditions hold

(a) $\alpha_1 - \alpha^{(\pi)}_{\rho^u,\pi_{cusp}} \leq 1$ *if* $\ell \geq 1$,

(b) $\alpha_i \leq \alpha^{(\pi)}_{\rho^u,\pi_{cusp}} + i; \quad i = 1, \ldots, \ell.$

Proof. (1) Consider some ρ_k such that $\rho_k^u \not\cong \tilde{\rho_k^u}$. Then [33, Theorem 4.2(i)] and [29, Theorem 7.5] imply that π is a subquotient of a representation of the form

$$u(\delta([\nu^{-(\ell_1-1)/2}\rho_k^u, \nu^{(\ell_1-1)/2}\rho_k^u]), \ell_2) \times \rho_1' \times \cdots \times \rho_i' \rtimes \pi_{cusp}, \tag{3.1}$$

or of the form

(3.2)

$$\nu^{\alpha}u(\delta([\nu^{-(\ell_1-1)/2}\rho_k^u, \nu^{(\ell_1-1)/2}\rho_k^u]), \ell_2) \times \nu^{\alpha}u(\delta([\nu^{-(\ell_1-1)/2}\rho_k^u, \nu^{(\ell_1-1)/2}\rho_k^u]), \ell_2) \\ \times \rho_i'' \times \cdots \times \rho_j'' \rtimes \pi_{cusp},$$

where all the representations $\rho'_{i'}$ and $\rho'_{j'}$ are irreducible and cuspidal, $0 < \alpha < \frac{1}{2}$ and ρ_k is in the cuspidal support of the first factor of (3.1) or the first two factors of (3.2).

Denote $\ell_0 = \ell_1 + \ell_2$. There can be more possibilities for above representations (3.1) or (3.2). In that case, we chose above representation with maximal possible ℓ_0.

Suppose that ρ_k is coming from (3.1). First consider the case when ℓ_0 is even. Then the absolute values of the exponents of cuspidal representations that show up in the cuspidal support of the first factor of (3.1) are containing the following sequence

$$0, 1, 1, 2, 2, \ldots \tfrac{\ell_0}{2} - 2, \tfrac{\ell_0}{2} - 2, \tfrac{\ell_0}{2} - 1, \tfrac{\ell_0}{2} - 1.$$

(i.e., the exponents which are ≥ 1 show up at least two times each, and the exponent 0 shows up at least once). Observe that if we denote the above sequence by $\beta_1, \ldots, \beta_l$, then $\beta_i = \frac{i}{2}$ for even indexes. Clearly, $|\beta_1| = 0 \leq \frac{1}{2}$. For odd $i > 1$ we obviously have $|\beta_i| = |\beta_{i-1}| = \frac{i-1}{2} < \frac{i}{2}$.

Obviously $|e(\rho_k)| \leq \beta_k$ if $k \leq \ell_0$. Therefore, $|e(\rho_k)| \leq \frac{k}{2}$. If $k > \ell_0$, then the maximality of ℓ_0 implies $|e(\rho_k)| \leq |e(\rho_{\ell_0})| \leq \frac{\ell_0}{2} \leq \frac{k}{2}$. This completes the proof of the claim of (1) in this case.

Similarly goes the case when ℓ_0 is odd. Then absolute values of the exponents that show up in the case of the first factor of (3.1) now contain the sequence

$$\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \ldots \tfrac{\ell_0}{2} - 2, \tfrac{\ell_0}{2} - 2, \tfrac{\ell_0}{2} - 1, \tfrac{\ell_0}{2} - 1.$$

Again we denote the above sequence by $\beta_1,\dots,\beta_l$, and now holds $\beta_i=\frac{i}{2}$ if i is odd. For even i we have as above $|\beta_i|=|\beta_{i-1}|=\frac{i-1}{2}<\frac{i}{2}$. Now we finish the proof of (1) in this case in the same way as in the previous case.

Suppose now that ρ_k is in the cuspidal support of the first two factors in (3.2). Consider first the case of even ℓ_0. In this case the first two factors of (3.2) give exponents which contain the following sequence

$$\alpha, 1-\alpha, 1+\alpha, 2-\alpha, 2+\alpha, \dots, \tfrac{\ell_0}{2}-1-\alpha, \tfrac{\ell_0}{2}-1+\alpha.$$

Observe that each term of the above sequence is less then the corresponding term of the following sequence $\frac{1}{2},1,\frac{3}{2},2,\dots,\frac{\ell_0}{2}-1,\frac{\ell_0}{2}-\frac{1}{2}$. Now we end the proof in the same way as in previous cases.

At the end consider the case of odd ℓ_0. In this case the first two factors of (3.2) give a sequence of exponents that contains the following exponents

$$\tfrac{1}{2}-\alpha, \tfrac{1}{2}+\alpha, \tfrac{3}{2}-\alpha, \tfrac{3}{2}\alpha, 2+\alpha, \dots, \tfrac{\ell_0-1}{2}-\alpha, \tfrac{\ell_0-1}{2}+\alpha.$$

Again we use the sequence $\frac{1}{2},1,\frac{3}{2},2,\dots,\frac{\ell_0}{2}-1,\frac{\ell_0}{2}-\frac{1}{2}$ in the same way as above, and complete the proof.

(2) The first inequality in (2), $\alpha_i-\alpha_{i-1}\le 1$, is proved in [33, Proposition 4.6].

Now we shall prove the first inequality in (ii). Suppose that for all ρ_i holds $|e(\rho_i)|>\alpha_{\rho^u,\pi_{cusp}}$ (we here assume $\alpha_{\rho^u,\pi_{cusp}}\ge\frac{1}{2}$). Denote

$$\rho_i'=\begin{cases}\tilde\rho_i, & e(\rho_i)<0\\ \rho_i, & e(\rho_i)>0.\end{cases}$$

Then by (ii) of Proposition 3.1, we can write $\pi\cong\tau\rtimes\pi'$, where cuspidal support of τ is contained in $\{\rho_i',\dots,\rho_n'\}$, and no one of the representations of $\{\rho_i',\dots,\rho_n'\}$ or their contragredients is a factor of π'. Moreover, for any factor μ of π' we have $\mu^u\ncong\rho^u$. We denote the Hermitian contragredient $\bar{\tilde\pi}$ of π by π^+. Since π is unitarizable, it is a Hermitian representation, i.e., it holds $\pi\cong\pi^+$. Thus $\tau\rtimes\pi'\cong\tau^+\rtimes\pi'^+\cong\bar\tau\rtimes\pi'^+$.

Observe that a representation in the cuspidal support of τ is of the form $\nu^\alpha\rho^u$, with $\alpha>0$. Now the complex conjugate of $\nu^\alpha\rho^u$ is isomorphic to $\nu^\alpha\bar{\rho^u}\cong\nu^\alpha\tilde{\tilde{\rho^u}}\cong\nu^\alpha\rho^u$. From this follows that τ and $\bar\tau$ have the same cuspidal supports. Now the unicity claimed in (ii) of Proposition 3.1 implies $\tau\cong\bar\tau$ and $\pi'\cong\pi'^+$.

Further, by (i) of Proposition 3.1 $\nu^\alpha\tau\rtimes\pi'$ is irreducible for any $\alpha\ge 0$. Also $(\nu^\alpha\tau\rtimes\pi')^+\cong(\nu^\alpha\tau)^+\rtimes\pi'^+\cong(\nu^\alpha\tau)^-\rtimes\pi'\cong\nu^\alpha\bar\tau\rtimes\pi'\cong\nu^\alpha\tau\rtimes\pi'$. Thus, $\nu^\alpha\tau\rtimes\pi'$, $\alpha\ge0$ is a family of irreducible Hermitian representations. Since it is continuous family and for $\alpha=0$ we have unitarizability, the whole family consists of unitarizable representations (see the construction (b) from [31, Section 3]. This is impossible since this family is not bounded (see [30]). Thus, $|e(\rho_i)|\le\alpha_{\rho^u,\pi_{cusp}}$ for at least one index i.

