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ON THE SPECTRAL PROPERTIES OF DEGENERATE NON-SELFADJOINT ELLIPTIC SYSTEMS OF DIFFERENTIAL OPERATORS

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ABSTRACT. Let $(Pu)(t) = -\frac{d}{dt}\left(t^{\alpha}(1-t)^{\alpha}R(t)\frac{du(t)}{dt}\right)$ be a non - selfadjoint differential operator on Hilbert space $H_{\ell} = L^2(0,1)^{\ell}$ with Dirichlet-type boundary conditions. Here, $0 \leq \alpha < 1$, and $R(t) \in C^2([0,1],\ End\ \mathbf{C}^{\ell})$ denotes for each $t \in [0,1]$ the matrix function $R(t)$. Let for each $t \in [0,1]$ the matrix function $R(t)$ has ℓ-simple non-zero eigenvalues $\mu_j(t) \in C^2[0,1],\ j = 1, \ldots, \ell$ arranged in the complex plane in the following way: $\mu_1(t), \ldots, \mu_{\ell}(t) \in \mathbf{C} \backslash \Phi$ where $\Phi = \{z \in \mathbf{C} :\ |arg\ z| \leq \varphi\},\ \varphi \in (0, \pi)$. In this paper we investigate some spectral properties of the degenerate non-selfadjoint elliptic differential operators P acting on H_{ℓ}. In particular, we will determine the resolvent estimate of the operator P that satisfies Dirichlet-type boundary conditions in spaces H_1 and H_{ℓ}.

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1. **Introduction**

We consider the weighted Sobolev space $\mathcal{H}_\ell = W^1_{2,\alpha}(0,1)^\ell$ (ℓ-times) as the space of vector functions $u(t) = (u_1(t), \ldots, u_\ell(t))$ defined on $(0,1)$ with finite norm

$$|u|_+ = (\int_0^1 t^\alpha(1-t)^\alpha|\frac{du(t)}{dt}|^2_{\mathbf{C}^\ell}dt + \int_0^1 |u(t)|^2_{\mathbf{C}^\ell})^{1/2}.$$

Here, $0 \le \alpha < 1$, and the notations $|\frac{du(t)}{dt}|^2_{\mathbf{C}^\ell}$, $|u(t)|^2_{\mathbf{C}^\ell}$ stand for the norm in space $\mathbf{C}^\ell$. The above definition of the norm has been previously used. (See papers [9], [10].) Of course, this could also be done in matrix language, but at the cost of greater notational complexity. By $\overset{\circ}{\mathcal{H}}_\ell$ we denote the closure of $C_0^\infty(0,1)^\ell$ with respect to the above norm (i.e., $\overset{\circ}{\mathcal{H}}_\ell$ is the closure of $C_0^\infty(0,1)^\ell$ in $\mathcal{H}_\ell$). $C_0^\infty(0,1)$ denotes the space of infinitely differentiable functions with compact support in $(0,1)$. If $\ell = 1$, then $H = H_1$, $\mathcal{H} = \mathcal{H}_1$, and $\overset{\circ}{\mathcal{H}}=\overset{\circ}{\mathcal{H}}_1$.

To get a feeling for the history of the subject under study, refer to papers [1-3]. In [1] the authors consider d ifferential operator

$$Pu = -(t^\alpha A(t)u'(t))' + Q(t)u(t),\ 0 \le \alpha < 2,$$

with Dirichlet-type boundary conditions in the space $H = L^2((0,T); C)$, and find the distribution function of a series of eigenvalues of the operator P.(See MR 91e:34088.) In [2] the authors consider a certain matrix elliptic differential operator A and calculate the principle term in the asymptotic expansion of the function $N_\Phi(t)$, representing the distribution of eigenvalues of the operator A in the sector Φ. (See MR 91m:35175.) In [3] the author studies the distribution of eigenvalues of the operator in H^n defined by

$$(Py)(t) = -(t^\alpha A(t)y'(t))' + C(t)y(t),$$

with matrix coefficients $A(t) \in C^\infty([0,1],\ End\ \mathbf{C}^n)$, $C(t) \in C([0,1], End\ \mathbf{C}^n)$, (See MR 93k:34179.) In this paper, we consider the differential operator

