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RECURRENCES AND EXPLICIT FORMULAE FOR THE EXPANSION AND CONNECTION COEFFICIENTS IN SERIES OF THE PRODUCT OF TWO CLASSICAL DISCRETE ORTHOGONAL POLYNOMIALS

H.M. AHMED

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Abstract. Suppose that for an arbitrary function $f(x, y)$ of two discrete variables, we have the formal expansions.

$$f(x,y)=\sum_{m,n=0}^{\infty} a_{m,n}\, P_m(x)P_n(y),$$

$$\nabla_x^p\nabla_y^q f(x,y)=f^{(p,q)}(x,y)=\sum_{m,n=0}^{\infty} a_{m,n}^{(p,q)}\, P_m(x)P_n(y),\ a_{m,n}^{(0,0)}=a_{m,n},$$

where $P_n(x)$, $n=0,1,2,\ldots$ are the Hahn, Meixner, Kravchuk and Charlier polynomials.

We prove formulae which give $a_{m,n}^{(p,q)}$, as a linear combination of $a_{i,j}$, i, $j=0,1,2,\ldots$. Using the moments of a discrete orthogonal polynomial,

$$x^m P_j(x)=\sum_{n=0}^{2m} a_{m,\,n}(j)P_{j+m-n}(x),$$

we find the coefficients $b_{i,j}^{(p,q,\ell,\,r)}$ in the expansion

$$x^\ell y^r\, \nabla_x^p\nabla_y^q\, f(x,y)=x^\ell y^r f^{(p,q)}(x,y)=\sum_{i,j=0}^{\infty} b_{i,j}^{(p,q,\ell,\,r)}P_i(x)P_j(y).$$

We give applications of these results in solving partial difference equations with varying polynomial coefficients, by reducing them to recurrence relations (difference equations) in the expansion coefficients of the solution.

Keywords: Hahn, Meixner, Kravchuk and Charlier polynomials, expansion coefficient, recurrence relations, linear difference equations, connection coefficients.
MSC(2010): Primary: 33C45; Secondary: 65Q30, 68W30.

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1. **Introduction**

Classical orthogonal polynomials of a discrete variable (the Hahn, Meixner, Kravchuk, and Charlier polynomials) which are solutions of difference equations of hypergeometric type on uniform lattices [32, Eq. (2.1.4)] are widely used in various fields of physics and mathematics (see [20, 29, 32, 39, 41, 53]). These polynomials have long been used in two dimensional problems of approximation theory and numerical analysis. In the last three decades, many authors aimed to generalize the construction of the theory of classical orthogonal polynomials of a discrete variable as solutions of a difference equation of hypergeometric type from one to several variables, see for instance, Rodal [33–35], Tratnik [42, 43], Van Diejen [44] and Xu [45, 46]. They presented a multivariable generalization for all the discrete families of the Askey-Wilson tableau (see [24]), giving for each family an hypergeometric representation and the orthogonality weight function, proving that these are orthogonal with respect to subspaces of lower degree and biorthogonal within a given subspace. These results motivate the researchers interested in multidimensional mathematical physics problems to use expansions in terms of orthogonal polynomials of several discrete variables (see, for instance [29, 41, 53]). According to this situation, the generation of recurrence relations for the expansion coefficients of multivariable orthogonal polynomials like in the one continuous/discrete variable case, see, Ahmed [1], Ahmed and El-Soubhy [2], Area et al. [8], Doha [13–15], Doha and Ahmed [17–19], Godoy et al. [21–23], Koepf [25], Lewanowicz [26, 27], Lewanowicz and Woźny [28], Ronveaux et al. [36–38] and Woźny [51] is a problem of great interest.

Up to now and to the best of the author's knowledge, explicit formulae for the expansion coefficients of general-order difference derivatives of an arbitrary function of two discrete variables and for the evaluation of the expansion coefficients of the moments of high-order difference derivatives of such function in terms of the product of two classical orthogonal polynomials of a discrete variable (the Hahn, Meixner, Kravchuk and Charlier polynomials), similar to those obtained by Doha [10–12] and Doha et al. [16], for classical orthogonal polynomials of continuous variable (the Chebyshev, Legendre, ultraspherical and Jacobi), respectively, are not well-known and traceless in the literature. Another motivation is that the theoretical and numerical analysis of numerous physical and mathematical problems very often require the expansion of an arbitrary polynomial or the expansion of an arbitrary function of two discrete variables with its difference derivatives and moments into a set of orthogonal polynomials. This is also true for the product of two Hahn, Meixner, Kravchuk and Charlier polynomials. To be precise, these polynomials form a set of orthogonal polynomials on the set $[0, N] \times [0, N]$, $[0, \infty) \times [0, \infty)$, $[0, N] \times [0, N]$ and $[0, \infty) \times [0, \infty)$, respectively.