The proof of (a) goes in a similar way (we suppose that (a) does not hold, and then we can construct complementary series which go to infinity, which is impossible). Now (b) follows from (a), and the first inequality in (2) ($\alpha_i - \alpha_{i-1} \leq 1$).

For (i), we first prove $\alpha_i \leq 1/2$ (for the proof, we suppose $\alpha_i > 1/2$, and form as above complementary series which go to infinity, which is impossible). After this, the first inequality of (2) implies the rest of (i). $\square$

From the last proposition we see that one of the key information for understanding where unitarizability can show up in the parabolically induced representations is contained in bounds that we can get on cuspidal reducibility points $\alpha_{\rho,\pi_{cusp}}$. Now we shall turn our attention to that reducibility points.

One could get a bound in the following way. If the reducibility point is strictly positive, then we have a complementary series. The end of complementary series is a representation of length two, and both irreducible subquotients are unitarizable. Therefore, they have bounded matrix coefficients. Now using Casselman's asymptotics of matrix coefficients, one would get an explicit bound for the reducibility point. This bound may not be very accurate. One can get much more accurate estimate using the recent work of J. Arthur and C. Mœglin. We shall use this approach. The references to their work, what we shall use, are now complete. A first general consequence of their work is that always

$$\alpha_{\rho,\pi_{cusp}} \in (1/2)\mathbb{Z}.$$

Now we shall recall of definition of Jordan blocks $\mathrm{Jord}(\sigma)$ of an irreducible square integrable representation σ of S_q. We give a slightly different (but equivalent) definition then C. Mœglin. In $\mathrm{Jord}(\sigma)$ are irreducible selfdual square integrable representations of general linear groups. Such a representation $\tau = \delta(\rho, k)$ belongs to $\mathrm{Jord}(\sigma)$ if and only if

$$\delta(\rho, k) \rtimes \sigma$$

is irreducible, and

$$\delta(\rho, l) \rtimes \sigma$$

is reducible for some l of the same parity as k (C. Mœglin considers instead of a representation $\tau = \delta(\rho, k)$, the pair (ρ, k) which parameterize the square integrable representation).

C. Mœglin has proved that for an irreducible square integrable representation σ of S_q the following relation holds

$$\sum_{\delta(\rho,k)\in \mathrm{Jord}(\sigma)} k n_\rho = q^*, \tag{3.3}$$

where n_ρ is determined by the condition that ρ is a representation of $GL(n_\rho, F)$, and further where q^* is the dimension of the vector space on which the dual group ${}^L(S_q)^0$ acts (for $Sp(2q, F)$, it is $q^* = 2q + 1$, and $2q$ in the case of $SO(2n + 1, F)$). For fixed ρ, k's such that $\delta(\rho, k) \in \mathrm{Jord}(\sigma)$ are always of the same parity.

Denote

$$\max_\rho(\sigma) = \max\{k; \delta(\rho,k) \in \mathrm{Jord}(\sigma)\}$$

in the case if the set on the right hand side is non-empty. Otherwise, $\max_\rho(\sigma)$ is not defined.

C. Mœglin has proved that if σ is cuspidal and $\delta(\rho,k) \in \mathrm{Jord}(\sigma)$ is such that $k \geq 3$, then $\delta(\rho,k-2) \in \mathrm{Jord}(\sigma)$. Taking this into account, the equality (3.3) for cuspidal representation σ of S_q becomes

(3.4)

$$\sum_{\rho\,;\,\max_\rho(\sigma)\in 2\mathbb{Z}} \tfrac{\max_\rho(\sigma)(\max_\rho(\sigma)+2)}{4} n_\rho + \sum_{\rho\,;\,\max_\rho(\sigma)\in 1+2\mathbb{Z}} \tfrac{(\max_\rho(\sigma)+1)^2}{4} n_\rho = q^*.$$

Now the basic assumption tells $\alpha_{\rho,\sigma} = (\max_\rho(\sigma)+1)/2$ if $\max_\rho(\sigma)$ is defined (see [19]). Then $\max_\rho(\sigma) = 2\alpha_{\rho,\sigma} - 1$, and this implies

$$\sum_{\rho\,;\,\alpha_{\rho,\sigma}\geq 1\,;\,\alpha_{\rho,\sigma}\notin\mathbb{Z}} (\alpha_{\rho,\sigma}^2 - \tfrac{1}{4}) n_\rho + \sum_{\rho\,;\,\alpha_{\rho,\sigma}\geq 1\,;\,\alpha\in\mathbb{Z}} \alpha_{\rho,\sigma}^2 n_\rho = q^*.$$

This directly implies the following

Lemma 3.3. *Let ρ be an irreducible cuspidal self dual representation of $GL(p,F)$ and let σ be an irreducible cuspidal representation of S_q such that $\alpha_{\rho,\sigma} \geq 1$. Then*

$$\alpha_{\rho,\sigma}^2 \leq \begin{cases} \frac{q^*}{p}, & \alpha_{\rho,\sigma} \in \mathbb{Z}; \\ \frac{q^*}{p} + \frac{1}{4}, & \alpha_{\rho,\sigma} \notin \mathbb{Z}. \end{cases}$$

Definition 3.4. Let π be an irreducible representation of S_n. Then π is a subquotient of a representation of the form

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma,$$

where ρ_i are irreducible cuspidal representations of general linear groups and σ is an irreducible cuspidal representation of some S_q such that $e(\rho_1) \geq e(\rho_2) \geq \cdots \geq e(\rho_k) \geq 0$. The n-tuple

$$(\underbrace{e(\rho_1),\dots,e(\rho_1)}_{n_{\rho_1}-\text{times}}, \underbrace{e(\rho_2),\dots,e(\rho_2)}_{n_{\rho_2}-\text{times}}, \dots, \underbrace{e(\rho_k),\dots,e(\rho_k)}_{n_{\rho_k}-\text{times}}, \underbrace{0,\dots,0}_{q-\text{times}})$$

is uniquely determined by π, and it is denoted by

$$||\pi||.$$

The trivial (one-dimensional) representation of a group G will be denoted by $\mathbf{1}_G$.

Below we shall restrict to the groups $Sp(2n,F)$ and $SO(2n+1,F)$ (but we expect that the following statements also hold for general classical groups, what should be relatively easy to check).

Lemma 3.5. *Let ρ be an irreducible selfual cuspidal representation of $GL(p,F)$ and let σ be an irreducible cuspidal representation of $Sp(2q,F)$ or $SO(2q+1,F)$.*

Consider the subgroup $X := \{(a,1,1,\dots,1,1,a^{-1}); a \in F^\times\}$ *of* S_{q+1}. *Denote* $e_q := e(\delta_{P_{min}}^{1/2}|X)$, *where* $\delta_{P_{min}}$ *denotes the modular character of* P_{min}[13]. *Then*

$$\alpha_{\rho,\sigma} \le e_q.$$

Proof. Consider first the case of symplectic groups. Then $e_q = q+1$. If $\alpha_{\rho,\sigma} \in \mathbb{Z}$, then by Lemma 3.3 holds $\alpha_{\rho,\sigma}^2 \le \frac{2q+1}{p} \le 2q+1 \le (q+1)^2$, which implies the inequality in the lemma. If $\alpha_{\rho,\sigma} \notin \mathbb{Z}$, then by Lemma 3.3 holds $(\alpha_{\rho,\sigma}^2 - \frac{1}{2})p \le 2q$ since $\alpha_{\rho,\sigma}^2 - \frac{1}{2}$ is an even number in that case. Thus $\alpha_{\rho,\sigma}^2 \le \frac{2q}{p} + \frac{1}{2} \le (q+1)^2$, which completes the proof in this case.