$$(Pu)(t) = -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha R(t)\frac{du(t)}{dt}\right), \quad (1.1)$$

be a degenerate non- selfadjoint differential operator on Hilbert space $H_\ell = L^2(0,1)^\ell$ with Dirichlet-type boundary conditions. Here, $0 \leq \alpha < 1$, and $R(t) \in C^2([0,1],\ End\,\mathbf{C}^\ell)$ denotes for each $t \in [0,1]$ the matrix function $R(t)$. Assume that $R(t)$ has simple non-zero eigenvalues in the complex plane, arranged in different locations in view of $\Phi \subset \mathbf{C}$, where $\Phi = \{z \in \mathbf{C} :\ |arg\ z| \leq \varphi\},\ \varphi \in (0,\pi)$. In this paper, we investigate some spectral properties of the degenerate non-selfadjoint elliptic differential operators P acting on H_ℓ. In particular, we will determine the resolvent estimate of the operator P which satisfies Dirichlet-type boundary conditions in spaces H_ℓ and H. Now, the domain of operator P is defined as follows:

$$D(P) = \{u \in \overset{\circ}{\mathcal{H}}_\ell \cap W^2_{2,loc}(0,1)^\ell\ :\ \frac{d}{dt}\left(t^\alpha(1-t)^\alpha R\frac{du}{dt}\right) \in H_\ell\}$$

(see [7]). Here $W^2_{2,\,loc}(0,1)^\ell = W^2_{2,\,loc}(0,1) \times \cdots \times W^2_{2,\,loc}(0,1)$ $(\ell - times)$ where $W^2_{2,\,loc}(0,1)$ the space of functions $u(t)$ $(0 < t < 1)$ satisfying the condition

$$\sum_{i=0}^{2}\int_{\varepsilon}^{1-\varepsilon}|u^{(i)}(t)|^2dt < \infty, \qquad \forall\varepsilon\ \in (0,\frac{1}{2}).$$

Here, and in the sequel, the value of the function arg $z \in (-\pi,\pi]$ and $\|T\|$ denotes the norm of the bounded arbitrary operator T acting on H or H_ℓ. The paper has four sections. Section 1 is devoted to the introduction.In Section 2, we have Theorem 2.1 on the resolvent estimate of the differential operator A, acting on H in a certain case (i.e., in this case, we will study Theorem 2.1 under assumption (2.2)). In Section 3, we have Theorem 3.1 on the resolvent estimate of the differential operator A, acting on H in the general case (i.e., in this case, we will study Theorem 3.1 in contrast to Theorem 2.1. In other words, Theorem 3.1 does not include assumption (2.2) of Theorem 2.1). It is necessary to make further remarks regarding Theorem 2.1, and Theorem 3.1. We see that Theorem 2.1 under the assumption (2.2), leads to an assertion including two estimates (2.3) and (2.4). Meanwhile Theorem 3.1, without including assumption (2.2) of Theorem 2.1, leads to similar assertion, but asserts only statement (2.3) of Theorem 2.1, which becomes (3.2). In Section 4, assume that for each $t \in [0,1]$ the matrix function $R(t)$ has ℓ-simple non-zero eigenvalues $\mu_j(t) \in C^2[0,1],\ j = 1, \dots, \ell$ arranged in the complex plane in the following way:

$$\mu_1(t), \dots, \mu_\ell(t) \in \mathbf{C}\backslash\Phi.$$

Then we will estimate the resolvent of the differential operator P, acting on space $H_\ell = L^2(0,1)^\ell$.

2. Resolvent Estimate of A in $\mathbf{H} = \mathbf{L_2(0.1)}$

Theorem 2.1. Let $\Phi \subset \mathbf{C}$ be some closed sector with the vertex at 0, and set $P = A,\ R(t) = \mu(t)$ in (1.1) in Section 1. Then, we obtain
$(Av)(t) = -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha\mu(t)\frac{dv(t)}{dt}\right)$, acting on $H = L_2(0,1)$.
Assume that

$$\mu(t) \in C^2[0,1], \quad \mu(t) \in \mathbf{C}\backslash\Phi, \qquad \forall t \in [0,1], \tag{2.1}$$

$$|arg\{\mu(t_1)\mu^{-1}(t_2)\}| \leq \frac{\pi}{8}, \quad (\forall\ t_1, t_2 \in [0,1]). \tag{2.2}$$