In this paper, some basic results about the product of two classical orthogonal polynomials (see [34, 45, 46]) of one single discrete variable :

$$\{F_{mn}(x,y) : F_{mn}(x,y) = P_m(x)P_n(y)\,, P_m(x) \in T\},$$

where $T = \{P_n(x) :$ Hahn, Meixner, Kravchuk and Charlier$\}$, are provided, mainly to show that the exact explicit formulae for the coefficients of differentiated expansions in $P_n(x)$ obtained by Doha and Ahmed [18, 19] can be extended to expansions in $F_{mn}(x,y)$.

This paper is organized as follows. In Section 2, we recall some relevant properties of the Hahn, Meixner, Kravchuk and Charlier polynomials. In Section 3, we give relevant properties of $F_{m,n}(x,y)$. In Section 4, two corollaries are proved: the first expresses the relation between the coefficients $a_{m,n}^{(p,q)}$ and $a_{m,n}$, whereas the second expresses the $P_m(x)P_n(y)$ expansion coefficients of the moments of general order difference derivatives of an arbitrary function of two discrete variables in terms of its $P_m(x)P_n(y)$ original expansion coefficients. Application of these corollaries for solving partial difference equations with varying coefficients, by reducing them to recurrence relations in the expansion coefficients of the solution, is discussed in Section 5. Two applications of the work developed which provides an algebraic symbolic approach (using Mathematica Version 8) in order to build recurrence relations for the coefficients appearing in the two problems

$$(x+y)^{\underline{n}} = \sum_{i+j\le n} a_{i,j}(n)\ P_i(x)P_j(y),$$

where $x^{\underline{n}} = (-1)^n(-x)_n$ (with $(a)_n = \Gamma(a+n)/\Gamma(a)$), denote the falling factorial polynomials, and

$$Q_n(x+y) = \sum_{i+j\le n} a_{i,j}(n)\,P_i(x)P_j(y), \quad Q \in T,$$

are discussed in Sections 6 and 7, respectively, and the analytical solutions of these recurrence relations are provided.

2. Some relevant properties of discrete polynomials

Let $\{P_n(x) :$ Hahn, Meixner, Kravchuk and Charlier$\}$ be one of the families of monic classical discrete orthogonal polynomials. These polynomials are solutions of the second-order difference equation of hypergeometric type [24, pp. 204-247],

$$[\sigma(x)\nabla\Delta + \tau(x)\,\Delta + \lambda_n\,]P_n(x) = 0, \tag{2.1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of degree not greater than 2 and 1, respectively, $\lambda_n = -n\Delta\tau(x) - \frac{1}{2}n(n-1)\Delta^2\sigma(x)$, and

$$\nabla y(x) = y(x) - y(x-1), \qquad \Delta y(x) = y(x+1) - y(x),$$

denote the backward and forward difference operators, respectively. Doha and Ahmed [18, 19] showed that the difference equation (2.1) can be written in the form

$$[\sigma(x-1)\nabla^2 + [\tau(x-1) - \lambda_n]\nabla + \lambda_n]\, P_n(x) = 0. \tag{2.2}$$

The orthogonality relation of the polynomials $P_n(x)$ is

$$\sum_{x_i=a}^{b-1} P_n(x_i)P_m(x_i)\rho(x_i) = \delta_{nm}\, d_n^2, \qquad x_{i+1} = x_i + 1,$$

where the weight function $\rho(x)$ defined on $[a, b-1]$ must be a solution of the Pearson-type difference equation:

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x),$$

provided that the following condition

$$\rho(x)\sigma(x)x^k\big|_{x=a,b} = 0,\ \forall k \geq 0,$$

holds. The monic classical discrete orthogonal polynomials $P_n(x)$ may be generated by using the Rodrigues formula

$$P_n(x) = \frac{B_n}{\rho(x)}\nabla^n\left[\rho_n(x)\right],$$

where the function $\rho_n(x)$ is defined in terms of the weight function $\rho(x)$ and the polynomial $\sigma(x)$,

$$\rho_n(x) = \rho(x+n)\prod_{m=1}^{n}\sigma(x+m).$$

The four referred families of monic discrete orthogonal polynomials: Hahn $\tilde{h}_n^{(\alpha,\beta)}(x;N)$, Meixner $\tilde{M}_n^{(\gamma,\mu)}(x)$, Kravchuk $\tilde{K}_n^{(p)}(x;N)$ and Charlier $\tilde{C}_n^{(\mu)}(x)$, have the following hypergeometric representations [24, pp. 204-247]:

$$\begin{aligned}
\tilde{h}_n^{(\alpha,\beta)}(x;N) &= \frac{(1-N)_n(\beta+1)_n}{(\lambda+n)_n}\ {}_3F_2\left[\begin{array}{c} -n, \lambda+n,\ -x \\ 1-N, \beta+1 \end{array};1\right],\\
\tilde{M}_n^{(\gamma,\mu)}(x) &= (\gamma)_n\left(\frac{\mu}{\mu-1}\right)^n\ {}_2F_1\left[\begin{array}{c} -n,\ -x \\ \gamma \end{array};1-\frac{1}{\mu}\right],\\
\tilde{K}_n^{(p)}(x;N) &= \frac{(-p)^n N!}{(N-n)!}\ {}_2F_1\left[\begin{array}{c} -n,\ -x \\ -N \end{array};\frac{1}{p}\right],\\
\tilde{C}_n^{(\mu)}(x) &= (-\mu)^n\ {}_2F_0\left[\begin{array}{c} -n,\ -x \\ - \end{array};-\frac{1}{\mu}\right].
\end{aligned}$$

The following two recurrence relations are of fundamental importance in developing the present work. These are (see [24, 25])

$$\begin{aligned}
&x\,P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \qquad (n \geq 0),\\
&P_{-1}(x) = 0;\ \ P_0(x) = 1,
\end{aligned} \tag{2.3}$$

and

$$P_n(x) = \frac{1}{n+1}\Delta\, P_{n+1}(x) + F_n\, \Delta\, P_n(x) + G_n \Delta\, P_{n-1}(x),\ n \geq 0. \tag{2.4}$$

Remark 2.1. For the sake of completeness, the expressions of $\sigma(x)$, $\tau(x)$, $\rho(x), \lambda_n, d_n^2$, β_n, γ_n, F_n and G_n, for each one of the referred monic classical discrete orthogonal families can be found in [18, 19], [24, pp. 204-247], [25, pp. 58-61], and [32, p. 44].

Lemma 2.2. *$P_n(x)$ has the ∇-difference representation*

$$P_n(x) = \frac{1}{n+1}\nabla\, P_{n+1}(x) + (F_n+1)\nabla\, P_n(x) + G_n \nabla\, P_{n-1}(x),\ n \geq 0. \tag{2.5}$$

Proof. By replacing x by $x-1$ in (2.4) and using the property

$$\Delta y(x-1) = \nabla y(x),$$

we obtain

$$P_n(x-1) = \frac{1}{n+1}\nabla\, P_{n+1}(x) + F_n \nabla\, P_n(x) + G_n \nabla\, P_{n-1}(x),\ n \geq 0.$$

Using the definition of the backward difference operator gives immediately relation (2.5) which completes the proof. $\square$

Now, suppose a function $f(x)$ of a discrete variable x is given, which is formally expanded in an infinite series of monic discrete classical orthogonal polynomials $P_n(x) \in \{P_n(x) : \text{Hahn, Meixner, Kravchuk and Charlier}\}$; in the case of Kravchuk and Hahn polynomials, which are orthogonal on a finite set, we assume that f is a polynomial. Let