In the case of odd orthogonal groups we have $e_q = q + \frac{1}{2}$. Now Lemma 3.3 implies $\alpha_{\rho,\sigma}^2 \le \frac{2q}{p} + \frac{1}{4} \le 2q + \frac{1}{4} \le (q+\frac{1}{2})^2$, which again implies the inequality in the lemma. $\square$

Theorem 3.6. *Suppose* $\mathrm{char}(F) = 0$. *Let* π *be an irreducible unitarizable representation of* $G = Sp(2n,F)$ *or* $G = SO(2n+1,F)$. *Then*

$$||\pi|| \le_s ||\mathbf{1}_G||,$$

and the equality holds if and only if π *is a twist by a character of the trivial representation or a twist by a character of the Steinberg representation (in the case of symplectic groups, we do not have non-trivial twists).*

Proof. If π is cuspidal, the theorem obviously holds (in particular, the theorem holds for $n = 0$). It remains to consider the case of non-cuspidal representation π of some S_n, $n \ge 1$.

Let π be a (non-cuspidal) unitarizable irreducible subquotient of $\rho_1 \times \dots \times \rho_k \rtimes \sigma$ such that $e(\rho_1) \ge e(\rho_2) \ge \dots \ge e(\rho_k) \ge 0$. Fix some ρ_{i_0}, and let $\rho'_1, \dots, \rho'_{k'}$ be a subsequence of all ρ_i such that $\rho_i^u \cong \rho_{i_0}^u$ (we continue to assume $e(\rho'_1) \ge e(\rho'_2) \ge \dots \ge e(\rho'_{k'})$).

Suppose $\rho_{i_0} \not\cong \tilde\rho_{i_0}$. Then (1) of Proposition 3.2 obviously implies

$$(\underbrace{e(\rho'_1),\dots,e(\rho'_1)}_{n_{\rho'_1}-\text{times}}, \underbrace{e(\rho'_2),\dots,e(\rho'_2)}_{n_{\rho'_2}-\text{times}}, \dots, \underbrace{e(\rho'_{k'}),\dots,e(\rho'_{k'})}_{n_{\rho'_{k'}}-\text{times}}) \le_s (\tfrac{r}{2}, \tfrac{r-1}{2}, \dots, \tfrac{3}{2}, 1, \tfrac{1}{2})$$

for appropriate r.

Suppose now $\rho_{i_0} \cong \tilde\rho_{i_0}$. Since by the above lemma $\alpha_{\rho,\sigma} \le e_q$, now (2) of Proposition 3.2 obviously implies

$$(\underbrace{e(\rho'_1),\dots,e(\rho'_1)}_{n_{\rho'_1}-\text{times}}, \underbrace{e(\rho'_2),\dots,e(\rho'_2)}_{n_{\rho'_2}-\text{times}}, \dots, \underbrace{e(\rho'_{k'}),\dots,e(\rho'_{k'})}_{n_{\rho'_{k'}}-\text{times}}, \underbrace{0,\dots,0}_{q-\text{times}})$$
$$\le_s (r, r-1, \dots, \epsilon+2, \epsilon+1, \epsilon)$$

[13]Recall that $e_q = q+1$ in the case of the symplectic groups, and $e_q = q + \frac{1}{2}$ in the case of odd-orthogonal groups.

for appropriate r and $\epsilon = 1$ (resp. $\epsilon = \frac{1}{2}$) if $S_n = Sp(2n, F)$ (resp. $S_n = SO(2n+1, F)$).

The two above relations directly imply the inequality of the theorem.

Regarding equality, let us first consider the symplectic case. To get the equality, in all above inequalities we must have always equalities. This implies that we must have $q = 0$, $n_{\rho_i} = 1$ for all i, $k = k'$ and $\rho_k = \nu \mathbf{1}_{F^\times}$. This further implies that we must have $\rho_i = \nu^{k+1-i} 1_{F^\times}$ for all the other indexes. At the corresponding induced representation we have precisely two unitarizable subquotients by [3], the trivial and the Steinberg representation.

Similarly in the odd-orthogonal case, to get equalities at all the steps, we must have $q = 0$, $n_{\rho_i} = 1$ for all i, $k = k'$, $\rho_k = \nu^{1/2}\psi \mathbf{1}_{F^\times}$, with $\psi^2 \equiv 1$. This further implies $\rho_i = \nu^{k+1/2-i}\psi \mathbf{1}_{F^\times}$ for all the other indexes. At the corresponding induced representation we have precisely two unitarizable subquotients, the irreducible quotient and the irreducible subrepresentation (besides [3], see also [9] and [8]). □

It is evident from the proof of the above theorem that we can give much more accurate upper bound if we know by which parabolic subgroup is supported irreducible unitary representation. The following theorem is a result in that direction, which has the same proof as the previous theorem:

Theorem 3.7. *Assume* $\mathrm{char}(F) = 0$. *Let π be an irreducible unitarizable representation of S_n supported by a parabolic subgroup whose Levi factor is isomorphic to*

$$GL(p_1, F)^{n_1} \times \dots GL(p_k, F)^{n_k} \times S_q,$$

where $p_i \neq p_j$ for $i \neq j$. Let π be an irreducible unitarizable subquotient of $\rho_1 \times \dots \times \rho_l \rtimes \sigma$ (then $l = n_1 + \dots + n_k$). We can chose ρ_i's such that all $e(\rho_i) \geq 0$. Fix some index i_0, and let $\rho'_1, \dots, \rho'_{n_{i_0}}$ be a subsequence of all ρ_i which are representations of $GL(p_{i_0}, F)$. After a renumeration, we can assume $e(\rho'_1) \geq e(\rho'_2) \geq \dots \geq e(\rho'_{n_{i_0}}) (\geq 0)$. Then

$$(e(\rho'_1), e(\rho'_2), \dots, e(\rho'_{n_{i_0}})) \leq_s (r, r-1, \dots, c+1, c)$$

for appropriate r, where

$$c = \max\{t \in (1/2)\mathbb{Z}; t \leq \sqrt{\tfrac{q^*}{p_{i_0}} + \tfrac{1}{4}}\}.$$

Example 3.8. Consider an example of the symplectic group where $k = 1$, $p_1 = 2$, $n_1 = 5$ and $q = 6$. The above theorem gives the bound $(\frac{13}{2}, \frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{5}{2})$. In other words,

$$||\pi|| \leq_s (\tfrac{13}{2}, \tfrac{13}{2}, \tfrac{11}{2}, \tfrac{11}{2}, \tfrac{9}{2}, \tfrac{9}{2}, \tfrac{7}{2}, \tfrac{7}{2}, \tfrac{5}{2}, \tfrac{5}{2}, 0, 0, 0, 0, 0, 0).$$

This is much sharper estimate then given by Theorem 3.6, which gives the bound

$$||\pi|| \leq_s (16, 15, \dots, 2, 1).$$

At the end, the following theorem gives upper bounds for individual Bernstein components.