Then, for sufficiently large numbers in modulus $\lambda \in \Phi$, the inverse operator
$(A - \lambda I)^{-1}$ exists and is continuous in the space $H = L^2(0,1)$, and the following estimates hold

$$\|(A-\lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1}\ (\lambda \in \Phi, \quad |\lambda| > C_\Phi), \tag{2.3}$$

$$\|t^{\alpha/2}(1-t)^{\alpha/2}\ \frac{d}{dt}(A-\lambda I)^{-1}\| \leq M'_\Phi|\lambda|^{-\frac{1}{2}}\ (\lambda \in \Phi, \quad |\lambda| > C_\Phi), \tag{2.4}$$

where the numbers $M_\Phi,\ M'_\Phi$ and $C_\Phi > 0$ are sufficiently large numbers depending on Φ where $\Phi = \{z \in \mathbf{C}:\ |arg\ z| \leq \varphi\},\ \varphi \in (0,\pi)$.

Proof. Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.3). As in Section 1, for the closed extension of the operator A, (for more explanations, see chapter 6 of [7]), we need to extend its domain to the

$$D(A) = \{v \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\text{loc}}(0,1):\quad (t^\alpha(1-t)^\alpha\mu v')' \in H\}.$$

Let the operator A now satisfy (2.1) and (2.2), then, there exists a real $\gamma \in (-\pi, \pi]$, such that for the complex number $e^{i\gamma}$ we have $|e^{i\gamma}| = 1$, and so

$$c' \leq Re\{e^{i\gamma}\mu(t)\},\ \ c'|\lambda| \leq -Re\{e^{i\gamma}\lambda\}, \quad c' > 0\ \forall\, t \in [0,1],\ \lambda \in \Phi. \tag{2.5}$$

For $v \in D(A)$ we will have

$$c' \int_0^1 t^\alpha (1-t)^\alpha |v'(t)|^2 dt \leq Re \int_0^1 e^{i\gamma} t^\alpha (1-t)^\alpha \mu |v'(t)|^2\, dt$$
$$= Re\{e^{i\gamma}(Av\ ,\ v)\}. \tag{2.6}$$

Here the symbol (,) denotes the inner product in H. Notice that the equality in (2.6) above obtains by the well-known theorem of the m-sectorial operators, (For further explanations see the well-known Theorem 2.1, chapter 6 of [7].) By (2.5), we have $c'|\lambda| \leq -Re\{e^{i\gamma}\lambda\},\ c' > 0,\ \forall\lambda \in \Phi$. Multiplying the latter inequality by $\int_0^1 |v(t)|^2\, dt = (v\ ,\ v) = \|v\|^2 > 0$, then

$$c'|\lambda| \int_0^1 |v(t)|^2\, dt \leq -Re\{e^{i\gamma}\lambda\}(v\ ,\ v).$$

By the latter inequality and (2.6), and by considering $c' = 1/M$, it follows that

$$\begin{aligned} & \int_0^1 t^\alpha (1-t)^\alpha |v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2 dt \\ \leq\ & MRe\{e^{i\gamma}(A, v) - e^{i\gamma}\lambda(v, v)\} \\ =\ & MRe\{e^{i\gamma}((A-\lambda I)v, v)\} \\ \leq\ & M\|e^{i\gamma}\|\|v\|\|(A-\lambda I)v\| \\ =\ & M\|v\|\|(A-\lambda I)v\|. \end{aligned} \tag{2.7}$$

Or

$$\int_0^1 t^\alpha (1-t)^\alpha |v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2\, dt \leq M\|v\|\ \|(A\ -\ \lambda\ I)v\|.$$