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{2.6}$$

and for the *pth* backward-difference derivatives of $f(x)$, i.e., $\nabla^p f(x)$,

$$f^{(p)}(x) = \nabla^p f(x) = \sum_{n=0}^{\infty} a_n^{(p)} P_n(x),\ a_n^{(0)} = a_n, \tag{2.7}$$

then it is possible to derive a recurrence relation involving the expansion coefficients of successive backward-difference derivatives of $f(x)$. Let us write

$$\nabla \left[\sum_{n=0}^{\infty} a_n^{(p-1)} P_n(x)\right] = \sum_{n=0}^{\infty} a_n^{(p)} P_n(x),$$

then the use of identity (2.5) leads to the recurrence relation

$$\frac{1}{n} a_{n-1}^{(p)} + (F_n+1) a_n^{(p)} + G_{n+1} a_{n+1}^{(p)} = a_n^{(p-1)},\quad p \geq 1,\, n \geq 1. \tag{2.8}$$

Lemma 2.3.

$$\nabla^p P_n(x) = \sum_{k=0}^{n-p} C_{p,k}(n) P_k(x), \ \ n \geq 0, p \geq 0, \tag{2.9}$$

if and only if

$$a_n^{(p)} = \sum_{k=0}^{\infty} C_{p,n}(n+p+k)\, a_{n+p+k}, \ \ n \geq 0, p \geq 0, \tag{2.10}$$

where the expansion coefficients $C_{p,k}(n)$ *are assumed to be known.*

Proof. Suppose we are given the expansion (2.9), then by applying the operator ∇^p to the expansion (2.6), we obtain

$$\nabla^p f(x) = \sum_{n=p}^{\infty} a_n \nabla^p P_n(x). \tag{2.11}$$

Substituting (2.9) into (2.11), expanding and collecting similar terms, we obtain

$$\nabla^p f(x) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} C_{p,\,n}(n+p+k)\, a_{n+p+k} \right] P_n(x). \tag{2.12}$$

Identifying (2.7) with (2.12) gives immediately (2.10).

On the other hand, suppose we have (2.10). Substituting (2.10) into (2.7) gives (2.12). Expanding (2.12) and collecting similar terms and identifying the result with (2.11), we get (2.9) which completes the proof. □

Using Lemma 2.3, for the case of Hahn polynomials, Doha and Ahmed [19] proved that the solution of (2.8) is,

$$a_n^{(p)} = \sum_{k=0}^{\infty} (n+k+p+\lambda)_p\, C_{n+k,n}(\alpha+p, \beta+p, \alpha, \beta, N-p, N, -p)\, a_{n+p+k}, \qquad n \geq 0, p \geq 0,$$

where

$$\begin{aligned} C_{n,i}(\gamma,\delta,\alpha,\beta,M,N,\,s) = \binom{n}{i} &\frac{(1-M+i)_{n-i}(1+\delta+i)_{n-i}}{(\mu+n+i)_{n-i}} \\ &\times \sum_{k=0}^{n-i} \frac{(i-n)_k}{k!} \frac{(\mu+n+i)_k (1-N+i)_k (1+\beta+i)_k}{(2i+\lambda+1)_k (1-M+i)_k (1+\delta+i)_k} \\ &\times {}_3F_2 \left[\begin{matrix} -(n-i-k), \mu+n+k+i, -s \\ 1-M+i+k, 1+i+\delta+k \end{matrix} ;1 \right], \end{aligned} \tag{2.13}$$

with $a_n = 0$, $n > N$, and $\mu = \gamma + \delta + 1$ and $\lambda = \alpha + \beta + 1$. While for the case of Meixner, Kravchuk and Charlier polynomials [18]

$$a_n^{(p)} = \sum_{k=0}^{\infty} C_{p,n}(n+p+k)\, a_{n+p+k}, \quad n \geq 0, p \geq 0,$$

where

$$C_{p,k}(n) = \frac{(-1)^{n-p-k} n!}{(p-1)!} \frac{(n-k-1)!}{(n-p-k)!k!} (\theta+1)^{n-p-k}, \tag{2.14}$$

with $\theta = \frac{\mu}{1-\mu}$, $-p$, 0 , respectively.