Theorem 3.9. *Let* $\mathrm{char}(F)=0$. *Fix an irreducible cuspidal representation* σ *of* S_q. *Let* ρ_i *be irreducible unitarizable cuspidal representations of* $GL(p_i,F)$, $i=1,\dots,k$, *such that* $\rho_i \not\cong \nu^\beta\rho_j$ *for any* $i\neq j$ *and any* $\beta\in\mathbb{C}$, *and let* $n_1,\dots,n_k$ *be positive integers.*

Let $\beta_{i,j}, 1\leq i\leq k, 1\leq j\leq n_i$ *be a set of complex numbers such that the representation*

$$\nu^{\beta_{1,1}}\rho_1\times\ldots\times\nu^{\beta_{1,n_1}}\rho_1\times\ldots$$
$$\ldots\times\nu^{\beta_{k,1}}\rho_1\times\ldots\times\nu^{\beta_{k,n_k}}\rho_1\rtimes\sigma$$

contains an irreducible unitarizable subquotient. Fix some index i. *After a renumeration we can assume that for real parts of complex exponents hold* $|\Re(\beta_{i,1})|\geq\cdots\geq|\Re(\beta_{i,n_i})|$. *Denote by* X_i *the set of all unramified characters* χ *of* $GL(p_i,F)$ *such that* $\chi\rho_i$ *is a self dual representation. Then* X_i *is a finite set. If* $X_i\neq\emptyset$, *set*

$$c_i=\frac{1+\max\{\mathrm{card}(\mathrm{Jord}_{\chi\rho_i}(\sigma));\chi\in X_i\}}{2},$$

and then

$$(|\Re(\beta_{i,1})|,\dots,|\Re(\beta_{i,n_i})|)\leq_s(c_i+n_i-1,c_i+n_i-2,\dots,c_i+1,c_i).$$

Further

$$(|\Re(\beta_{i,1})|,\dots,|\Re(\beta_{i,n_i})|)\leq_s(\tfrac{n_i}{2},\tfrac{n_i-1}{2},\dots,1,\tfrac{1}{2})$$

if $X_i=\emptyset$.

4. Unramified unitary dual of $Sp(2n,F)$

The ring of integers in F is denoted by $\mathcal{O}_F$. We fix an uniformizing element of $\mathcal{O}_F$ and denote it by ϖ_F. The normalized absolute value on F is denoted by $|\ \ |_F$. Then $|\varpi_F|_F=\mathrm{card}(\mathcal{O}_F/\varpi_F\mathcal{O}_F)^{-1}$.

Using the determinant homomorphism, we identify characters of $F^\times=GL(1,F)$ with characters of $GL(n,F)$. If φ is a character of $GL(n,F)$, then there exist a unique unitary character φ^u of $GL(n,F)$ and $e(\varphi)\in\mathbb{R}$ such that

$$\varphi=\nu^{e(\varphi)}\varphi^u.$$

The subgroup of all diagonal matrices in $Sp(2n,F)$ will be denoted by A. The mapping $\mathrm{diag}(a_1,\dots,a_n,a_n^{-1},\dots,a_1^{-1})\mapsto(a_1,\dots,a_n)$ is an isomorphism of A and $(F^\times)^n$, and using this isomorphism we identify these two groups. The subgroup of all upper triangular unipotent matrices in $Sp(2n,F)$ will be denoted by N.

In $GL(n,F)$ we fix the maximal compact subgroup $GL(n,\mathcal{O}_F)$ and in $Sp(2n,F)$ the maximal compact subgroup $K_{\max}=Sp(2n,F)\cap GL(2n,\mathcal{O}_F)$. An irreducible representation (π,V) of $GL(n,F)$ or $Sp(2n,F)$ is called unramified if V contains a non-trivial vector invariant for the action of the fixed maximal compact subgroup. Then the space of invariant vectors for the maximal compact subgroup is one dimensional.

For the group $G(F)$ of F-rational points of a reductive group G defined over F, we denote the set of equivalence classes of irreducible smooth representations by $\widetilde{G(F)}$. The subset of unitarizable classes in $\widetilde{G(F)}$ is denoted by $\widehat{G(F)}$. If a maximal compact subgroup in G is fixed, then we denote by $\widetilde{G(F)}^{\mathbf{1}}$ the set of all unramified classes in $\widetilde{G(F)}$. We denote by $\widehat{G(F)}^{\mathbf{1}}$ the unramified classes in $\widehat{G(F)}$, and call it the unramified unitary dual.

If we have a smooth representation π of $Sp(2n,F)$, we denote by

$$s_{(\underbrace{1,\dots,1,1}_{n\text{ times}})}(\pi)$$

the (normalized) Jacquet module of π with respect to $P_{min} = AN$. It is a representation of A, which we have identified with $(F^\times)^n$. If τ is an irreducible subquotient of $s_{(\underbrace{1,\dots,1,1}_{n\text{ times}})}(\pi)$, using the above identification, we can write τ as $\tau_1 \otimes \dots \otimes \tau_n$, where τ_i are characters of $F^\times$. Now we shall recall of some definitions from [21] in the case of $Sp(2n,F)$.

Definition 4.1. Let π be an irreducible unramified representation of $Sp(2n,F)$. Then π is called negative if for any irreducible subquotient $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_n$ of the Jacquet module $s_{(\underbrace{1,\dots,1,1}_{n\text{ times}})}(\pi)$ we have

$$\begin{aligned}
&e(\varphi_1) \le 0,\\
&e(\varphi_1) + e(\varphi_2) \le 0,\\
&\qquad\vdots\\
&e(\varphi_1) + e(\varphi_2) + \ \ldots\ + e(\varphi_n) \le 0.
\end{aligned}$$

Further, π will be called strongly negative if all the above inequalities are strict.

By

$$\mathrm{Jord}'_{\mathrm{sn}}(n)$$

we will denote the collection of all possible finite sets $J{:=}\{(\chi_1,m_1),\dots,(\chi_k,m_k)\}$[14] such that χ_i are self dual unramified characters of $F^\times$ and m_i are odd positive integers which satisfy

$$\textstyle\sum_{i=1}^k m_i = 2n+1.$$

[14]One possibility would be to write instead of pairs (χ_i, m_i) unramified self dual characters $\chi_i \circ \det_{m_i}$ of $GL(m_i,F)$

There are precisely two self dual unramified characters of $F^\times$, the trivial character $\mathbf{1}_{F^\times}$ and the non-trivial self dual unramified character, which we denote by $\mathbf{sgn}_{F^\times}$. For a self dual unramified character χ of $F^\times$ and $J = \{(\chi_1, m_1), \dots, (\chi_k, m_k)\} \in \mathrm{Jord}'_{\mathrm{sn}}(n)$ we denote

$$J(\chi) = \{m_i; \chi_i = \chi\}.$$

If we write $J(\chi)$ for $J \in \mathrm{Jord}'_{\mathrm{sn}}(n)$, then χ will be always assumed to be unramified selfdual character of $F^\times$.

We denote by

$$\mathrm{Jord}_{\mathrm{sn}}(n)$$

the set of all $J \in \mathrm{Jord}'_{\mathrm{sn}}(n)$ such that $J(\mathbf{sgn}_{F^\times})$ has even cardinality.

Denote

$$J(\chi)' = \begin{cases} J(\chi), & \text{if } \chi = \mathbf{sgn}_{F^\times}; \\ J(\chi) \cup \{-1\}, & \text{if } \chi = \mathbf{1}_{F^\times}. \end{cases}$$

To a character χ of $F^\times$ and $r_1, r_2 \in \mathbb{R}$ such that $r_2 - r_1 \in \mathbb{Z}$, we attach representation

$$\langle [\nu^{r_1}\chi, \nu^{r_2}\chi] \rangle := \nu^{(r_2+r_1)/2}\chi \ \mathbf{1}_{GL(r_2-r_1+1,F)}$$

if $r_2 \geq r_1$ (we use here Zelevinsky notation: $\langle [\nu^{r_1}\chi, \nu^{r_2}\chi] \rangle$ is characterized as a unique irreducible subrepresentation of $\nu^{r_1}\chi \times \nu^{r_1+1}\chi \times \cdots \times \nu^{r_2}\chi$). Otherwise, we take $\langle [\nu^{r_1}\chi, \nu^{r_2}\chi] \rangle$ to be the trivial representation of the trivial group $GL(0, F)$ (we consider formally this group as 0×0 - matrices).