Since $\int_0^1 t^\alpha(1-t)^\alpha|v'(t)|^2dt$ is positive, we will have

$$|\lambda|\|v(t)\|^2 = |\lambda|\int_0^1 |v(t)|^2\, dt \le M\|v\|\|(A \quad - \lambda\, I)v\|, \qquad (2.8)$$

i.e.,

$$|\lambda|\|v\| \le M\|(A \quad - \lambda\, I)v\|.$$

The above relation ensures that the operator $(A \ - \ \lambda\, I)$ is a one-to- one operator, which implies that $ker(A \ - \ \lambda\, I) = 0$. Therefore, the inverse operator $(A - \ \ \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (2.3) of Theorem 2.1. To prove (2.3), we set $v = (A-\lambda I)^{-1}f,\ f \in H$ in (2.8), so that

$$|\lambda|\int_0^1 |(A{-}\lambda I)^{-1}f|^2\, dt \le M\|(A{-}\ \ \lambda I)^{-1}f\|\|(A\ \ {-}\lambda I)(A{-}\lambda I)^{-1}f\|.$$

Since $(A \ - \ \lambda\, I)(A-\lambda I)^{-1}f = I(f) = f$, it follows that

$$|\lambda|\int_0^1 |(A \quad - \lambda I)^{-1}f|^2\, dt \le M\|(A \quad - \lambda I)^{-1}f\||f|.$$

Therefore,

$$|\lambda|\|(A \quad - \lambda I)^{-1}(f)\|^2 \le M\|(A \quad - \lambda I)^{-1}(f)\||f|.$$

By canceling the positive term $\|(A-\lambda I)^{-1}(f)\|$ from both sides of the latter inequality, we will find

$$|\lambda|\|(A \quad - \lambda I)^{-1}(f)\| \le M|f|,$$

and since $\lambda \ne 0$, we imply that $\|(A-\lambda I)^{-1}(f)\| \le M|\lambda|^{-1}|f|$. The end result is

$$\|(A-\lambda I)^{-1}\| \le M_\Phi|\lambda|^{-1}.$$

This completes the proof of the estimate (2.3) from Theorem 2.1.

To prove the estimate (2.4) of Theorem 2.1. As in the first arguments to prove estimate (2.3) above, here, we drop the positive term $|\lambda|\int_0^1 |v(t)|^2\, dt$ from

$$\int_0^1 t^\alpha(1-t)^\alpha|v'(t)|^2dt + |\lambda|\int_0^1 |v(t)|^2\, dt \le M|v|\ \|(A \ - \ \lambda\, I)v\|.$$

It follows that

$$\int_0^1 t^\alpha(1-t)^\alpha|v'(t)|^2dt \le M|v|\ \|(A \ - \ \lambda\, I)v\|.$$

Set $v=(A-\lambda I)^{-1}f,\ f\in H$ in the latter inequality, and as above, by proceeding with similar calculations, we then obtain:

$$\int_0^1 t^\alpha(1-t)^\alpha|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2dt$$
$$\leq M\|(A-\lambda I)^{-1}f\|\|(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f\|.$$

Since $(A\ -\ \lambda\ I)(A-\lambda I)^{-1}f=f$, and

$$\int_0^1 t^\alpha(1-t)^\alpha|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2dt\leq M\|(A-\lambda I)^{-1}f\||f|,$$

consequently by (2.3) we have $\|(A-\lambda I)^{-1}f\|\leq M|f||\lambda|^{-1}$. Then,

$$\int_0^1 t^\alpha(1-t)^\alpha|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2dt$$
$$\leq M\|(A-\lambda I)^{-1}f\||f|\leq MM|\lambda|^{-1}|f|^2.$$

Therefore,

$$\int_0^1 t^\alpha(1-t)^\alpha|\frac{d}{dt}(A-\lambda I)^{-1}f(t)|^2dt\leq M_\Phi|\lambda|^{-1}|f|^2;$$

i.e., $\|t^{\alpha/2}(1-t)^{\alpha/2}\ \frac{d}{dt}(A-\lambda I)^{-1}f\|^2\leq M_\Phi|\lambda|^{-1}|f|^2$. i.e.,

$$\|t^{\alpha/2}(1-t)^{\alpha/2}\ \frac{d}{dt}(A-\lambda I)^{-1}f\|\leq M'_\Phi|\lambda|^{-\frac{1}{2}}|f|.$$

Consequently,

$$\|t^{\alpha/2}(1-t)^{\alpha/2}\ \frac{d}{dt}(A-\lambda I)^{-1}\|\leq M'_\Phi|\lambda|^{-\frac{1}{2}}.$$

This estimate completes the proof of (2.4); Theorem 2.1 is thereby proved. □

3. Resolvent Estimate of A in $\mathbf{H}=\mathbf{L_2(0,1)}$ in General Case

In this section, we will derive a new general theorem by dropping assumption (2.2) from Theorem 2.1 in Section 2.