Also, for each case of $P_j(x)$, they (see [18,19]) proved by the aid of recurrence relation (2.3) that the coefficients $a_{m,n}(j)$ in the expansion

$$x^m P_j(x) = \sum_{n=0}^{2m} a_{m,\,n}(j) P_{j+m-n}(x), \; j \geq 0, \; m \geq 0, \tag{2.15}$$

satisfy the recurrence relation

$$\begin{aligned} a_{m,n}(j) &= a_{m-1,n}(j) + \beta_{j+m-n} a_{m-1,n-1}(j) + \gamma_{j+m-n+1} a_{m-1,n-2}(j), \\ &\quad n = 0, 1, ..., 2m, \end{aligned}$$

with $a_{0,0}(j) = 1$, $a_{m-1,-\ell}(j) = 0$, $\forall \ell > 0$, $a_{m-1,r}(j) = 0$, $r = 2m-1, 2m$.

Remark 2.4. The explicit expression of $a_{m,n}(j)$ for the case of Hahn polynomials, is given in Doha and Ahmed [19, p. 792], while for the case of Meixner, Kravchuk and Charlier polynomials, the explicit expressions of $a_{m,n}(j)$ are given in Doha and Ahmed [18, p. 335].

Using formula (2.15), for each case of $P_j(x)$, Doha and Ahmed [18,19] proved the following theorem which expresses the $P_n(x)$ expansion coefficients of the moments of a general order difference derivative of an arbitrary function of a discrete variable in terms of its $P_n(x)$ original expansion coefficients.

Theorem 2.5 ([18, 19]). *Assume that $f(x)$, $f^{(p)}(x)$ and $x^m P_j(x)$ have the expansions (2.6), (2.7) and (2.15) respectively, and assume also that*

$$x^m \left(\sum_{i=0}^{\infty} a_i^{(p)} P_i(x) \right) = \sum_{i=0}^{\infty} b_i^{(p,m)} P_i(x) = I^{p,m},$$

say then the expansion coefficients $b_i^{p,m}$ *are given by*

$$b_i^{(p,m)} = \begin{cases} \sum_{k=0}^{m-1} a_{m\,,\,k+m-i}(k) a_k^{(p)} + \sum_{k=0}^{i} a_{m\,,\,k+2m-i}(k+m) a_{k+m}^{(p)}, & 0 \le i \le m, \\ \sum_{k=i-m}^{m-1} a_{m\,,\,k+m-i}(k) a_k^{(p)} + \sum_{k=0}^{i} a_{m\,,\,k+2m-i}(k+m) a_{k+m}^{(p)}, & m+1 \le i \le 2m-1, \\ \sum_{k=i-2m}^{i} a_{m\,,\,k+2m-i}(k+m) a_{k+m}^{(p)}, & i \ge 2m. \end{cases}$$

Corollary 2.6. *It is not difficult to show that*

$$b_i^{(p,m)} = \sum_{r=0}^{2m} a_{m,\,r}(r+i-m)\, a_{r+i-m}^{(p)},\ i \ge 0.$$

3. Some properties of product of classical discrete orthogonal polynomials

Comparing to the theory in one variable, the structure of discrete orthogonal polynomials in several variables is much more complicated. Some basic results are obtained in Area et al. [4–7, 9], Rodal et al. [35] and Xu [46]; the relevant ones will be recalled in this section. The orthogonal polynomials that satisfy the second-order partial difference equation

$$\begin{aligned} &A_{1,1}(x,y)\Delta_x\nabla_x u + A_{1,2}(x,y)\Delta_x\nabla_y u + A_{2,1}(x,y)\Delta_y\nabla_x u \\ &+ A_{2,2}(x,y)\Delta_y\nabla_y u\, B_1(x,y)\Delta_x u + B_2(x,y)\Delta_y u - \lambda u = 0, \end{aligned} \tag{3.1}$$

will be discrete orthogonal polynomials of two variables, where $A_{i,j}$ are polynomials of second degree and B_i are polynomials of the first degree, λ is a real number and

$$\nabla_x u(x,y) = u(x,y) - u(x-1,y), \qquad \nabla_y u(x,y) = u(x,y) - u(x,y-1).$$