For $J \in \mathrm{Jord}_{\mathrm{sn}}(n)$ write $J(\chi)' = \{a^{(\chi)}_{2l_\chi}, a^{(\chi)}_{2l_\chi - 1}, \dots, a^{(\chi)}_1\}$, where

$$a^{(\chi)}_{2l_\chi} > a^{(\chi)}_{2l_\chi - 1} > \cdots > a^{(\chi)}_1$$

(if $J(\chi) = \emptyset$ we take $l_\chi = 0$). We define

$$\sigma(J)$$

to be the unique irreducible unramified subquotient of

$$\left(\underset{\chi}{\times} \left(\overset{l_\chi}{\underset{i=1}{\times}} \langle [\nu^{-(a^{(\chi)}_{2i}-1)/2}\chi, \nu^{(a^{(\chi)}_{2i-1}-1)/2}\chi] \rangle \right) \right) \rtimes \mathbf{1}_{Sp(0,F)},$$

where the first product runs over (two) unramified selfdual characters of $F^\times$.

G. Muić in [21] has proved the following explicit classifications of the strongly negative and the negative irreducible unramified representations:

Theorem 4.2. (i) *The mapping $J \mapsto \sigma(J)$ is a bijection from $\mathrm{Jord}_{sn}(n)$ on the set of all equivalence classes of strongly negative irreducible unramified representations of $Sp(2n, F)$.*
(ii) *Suppose $J \in \mathrm{Jord}_{sn}(m)$ and suppose that $\psi_1, \dots, \psi_l$ are unramified unitary characters of $GL(n_1, F), \dots, GL(n_l, F)$, respectively, such that $n_1 + \cdots +$*

$n_l + m = n$. Let π be the unique unramified irreducible subquotient (actually subrepresentation) of

$$\psi_1 \times \cdots \times \psi_l \rtimes \sigma(J).$$

Then π is an irreducible negative unramified representation of $G_n(F)$. Moreover, π determines J uniquely, and it determines characters $\psi_1, \ldots, \psi_l$ up to a permutation and changes $\psi_i \leftrightarrow \psi_i^{-1}$. Further, each irreducible negative unramified representation of $G_n(F)$ is equivalent to some representation π as above.

Remark 4.3. Sometimes is more convenient the following description of $\mathrm{Jord}_{\mathrm{sn}(n)}$. Since there are exactly two selfdual unramified characters of $F^\times$, $\mathbf{1}_{F^\times}$ and $\mathbf{sgn}_{F^\times}$ (the non-trivial unramified character of order two), to $J \in \mathrm{Jord}_{\mathrm{sn}}(n)$ we attach the ordered pair

$$(J(\mathbf{1}_{F^\times}), J(\mathbf{sgn}_{F^\times})),$$

where we consider $J(\mathbf{1}_{F^\times})$ and $J(\mathbf{sgn}_{F^\times})$ as partitions. This pair determines J, and the partitions satisfy the following properties.

For a partition p of n into sum of k positive integers we shall write $\vdash (p) = n$ and $\mathrm{card}(p) = k$. We shall write always members of partitions in descending order.

In this way, $\mathrm{Jord}_{\mathrm{sn}}(n)$ (and irreducible unramified strongly negative representations of $Sp(2n, F)$) are parameterized by pairs

$$(t, s),$$

where both t and s are partitions into different odd numbers, which satisfy $\vdash (t) + \vdash (s) = 2n + 1$ and $\mathrm{card}(s) \in 2\mathbb{Z}$. The corresponding strongly negative representation will be denoted by $\sigma(t, s)$.

Now we shall recall of classification of the unramified unitary duals obtained in [23]. Let us note that the classification in [23] is for fields of characteristic 0, what is not pointed out in [23] (for the classification in [23] we need unitarizability proved in [22], which is proved only in characteristic 0).

Theorem 4.4. (i) *Let φ_i be unramified characters of $GL(n_i, F)$ such that $e(\varphi_i) > 0$ for $i = 1, \ldots, m$, and let σ_{neg} be an irreducible negative unramified representation of $Sp(2(n - n_1 - \cdots - n_m), F)$ (we assume $n_1 + \cdots + n_m \leq n$). Denote*

$$\pi = \varphi_1 \times \cdots \times \varphi_m \rtimes \sigma_{neg}.$$

For any φ showing up among $\varphi_1^u, \ldots, \varphi_m^u$, denote by $\mathbf{e}_\pi(\varphi)$ the multiset of exponents $e(\varphi_i)$ for those i such that $\varphi_i^u \cong \varphi$, and suppose that the following conditions hold:

(1) $\mathbf{e}_\pi(\tilde{\varphi}) = \mathbf{e}_\pi(\varphi)$.
(2) *If either $\varphi \not\cong \tilde{\varphi}$, or $\varphi = \tilde{\varphi}$ and $\nu^{\frac{1}{2}}\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ reduces, then $\alpha < \frac{1}{2}$ for all $\alpha \in \mathbf{e}_\pi(\varphi)$.*

(3) *If* $\tilde{\varphi} \cong \varphi$ *and* $\nu^{\frac{1}{2}}\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ *is* *irreducible, then all exponents in* $\mathbf{e}_\pi(\varphi)$ *are* < 1. *If we write* $\mathbf{e}_\pi(\varphi) = \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l\}$ *in a way that*

$$0 < \alpha_1 \leq \cdots \leq \alpha_k \leq \frac{1}{2} < \beta_1 \leq \cdots \leq \beta_l < 1,$$

then first $\beta_1 < \ldots < \beta_l$ *(we can have* $k = 0$ *or* $l = 0$*). Further,*

(a) $\alpha_i + \beta_j \neq 1$ *for all* $i = 1, \dots, k$, $j = 1, \dots, l$ *and* $\alpha_{k-1} \neq \frac{1}{2}$ *if* $k > 1$.
(b) $\mathrm{card}\big(\{1 \leq i \leq k : \alpha_i > 1 - \beta_1\}\big)$ *is even if* $l > 0$.
(c) $\mathrm{card}\big(\{1 \leq i \leq k : 1 - \beta_j > \alpha_i > 1 - \beta_{j+1}\}\big)$ *is odd for* $j = 1, \dots, l-1$.
(d) $k + l$ *is even if* $\varphi \rtimes \sigma_{neg}$ *reduces.*

Then π *is an irreducible unitarizable unramified representations of* $Sp(2n, F)$.
(ii) *If we have an irreducible unitarizable unramified representation* π *of* $Sp(2n, F)$, *then there exist* $\varphi_1, \varphi_2, \dots, \varphi_m, \sigma_{neg}$ *as in* (i)*, which satisfy all the conditions in* (i)*, such that*

$$\pi \cong \varphi_1 \times \cdots \times \varphi_m \rtimes \sigma_{neg}.$$

Further, σ_{neg} *and the multiset* $(\varphi_1, \dots, \varphi_k)$ *are uniquely determined by* π *up to equivalence.*

To have an explicit classification, one needs to understand when $\nu^{\frac{1}{2}}\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ and $\varphi \rtimes \sigma_{neg}$ from above theorem reduce. Since in the above theorem φ is selfdual, we can write $\varphi = \langle[\nu^{-(p-1)/2}\chi, \nu^{(p-1)/2}\chi]\rangle$ where $p \in \mathbb{Z}_{>0}$ and χ is a selfdual unramified character of $F^\times$. Now the reducibility is described by the following results of G. Muić in [21]:

Proposition 4.5. *Let*

$$\varphi = \langle[\nu^{-(p-1)/2}\chi, \nu^{(p-1)/2}\chi]\rangle,$$

where $p \in \mathbb{Z}_{>0}$ *and* χ *is a selfdual unramified character of* $F^\times$. *Suppose that* σ_{neg} *is an (unramified) irreducible subrepresentation of some*

$$\psi_1 \times \cdots \times \psi_s \rtimes \sigma(J),$$

where ψ_i *are unitary unramified characters of general linear groups and* $J \in \mathrm{Jord}_{sn}(q)$, $q \geq 0$. *Then*

(1) $\nu^{\frac{1}{2}}\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ *reduces if and only if* p *is even;*
(2) $\varphi \rtimes \sigma_{neg}$ *reduces if and only if* p *is odd,* $(\chi, p) \notin J$ *and* $\varphi \notin \{\psi_1, \dots, \psi_s\}$.