Theorem 3.1. Let Φ and A be defined as in Theorem 2.1, and let that except for assumption (2.2) of Theorem 2.1, all other assumptions are satisfied: Let the differential operator $(Av)(t) = -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha\mu(t)\frac{dv(t)}{dt}\right)$, acting on $H = L_2(0,1)$.
Assume that

$$\mu(t) \in C^2[0,1], \quad \mu(t) \in \mathbf{C}\backslash\Phi, \quad \forall t \in [0,1]. \tag{3.1}$$

Then, for a sufficiently large number in modulus $\lambda \in \Phi$, the inverse operator
$(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(0,1)$, and the following estimate holds:

$$\|(A - \lambda I)^{-1}\| \le M_\Phi |\lambda|^{-1} \tag{3.2}$$

where M_Φ, $C_\Phi > 0$ are sufficiently large numbers depending on Φ and $|\lambda| > C_\Phi$.

Proof. To prove the assertion of Theorem 3.1, we construct the functions $\varphi_{(1)}(t), \dots, \varphi_{(\rho)}(t)$, $\mu_{(1)}(t), \dots, \mu_{(\rho)}(t)$ so that each one of the functions $\mu_{(j)}(t)$ $j = 1, \dots, \rho$ ($t \in supp\ \varphi_{(j)}$), as the function $\mu(t)$ in Theorem 2.1 satisfies (2.2). Therefore, let

$$\mu_{(1)}(t), \dots, \mu_{(\rho)}(t), \quad \varphi_{(1)}(t), \dots, \varphi_{(\rho)}(t) \in C^\infty[0,1]$$

satisfy

$$0 \le \varphi_{(j)}(t),\ j = 1, \dots, \rho, \quad \varphi^2_{(1)}(t) + \dots + \varphi^2_{(\rho)}(t) \equiv 1 \quad (0 \le t \le 1)$$

$$\frac{d}{dt}\varphi_{(j)}(t) \in C_0^\infty(0,1), \quad \mu_{(j)}(t) = \mu(t), \quad \forall t \in supp\ \varphi_{(j)}$$

$$\mu_{(j)}(t) \in \mathbf{C}\backslash S, \quad \forall t \in [0,1], \quad j = 1, \dots, \rho.$$

$$|\arg\{\mu_{(j)}(t_1)\mu^{-1}_{(j)}(t_2)\}| \le \frac{\pi}{8}, \quad (\forall\, t_1, t_2 \in supp\ \varphi_{(j)}), \quad j = 1, \dots, \rho.$$

In view of Theorem 2.1, and by (2.3) and (2.4) set $(A_{(j)}v)(t) = A(t)$, we will have the differential operator

$$(A_{(j)}v)(t) = -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha\mu_{(j)}(t)\frac{dv(t)}{dt}\right)$$

acting on $H = L_2(0,1)$ where

$$D(A_{(j)}) = \{v \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,loc}(0,1), \quad \left(t^\alpha(1-t)^\alpha\mu_{(j)}v'\right)' \in H\}.$$

Due to the assertion of Theorem 2.1, for $0 \neq \lambda \in \Phi$ the inverse operator $(A-\lambda I)^{-1}$ exists and is continuous in space $H = L^2(0,1)$ and satisfies $\|(A_{(j)}-\lambda I)^{-1}\| \leq M_1|\lambda|^{-1}$,

$$\|.t^{\alpha/2}(1-t)^{\alpha/2}\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\| \leq M_1|\lambda|^{-\frac{1}{2}}, (0 \neq \lambda \in \Phi). \quad (3.3)$$

Let us introduce

$$T(\lambda) = \sum_{j=1}^{\rho} \varphi_{(j)}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}. \quad (3.4)$$

Here $\varphi_{(j)}$ is the multiplication operator in H by the function $\varphi_{(j)}(t)$. Consequently,

$$(A-\lambda I)T(\lambda)v = I_1 + I_2 + I_3 + I_4$$

where

$$I_1 = -\sum_{j=1}^{\rho}[t^\alpha(1-t)^\alpha\mu(\varphi_{(j)})'_t]'_t(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}v,$$

$$I_2 = -\sum_{j=1}^{\rho} t^\alpha(1-t)^\alpha\mu(\varphi_{(j)})'_t\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}v,$$

$$I_3 = -\sum_{j=1}^{\rho}\varphi_{(j)}[t^\alpha(1-t)^\alpha\mu\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}v]'_t,$$

$$I_4 = -\lambda\sum_{j=1}^{\rho}\varphi_{(j)}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}v.$$

As $\mu_{(j)}(t) = \mu(t)$ $(\forall t \in supp\, \varphi_{(j)})$, replace $\mu(t)$ by $\mu_{(j)}(t)$ in the sum I_3. Then, in view of $\sum_{j=1}^{\rho}\varphi_{(j)}{}^2(t) \equiv 1$, we will have

$$\begin{aligned} I_3 + I_4 &= -\sum_{j=1}^{\rho}\varphi_{(j)}[t^\alpha(1-t)^\alpha\mu_{(j)}\frac{d}{dt}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}v)'_t \\ &+ \lambda(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}v] \\ &= \sum_{j=1}^{\rho}\varphi_{(j)}(A_{(j)}-\lambda I)[A_j-\lambda I)^{-1}]\varphi_{(j)}v = \sum_{j=1}^{\rho}\varphi_{(j)}{}^2 v = v. \end{aligned}$$

i.e., $I_3 + I_4 = v$. When considering $I_1 + I_2 = G(\lambda)v$, then

$(A-\lambda I)T(\lambda)v = v + G(\lambda)v$. Or equivalently

$$(A-\lambda I)T(\lambda) = I + G(\lambda). \tag{3.5}$$

using the fact that ${\varphi_{(j)}}'_t \in C^{\infty}(0,1)$, and by (3.3), we can estimate I_1, I_2 as follows:

$$|I_1| \leq M \sum_{j=1}^{\rho} |(A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v| \leq M|\lambda|^{-1}|v|,$$

$$|I_2| \leq M \sum_{j=1}^{\rho} |\frac{d}{dt}(A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v| \leq M\text{"}|\lambda|^{-\frac{1}{2}}|v|.$$

Using these estimates, and in view of $G(\lambda)$ above, we will have
$\|G(\lambda)(v)\| \leq |I_1| + |I_2| \leq M|\lambda|^{-1}|v| + M\text{"}|\lambda|^{-\frac{1}{2}}|v|$,
λ is a sufficiently large number, implying that $|\lambda|^{-1} \leq |\lambda|^{-\frac{1}{2}}$ then

$$\|G(\lambda)\| \leq M'_{\Phi}|\lambda|^{-\frac{1}{2}}. \tag{3.6}$$

By this λ, we can also have $\|G(\lambda)\| \leq \frac{1}{2} < 1$, where $\lambda \in \Phi$. Now, by this, and using the well-known theorem in operator theory, we conclude that $I + G(\lambda)$ and hence, by (3.5), $(A - \lambda I)T(\lambda)$ are invertible. Therefore $((A - \lambda I)T(\lambda))^{-1}$ exists, and by (3.5)

$$(T(\lambda))^{-1}(A-\lambda I)^{-1} = (I \ + \ G(\lambda))^{-1}. \tag{3.7}$$

Adding $+I$ and $-I$ to the right side of (3.7) it follows that

$$(T(\lambda))^{-1}(A-\lambda I)^{-1} = (I \ + \ G(\lambda))^{-1} \ -I \ +I.$$

Setting $F(\lambda) = (I \ + \ G(\lambda))^{-1} \ - \ I$ implies that

$$(T(\lambda))^{-1}(A-\lambda I)^{-1} = I \ + \ F(\lambda).$$

In view of $\|G(\lambda)\| \leq \frac{1}{2} < 1$, and (3.6), by applying the geometric series for $F(\lambda)$:

$$\begin{aligned}\|F(\lambda)\| \leq \sum_{i=1}^{+\infty} \|G^k(\lambda)\| &\leq \|G(\lambda)\|(1 + \|G^k(\lambda)\| + \|G^k(\lambda)\|^2 + \dots) \\ &\leq \|G(\lambda)\|(1 + 1/2 + 1/4 + \dots) \\ &\leq 2M'_{\Phi}|\lambda|^{-1/2}\end{aligned}$$

i.e.,

$$\|F(\lambda)\| \leq M1_{\Phi}|\lambda|^{-1}.$$

Now, by (3.3) and (3.4),

$$\begin{aligned}\|T(\lambda)\| &= \|\sum_{j=1}^{\rho}\varphi_{(j)}(A_{(j)}-\lambda I)^{-1}\varphi_{(j)}\| \\ &\leq M"_{\Phi}\|(A_{(j)}-\lambda I)^{-1}\| \\ &\leq M"_{\Phi}M_{\Phi}|\lambda|^{-1} = M2_{\Phi}|\lambda|^{-1};\end{aligned}$$

i.e.,

$$\|T(\lambda)\| \leq M2_{\Phi}|\lambda|^{-1}.$$

By this and (3.7), it follows that

$$\begin{aligned}\|(A-\lambda I)^{-1}\| &= \|T(\lambda)\|\|I\ +\ F(\lambda)\| \\ &\leq M2_{\Phi}|\lambda|^{-1}\|(1+M1_{\Phi}|\lambda|^{-1}) \\ &\leq M2_{\Phi}|\lambda|^{-1}+M1_{\Phi}M2_{\Phi}|\lambda|^{-2}.\end{aligned}$$

Since $|\lambda|^{-2} \leq |\lambda|^{-1}$, then,

$$\|(A-\lambda I)^{-1}\| \leq M_{\Phi}|\lambda|^{-1}, \quad (|\lambda| \geq C_{\Phi}, \qquad \lambda \in \Phi).$$

This estimate completes the proof of (3.2); Theorem 3.1 is thereby proved. □

4. Asymptotic resolvent of degenerate elliptic differential operators in $\mathbf{L}_\ell(\mathbf{0},\mathbf{1})$

Let P and R be as defined in Section 1. Suppose that there exists some closed sector $S \subset \mathbf{C}$, with the origin at zero, free from eigenvalues of the matrix $R(t)$, $(0 \leq t \leq 1)$. Consider those eigenvalues $\mu_1(t),\dots,\mu_\ell(t)$ of the matrix $R(t)$, such that $\mu_j(t) \in C^2[0,1]$, and convert the matrix $R(t)$ to the form

$$R(t) = U(t)\Lambda(t)U^{-1}(t), \text{ where } U(t),U^{-1}(t) \in C^2([0,1], End\ \mathbf{C}^\ell)$$

and

$$\Lambda(t) = diag\{\mu_1(t),\dots,\mu_\ell(t)\}.$$

Consider the space $H_\ell = H \oplus \cdots \oplus H$ (ℓ-times), and so consider in the space H_ℓ the operator 0

$$B(\lambda) = diag\{(P_1-\lambda I)^{-1},\dots,(P_\ell-\lambda I)^{-1}\}, \tag{4.1}$$

where the operator P_j satisfies $(P_j v)(t) = -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha \mu_j \frac{dv(t)}{dt}\right)$,

$$D(P_j) = \{v \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\,loc}(0,1) : \ \frac{d}{dt}\left(t^\alpha(1-t)^\alpha \mu_j \frac{du}{dt}\right) \in H\}. \tag{4.1'}$$