5. Bounding the trivial representation from the rest of the unitary dual of $Sp(2n, F)$

Denote

$$\mathbb{R}^q_\downarrow = \{x = (x_1, x_2, \dots, x_{q-1}, x_q) \in \mathbb{R}^q; x_1 \geq x_2 \geq \dots, x_{q-1} \geq x_q\}.$$

For $x \in \mathbb{R}^q$ we denote by

$$x_\downarrow$$

a unique $y \in \mathbb{R}^q_\downarrow$ such that the sequences $x_1, x_2, \dots, x_{q-1}, x_q$ and $y_1, y_2, \dots, y_{q-1}, y_q$ coincide up to a permutation. For $x \in \mathbb{R}^q$ we denote

$$|x| = (|x_1|, |x_2|, \dots, |x_{q-1}|, |x_q|).$$

We have defined two orderings on $\mathbb{R}^q$: $x \leq_w y$ if $\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i$ for all $j \in \{1, \dots, q\}$, and $x \leq_s y$ if $x_j \leq y_j$ for all $j \in \{1, \dots, q\}$. Then obviously the following simple properties hold

$$x \leq_w y \quad \& \quad x' \leq_w y' \quad \Longrightarrow \quad x + x' \leq_w y + y',$$

$$x \leq_s y \quad \& \quad x' \leq_s y' \quad \Longrightarrow \quad x + x' \leq_s y + y',$$

$$x \leq_s y \Longrightarrow x \leq_w y,$$

$$x \leq_w |x|, \qquad x \leq_s |x|,$$

$$x \leq_w x_\downarrow.$$

The last inequality holds since the sum of the first j coordinates of $x_\downarrow$ is greater then or equal to the sum of any j coordinates of $x_\downarrow$ (or x). Note that $x \leq_s x_\downarrow$ does not hold in general.

For $x \in \mathbb{R}^q$ and $y \in \mathbb{R}^p$ we denote

$$x{}^\frown y = (x_1, x_2, \dots, x_{q-1}, x_q, y_1, y_2, \dots, y_{p-1}, y_p) \in \mathbb{R}^{q+p}.$$

Lemma 5.1. (i) *Let $x \in \mathbb{R}^q_\downarrow$ and $y \in \mathbb{R}^q$. Then $x \geq_s y$ implies $x \geq_s y_\downarrow$.*
(ii) *For $x, x' \in \mathbb{R}^q_\downarrow$ and $y, y' \in \mathbb{R}^p_\downarrow$ holds*

$$x \;\geq_s\; x',\; y \;\geq_s\; y' \Longrightarrow (x{}^\frown y)_\downarrow \;\geq_s\; (x'{}^\frown y')_\downarrow.$$

Proof. (i) The assumption is that $x_1 \geq x_2 \geq \dots, x_{q-1} \geq x_q$ and $x_i \geq y_i$, $i = 1, \dots, q$. Suppose that $y_i < y_j$ for some $i < j$. Denote by $y' \in \mathbb{R}^q$ the element which one gets from y switching positions of y_i and y_j. Now $x_i \geq x_j \geq y_j$, and $x_j \geq y_j > y_i$. This implies $x \geq_s y'$. Further, if there are some $i < j$ such that $y'_i < y'_j$, one defines y'' in analogous way as was defined y' from y, and gets in the same way that $x \geq_s y''$. Repeating this procedure as long as it is possible (one can do it at most finitely many times), one will get $x \leq_s y^{(n)}$. Since $y_\downarrow = y^{(n)}$, the proof of is complete.

Denote $z = x{}^\frown y, z' = x'{}^\frown y'$. Obviously, $z \geq_s z'$. Let σ be a permutation of $\{1, \dots, q+p\}$ such that $z_\downarrow = (z_{\sigma(1)}, \dots, z_{\sigma(q+p)})$. Denote $z'' =$

$(z'_{\sigma(1)},\dots,z'_{\sigma(q+p)})$. Observe that $z \geq_s z'$ implies $z_\downarrow \geq_s z''$. Now (i) obviously implies $z_\downarrow \geq_s z''_\downarrow$. Since $z'_\downarrow = z''_\downarrow$, we get $z_\downarrow \geq_s z'_\downarrow$. This completes the proof of (ii). □

Let π be an unramified irreducible representation of $GL(q,F)$. Then π is a subquotient of some

$$\chi_1 \times \cdots \times \chi_q,$$

where χ_i are unramified characters of $F^\times$. The sequence $e(\chi_1),\dots,e(\chi_n)$ is determined by π up to a permutation. We denote

$$\mathbf{e}(\pi) := (e(\chi_1),\dots,e(\chi_q))_\downarrow.$$

Let $\tau_1,\dots,\tau_l$ be irreducible unramified representations of general linear groups, and let π be such a representation of a classical group. Then $\tau_1 \times \cdots \times \tau_l \rtimes \sigma$ contains a unique unramified irreducible subquotient. Denote it by π'. Then we define $||\tau_1 \times \cdots \times \tau_l \rtimes \sigma||$ to be $||\pi'||$. Observe that

$$||\tau_1 \times \cdots \times \tau_l \rtimes \sigma|| = (|\mathbf{e}(\tau_1)|\bar{\ }\dots\bar{\ }|\mathbf{e}(\tau_l)|\bar{\ }||\sigma||)_\downarrow.$$

For $u,v \in \mathbb{R}$ such that $v-u$ is a non-negative integer, we denote

$$[u,v]_\downarrow = (v, v-1,,\dots,u+1,u) \in \mathbb{R}^{v-u+1}.$$

Proposition 5.2. *Let π be a strongly negative unramified representation of $Sp(2q,F)$. Then*

$$||\pi|| \leq_s [1,q]_\downarrow,$$

where the equality holds if and only if π is the trivial representation of $Sp(2q,F)$. If π is a non-trivial strongly negative unramified representation, then

$$||\pi|| \leq_s [0,q-1]_\downarrow.$$

Proof. By [21], $\pi = \mathrm{Jord}(a,b)$, for some partitions a and b into different odd positive integers such that $\vdash(a) + \vdash(b) = 2q+1$ and that the number of integers in b is even. We shall prove the proposition by the induction with respect to the sum of numbers of integers entering partitions a and b (obviously, this number is always odd). We shall denote this number by m.

Consider first the case when that number is one. Then $(a,b) = ((2q+1),\emptyset)$ and π is the trivial representation $\mathbf{1}_{Sp(2q,F)}$. Obviously, $||\mathbf{1}_{Sp(2q,F)}|| = [1,q]_\downarrow$.

We go now to the inductive step. Suppose now $m \geq 3$. Then at least one of partitions a or b has at least 2 integers. We shall consider the case when a has at lest two integers (the other case goes completely analogously).