According to the results obtaining from Section 3, for sufficiently large absolute values of $\lambda \in S$, the operator $B(\lambda)$ exists and is continuous. Consider the operator $\Gamma(\lambda) = UB(\lambda)U^{-1}$, in which $(Uu)(t) = U(t)u(t),\ (u \in H_\ell)$. We have

$$\begin{aligned}(P-\lambda I)\Gamma(\lambda)u &= -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha R(t)\frac{d}{dt}(U(t)B(\lambda)U^{-1}(t)u(t))\right)\\ &= T_1 + T_2 + T_3,\end{aligned}$$

where, T_1 is equal to the following:

$$\begin{aligned}&-\frac{d}{dt}\left(t^\alpha(1-t)^\alpha R(t)U(t)\frac{d}{dt}B(\lambda)U^{-1}(t)u(t))\right)\\ =\ & -\frac{d}{dt}(t^\alpha(1-t)^\alpha U(t)\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u(t))\\ =\ & -U\frac{d}{dt}(t^\alpha(1-t)^\alpha\Lambda(t)\frac{d}{dt}B(\lambda)U^{-1}u)\\ & -U'(t)t^\alpha(1-t)^\alpha\Lambda\frac{d}{dt}B(\lambda)U^{-1}u\\ =\ & \lambda UB(\lambda)U^{-1}u - U'(t)t^\alpha(1-t)^\alpha\Lambda\frac{d}{dt}B(\lambda)U^{-1}u + UU^{-1}u,\end{aligned}$$

$$T_2 = -\frac{d}{dt}\left(t^\alpha(1-t)^\alpha RU'B(\lambda)U^{-1}u\right), \quad T_3 = -\lambda U(t)B(\lambda)U^{-1}u.$$

Here we use the equality

$$-\frac{d}{dt}(t^\alpha(1-t)^\alpha\Lambda\frac{d}{dt}B(\lambda)V) = V + \lambda B(\lambda)V, \quad V = U^{-1}u.$$

Since $(t^\alpha(1-t)^\alpha)^2 \le M\ t^\alpha(1-t)^\alpha$, and from the above relations and the estimate (3.9), we will have

$$(P-\lambda I)\Gamma(\lambda) = I + T_1^0 + T_2^0, \quad \text{where } T_2^0 = t^\alpha(1-t)^\alpha RU'B(\lambda)U^{-1},$$

$$\|T_1^0\| \le M|\lambda|^{-1/2} \quad (\lambda \in Q,\ \ |\lambda| \ge\ c).$$

For estimates of the operator T_2^0, the Hardy-type inequality is useful (see [5]). Thus, for a sufficiently large absolute-value of $\lambda \in S$, the estimate $\|T_2^0\| \leq M'|\lambda|^{-1/2}$ is true, and so we have

$$(P-\lambda I)\Gamma(\lambda) = I + \mathcal{F}(\lambda), \quad \|\mathcal{F}(\lambda)\| \leq M|\lambda|^{-\frac{1}{2}} \quad (\lambda \in S, \ |\lambda| > c'). \tag{4.2}$$

According to the assumptions made for $\lambda \in S$, $|\lambda| > c$, the range of the operator $P - \lambda I$ coincides with H_ℓ. The operator P^* and its domain $D(P^*)$ have the same structures as P, $D(P)$. Therefore, for a sufficiently large absolute value of $\lambda \in S$, the range of the operator $P^* - \lambda I$ coincides with H_ℓ, and consequently, $ker(P-\lambda I) = 0$. This equality by (4.2) proves the existence of the continuous operator $(P-\lambda I)^{-1}$, which satisfies

$$(P-\lambda I)^{-1} = \Gamma(\lambda)(I + \mathcal{Y}(\lambda)) \tag{4.3}$$

in which the operational-function $\mathcal{Y}(\lambda)$ has the estimate

$$\|\mathcal{Y}(\lambda)\| \leq M|\lambda|^{-\frac{1}{2}} \quad (\lambda \in S, \ |\lambda| > c_0) \tag{4.4}$$

Recall that

$$\Gamma(\lambda) = UB(\lambda)U^{-1}, \ B(\lambda) = diag\{(P_1 - \lambda I)^{-1}, \dots, (P_\ell - \lambda I)^{-1}\} \tag{4.5}$$

As in (3.9), $Q = S$, $\quad A = P_j$, $j = 1, \dots, \ell$. By (4.3)-(4.5), we get $\|(P_j - \lambda I)^{-1}\| \leq M|\lambda^{-1}$,
$\|t^{\alpha/2}(1-t)^{\alpha/2}\frac{d}{dt}(P_j - \lambda I)^{-1}\| \leq M|\lambda|^{-\frac{1}{2}} \quad (\lambda \in S, \ |\lambda| \geq c)$, which proves Theorem 2.1.

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