Write $a = (a_1,a_2)\bar{\ }a'$ in a way that $a_1 > a_2$. Then $a_1 + a_2 + \vdash(a') + \vdash(b) = 2q+1$. Define an integer q' by the requirement $\vdash(a') + \vdash(b) = 2q'+1$. Now π is a non-trivial representation, and we have

$$||\pi|| = ||\mathrm{Jord}(a,b)|| = ||\langle[\nu^{-\frac{(a_2-1)}{2}}\mathbf{1}_{F^\times}, \nu^{\frac{(a_1-1)}{2}}\mathbf{1}_{F^\times}]\rangle \rtimes \mathrm{Jord}(a',b))||$$

$$= ([1,\tfrac{(a_2-1)}{2}]_\downarrow\bar{\ }[0,\tfrac{(a_1-1)}{2}]_\downarrow\bar{\ }||\mathrm{Jord}(a',b)||)_\downarrow.$$

Now using the above lemma and the inductive assumption, we get

$$||\pi|| \leq ([1, \tfrac{(a_2-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [0, \tfrac{(a_1-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [1, q']_{\downarrow})_{\downarrow}.$$

$$= ([1, \tfrac{(a_2-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [1, q']_{\downarrow} \overline{\underline{\ \ }} [0, \tfrac{(a_1-1)}{2}]_{\downarrow})_{\downarrow}.$$

Again using the above lemma we get

$$||\pi|| \leq_s [0, q-1]_{\downarrow}.$$

This completes the proof of the proposition. □

Proposition 5.3. *Let π be a negative unramified representation of $Sp(2q, F)$ which is not strongly negative. Then at least one of the following two inequalities hold:*

$$||\pi|| \leq_s [0, q-1]_{\downarrow}$$

or

$$||\pi|| \leq_s [2, q-2]_{\downarrow} \overline{\underline{\ \ }} (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2})^{15}.$$

Proof. Observe that both right bounds are $\leq_s [1, q]_{\downarrow}$ (we shall use this evident fact below).

We shall prove the proposition by induction with respect to the rank q. By [21], we can write $\pi = \mathfrak{z}([-\frac{(c-1)}{2}, \frac{(c-1)}{2}]^{(\chi)}) \times \pi'$, where π' is a negative representation of $Sp(2q', F)$ (then $2c + q' = q$), c is a positive integer and χ is a unitary unramified character.

Consider first the case of odd c. Then

$$||\pi|| = ||\langle[\nu^{-\frac{(c-1)}{2}}\chi, \nu^{\frac{(c-1)}{2}}\chi]\rangle \rtimes \pi'||$$

$$= ([1, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [0, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} ||\pi'||)_{\downarrow}$$

Now using the above lemma and the inductive assumption, we get

$$||\pi|| \leq_s ([1, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [0, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [1, q']_{\downarrow})_{\downarrow}$$

$$([1, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [1, q']_{\downarrow} \overline{\underline{\ \ }} [0, \tfrac{(c-1)}{2}]_{\downarrow})_{\downarrow}.$$

Again using the above lemma we get

$$||\pi|| \leq_s [0, q-1]_{\downarrow}.$$

Now consider the case of even c. Then

$$||\pi|| = ||\langle[\nu^{-\frac{(c-1)}{2}}\chi, \nu^{\frac{(c-1)}{2}}\chi]\rangle \rtimes \pi')||$$

$$= ([\tfrac{1}{2}, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [\tfrac{1}{2}, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} ||\pi'||)_{\downarrow}.$$

Now using the above lemma we get

$$||\pi|| \leq_s ([\tfrac{1}{2}, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [\tfrac{1}{2}, \tfrac{(c-1)}{2}]_{\downarrow} \overline{\underline{\ \ }} [1, q']_{\downarrow})_{\downarrow}$$

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[15] For $q = 2$ we take the right hand side of this inequality to be $(\frac{1}{2}, \frac{1}{2})$.

$$([\tfrac{3}{2},\tfrac{(c-1)}{2}]_{\downarrow}\bar{\_}[\tfrac{3}{2},\tfrac{(c-1)}{2}]_{\downarrow}\bar{\_}[1,q']_{\downarrow}\bar{\_}(\tfrac{1}{2},\tfrac{1}{2}))_{\downarrow}.$$

Suppose $q' \geq 1$. Then Lemma 5.1 obviously imples

$$||\pi|| \leq_s [1,q-2]_{\downarrow}\bar{\_}(\tfrac{1}{2},\tfrac{1}{2}).$$

Observe

$$[1,q-2]_{\downarrow}\bar{\_}(\tfrac{1}{2},\tfrac{1}{2}) \leq_s [2,q-2]_{\downarrow}\bar{\_}(\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}).$$

Therefore the second inequality holds in this case.

Suppose $q' = 0$. Then $q = 2c$ and we have

$$||\pi|| = (\tfrac{(c-1)}{2},\tfrac{(c-1)}{2},\tfrac{(c-3)}{2},\tfrac{(c-3)}{2},\dots,\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}) \leq_s (q-2,q-3,\dots,3,2,\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}).$$

Therefore the second inequality holds also in this case.

This completes the proof of the proposition. $\square$

Proposition 5.4. *Let $q \geq 2$ and let π be an irreducible unitarizable unramified representation of $Sp(2q,F)$ which is not negative. Then the following inequality holds*

$$||\pi|| \leq_s [1,q-1]_{\downarrow}\bar{\_}(\tfrac{1}{2}).$$

Further, if we write $||\pi|| = (x_1,\dots,x_q)$, then

$$x_1+\cdots+x_n \leq q(q-1)/2.$$

Proof. Let c be a positive integer and χ a unitary character. Below we shall use the following inequalities which follow from Lemma 5.1:

$$0<\alpha\leq\tfrac{1}{2} \implies |\mathbf{e}(\nu^{\alpha}\langle[\nu^{-\frac{(c-1)}{2}}\chi,\nu^{\frac{(c-1)}{2}}\chi]\rangle)|_{\downarrow} \leq_s \begin{cases} \big([\frac{1}{2},\frac{c}{2}]_{\downarrow}\bar{\_}[1,\frac{c-1}{2}]_{\downarrow}\big)_{\downarrow}, & c \text{ is odd;}\\ \big([1,\frac{c}{2}]_{\downarrow}\bar{\_}[\frac{1}{2},\frac{c-1}{2}]_{\downarrow}\big)_{\downarrow}, & c \text{ is even;}\end{cases}$$

$$\tfrac{1}{2}<\alpha<1 \implies |\mathbf{e}(\nu^{\alpha}\langle[\nu^{-\frac{(c-1)}{2}}\chi,\nu^{\frac{(c-1)}{2}}\chi]\rangle)|_{\downarrow} \leq_s \begin{cases} \big([1,\frac{c+1}{2}]_{\downarrow}\bar{\_}[\frac{1}{2},\frac{c}{2}-1]_{\downarrow}\big)_{\downarrow}, & c \text{ is odd;}\\ \big([\frac{1}{2},\frac{c+1}{2}]_{\downarrow}\bar{\_}[1,\frac{c}{2}-1]_{\downarrow}\big)_{\downarrow}, & c \text{ is even.}\end{cases}$$

In the case $c=1$ we take $[1,0]_{\downarrow} = [\frac{1}{2},-\frac{1}{2}]_{\downarrow}$ to be the empty set.

Observe that the above estimates imply that for odd $c \geq 3$ and $0<\alpha<1$ the following relation holds

$$|\mathbf{e}(\nu^{\alpha}\langle[\nu^{-\frac{(c-1)}{2}}\chi,\nu^{\frac{(c-1)}{2}}\chi]\rangle)|_{\downarrow} \leq_s [\tfrac{1}{2},\tfrac{2c-1}{2}]_{\downarrow}. \tag{5.1}$$

We shall prove the proposition by induction with respect to the rank q. Since π is not negative, Theorem 4.4 implies that π is a member of a complementary series. We start with a complementary series of the form

$$\pi = \nu^{\alpha}\langle[\nu^{-\frac{(c-1)}{2}}\chi,\nu^{\frac{(c-1)}{2}}\chi]\rangle \times \nu^{\alpha}\langle[\nu^{-\frac{(c-1)}{2}}\bar{\chi},\nu^{\frac{(c-1)}{2}}\bar{\chi}]\rangle \rtimes \pi',$$

where π' is a unitarizable representation of $Sp(2q',F)$ (then $2c+q'=q$), c is a positive integer, χ is a unitary unramified character and $0<\alpha<1/2$. We shall break analysis of this complementary series into two possibilities. The first case is when c is odd. Then by Propositions 5.2, 5.3 and the inductive assumption we get

$$||\pi|| \leq_s ([\tfrac{1}{2},\tfrac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[\tfrac{1}{2},\tfrac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,\tfrac{(c-1)}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,\tfrac{(c-1)}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,q']_{\downarrow})_{\downarrow}$$

The above inequality implies

$$||\pi|| \leq_s [1,q-2]_{\downarrow}\,{}^{-}_{-}\,(\tfrac{1}{2},\tfrac{1}{2})$$

which is obvious if $c=1$, and for $c \geq 3$ follows from

$$([\tfrac{3}{2},\tfrac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[\tfrac{3}{2},\tfrac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,\tfrac{(c-1)}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,\tfrac{(c-1)}{2}]_{\downarrow})_{\downarrow} \leq_s [q'+1,q-2]_{\downarrow}.$$

The inequality $||\pi|| \leq_s [1,q-2]_{\downarrow}\,{}^{-}_{-}\,(\frac{1}{2},\frac{1}{2})$ also implies that the second inequality in the proposition holds.

Consider now the case of even c. Then analogously we have

$$||\pi|| \leq_s ([1,\tfrac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,\tfrac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[\tfrac{1}{2},\tfrac{(c-1)}{2}]_{\downarrow}\,{}^{-}_{-}\,[\tfrac{1}{2},\tfrac{(c-1)}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,q']_{\downarrow})_{\downarrow}$$

Now from the above inequality we get again

$$||\pi|| \leq_s [1,q-2]_{\downarrow}\,{}^{-}_{-}\,(\tfrac{1}{2},\tfrac{1}{2}).$$

The above inequality also implies that the second inequality in the proposition holds.

The second possibility for the complementary series is that

$$\pi = \nu^{\alpha}\langle[\nu^{-\frac{(c-1)}{2}}\chi,\nu^{\frac{(c-1)}{2}}\chi]\rangle \rtimes \pi',$$

where π' is an irreducible unitarizable unramified representation of $Sp(2q',F)$ (then $c+q'=q$), c is a positive integer, χ is a self dual character of $F^{\times}$, $0<\alpha<1/2$ and $\nu^{\frac{1}{2}}\langle[\nu^{-\frac{(c-1)}{2}}\chi,\nu^{\frac{(c-1)}{2}}\chi]\rangle \rtimes \mathbf{1}_{Sp(0,F)}$ reduces. The last reducibility condition and Proposition 4.5 imply that c is an even number. Now the inductive assumption implies $||\pi|| \leq_s \big([1,\frac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[\frac{1}{2},\frac{c-1}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,q']_{\downarrow})\big)_{\downarrow}$. This further implies that $||\pi|| \leq_s [1,q-2]_{\downarrow}\,{}^{-}_{-}\,(1,\frac{1}{2})$, which implies the first inequality in the proposition. Further, $||\pi|| \leq_s \big([1,\frac{c}{2}]_{\downarrow}\,{}^{-}_{-}\,[\frac{1}{2},\frac{c-1}{2}]_{\downarrow}\,{}^{-}_{-}\,[1,q']_{\downarrow})\big)_{\downarrow}$. implies the second inequality if $c>2$ or if $c=2$ and $q>2$. For $c=2$ and $q=2$ one directly sees that $||\nu^{\alpha}\langle[\nu^{-\frac{1}{2}}\chi,\nu^{\frac{1}{2}}\chi]\rangle \rtimes \mathbf{1}_{Sp(0,F)}|| = (\frac{1}{2}+\alpha,\frac{1}{2}-a)$. Therefore, the second equity holds also in this case.

If the complementary series cannot be written in any of the above two forms, then Theorem 4.4 implies that it must be a complementary series of the form

$$\pi = \nu^{x_1}\varphi \times \dots \nu^{x_k}\varphi \rtimes \pi',$$

where π' is an irreducible unitarizable unramified representation of $Sp(2q',F)$, φ is a self dual character of some $GL(c,F)$ such that $\nu^{1/2}\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ is irreducible, and $0<x_1,\dots,x_k<1$ satisfy conditions of (3) from Theorem 4.4 (we shall use these conditions later). Observe that $q'+kc=q$ and that c must be odd by Proposition 4.5 since $\nu^{1/2}\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ is irreducible. We shall consider four cases.

First consider the case $c = 1$ and $k = 1$. Now condition (d) of Theorem 4.4 implies that $\varphi \rtimes \pi_{neg}$ is irreducible. If $\pi' \cong \mathbf{1}_{Sp(2(q-1),F)}$, then $\pi' = \pi_{neg}$ and $\chi \times \pi_{neg}$ would be reducible since $q \geq 2$ (see Proposition 4.5). Thus, $\pi' \ncong \mathbf{1}_{Sp(2(q-1),F)}$. Suppose $q \geq 3$. Now inductive assumption implies that $[1, q-2]_{\downarrow} {}^{-}_{-} (1, \frac{1}{2})$ is an upper bound. This implies the first inequality of the proposition. This implies also the second inequality in this case.

Suppose $q = 2$. If π_{neg} is a representation of $Sp(2, F)$, then $\pi_{neg} \ncong \mathbf{1}_{Sp(2(q-1),F)} = \mathbf{1}_{Sp(2,F)}$ as we have observed above. This implies that π_{neg} is a unitary principal series representation, and then obviously both inequalities of the proposition hold. Suppose now that π_{neg} is a representation of $Sp(0, F)$. Then π' is a complementary series of $Sp(2, F)$, i.e., $\pi' \cong \nu^{\alpha}\mathbf{1}_{F^\times} \rtimes \mathbf{1}_{Sp(0,F)}$ with $0 < \alpha < 1$. This implies $\varphi = sgn_{F^\times}$ since $k = 1$. But then $\chi \rtimes \pi_{neg}$ is reducible, which is not the case. Thus, this case cannot happen. This completes the proof of the case $c = 1$ and $k = 1$.

Consider now the case $c = 1$ and $k \geq 2$. Observe that by (c) of the classification Theorem 4.4, at least one x_i is $\leq \frac{1}{2}$. Thus, $[1, q']_{\downarrow} {}^{-}_{-} (1, \ldots, 1, \frac{1}{2})$ is an upper bound, which obviously implies the first inequality of the proposition. For the second one, observe that the above inequality implies that an upper bound for the left hand side of the second equality is $(q-1)(q-2)/2 + 3/2$. If $q > 2$, then this is obviously $\leq q(q-1)/2$, and therefore the second equity also holds. Suppose that $q = 2$. Then (b) of Theorem 4.4 implies that $x_1 + x_2 \leq 1$. This implies the second inequality in this case.

It remains to prove the proposition when $c \geq 3$ (recall, c must be odd for the complementary series that we consider). Consider first the case $k = 1$. Then using the inductive assumption and (5.1) we get a following upper bound

$$\left([1, q']_{\downarrow} {}^{-}_{-} [\tfrac{1}{2}, \tfrac{2c-1}{2}]_{\downarrow}\right)_{\downarrow}.$$

Recall $q' + c = q$. This obviously implies the first inequality. This inequality implies the second inequality if $q' \geq 1$ (since $c \geq 2$). It remains to consider the case $q' = 0$. But then $\varphi \rtimes \mathbf{1}_{Sp(0,F)}$ is reducible since c is odd (see Proposition 4.5), and we cannot have complementary series (by (d) of Theorem 4.4). This completes the proof for this case.

We are left with the last case $c \geq 2$ and $k \geq 2$. Then we must have at least one exponent between 0 and $\frac{1}{2}$ by (c) of Theorem 4.4. Now we have an upper bound as above, except that we have k times $[\frac{1}{2}, \frac{2c-1}{2}]_{\downarrow}$. This obviously implies that the first inequality of the proposition holds. One gets the second inequality from this upper bound since $q'(q'+1)/2 + kc^2/2 \leq q(q-1)/2$ (one directly gets this using that $q' + kc = q$ and $k \geq 2$). □

Now from Propositions 5.2, 5.3 and 5.4 directly follows Theorem 1.5 in the introduction.

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