In particular, for the special case $A_{1,2}(x,y) = A_{2,1}(x,y) = 0$, $A_{1,1}(x,y) = \sigma(x)$, $A_{2,2}(x,y) = \sigma(y)$, $B_1(x,y) = \tau(x)$, $B_2(x,y) = \tau(y)$ and $\lambda = -(n+m)$, Eq. (3.1) becomes

$$[\sigma(x)\Delta_x\nabla_x + \sigma(y)\Delta_y\nabla_y + \tau(x)\Delta_x + \tau(y)\Delta_y + (n+m)\,]F_{mn}(x,y) = 0, \tag{3.2}$$

where $F_{mn}(x,y) = P_m(x)P_n(y)$. These types of orthogonal polynomials satisfy the orthogonality relation

$$\sum_G F_{mn}(x_i,y_j)F_{ks}(x_i,y_j)\Omega(x_i,y_j) = \delta_{m\,k}\delta_{n\,s}\, d_m^2 d_n^2, \quad m,n,k,s = 0,1,2,..., \tag{3.3}$$

where $\Omega(x,y) = \rho(x)\rho(y),\ (x,y) \in G = \{(x,y) : x,\ y \in [a,b]\}$.
Replacing x and y by $x-1$ and $y-1$ in (3.2), respectively, and using

$$\begin{aligned}\Delta_x u(x-1,y) &= \nabla_x u(x,y),\\ \Delta_y u(x,y-1) &= \nabla_y u(x,y),\end{aligned}$$

we get

$$\begin{aligned}&[\sigma(x-1)\nabla_x^2 + \sigma(y-1)\nabla_y^2 + [\tau(x-1)-m]\nabla_x + [\tau(y-1)-n]\nabla_y\\ &\qquad\qquad +(m+n)]F_{mn}(x,y) = 0.\end{aligned}$$

Now, suppose a function $f(x,y)$ of two discrete variables x and y is given, which is formally expanded as follows

$$f(x,y) = \sum_{m,n=0}^{\infty} a_{mn} P_m(x) P_n(y), \tag{3.4}$$

and for the partial backward difference derivatives $f(x,y)$, i.e., $\nabla_x^p \nabla_y^q f(x,y)$,

$$f^{(p,q)}(x,y) = \nabla_x^p \nabla_y^q f(x,y) = \sum_{m,n=0}^{\infty} a_{mn}^{(p,q)} P_m(x) P_n(y),\ a_{mn}^{(0,0)} = a_{mn}. \tag{3.5}$$

In view of (2.5) with the two relations

$$\nabla_x \sum_{m,n=0}^{\infty} a_{mn}^{(p-1,q)} P_m(x) P_n(y) = \sum_{m,n=0}^{\infty} a_{mn}^{(p,q)} P_m(x) P_n(y),$$

and

$$\nabla_y \sum_{m,n=0}^{\infty} a_{mn}^{(p,q-1)} P_m(x) P_n(y) = \sum_{m,n=0}^{\infty} a_{mn}^{(p,q)} P_m(x) P_n(y),$$

it is not difficult to derive the recurrences

$$\frac{1}{m} a_{m-1,n}^{(p,q)} + (\,F_m + 1)\ a_{m,n}^{(p,q)} + G_{m+1}\, a_{m+1,n}^{(p,q)} = a_{m,n}^{(p-1,q)}, \quad m,\ p \geq 1,\ n,\ q \geq 0\,, \tag{3.6}$$

and

$$\frac{1}{n} a_{m,n-1}^{(p,q)} + (\,F_n + 1)\ a_{m,n}^{(p,q)} + G_{n+1}\, a_{m,n+1}^{(p,q)} = a_{m,n}^{(p,q-1)}, \quad n,\ q \geq 1,\ m,\ p \geq 0. \tag{3.7}$$

4. Relation between the coefficients $a_{m,n}^{(p,q)}$ and $a_{m,n}$ and explicit formula for the expansion coefficients of $x^{m_1} y^{m_2}\, \nabla_x^p \nabla_y^q f(x,y)$

The main objective of this section is to state and prove two theorems such that the first expresses the relation between the coefficients $a_{m,n}^{(p,q)}$ and $a_{m,n}$, whereas the second expresses the $P_m(x)P_n(y)$ expansion coefficients of the moments of general order difference derivatives of an arbitrary function of two discrete variables in terms of its $P_m(x)P_n(y)$ original expansion coefficients.

Theorem 4.1. *The coefficients $a_{mn}^{(p,q)}$ are related to the coefficients $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and the original coefficients a_{mn} by*

$$a_{mn}^{(p,q)} = \sum_{i=0}^{\infty} C_{p,m}(p+m+i) a_{p+m+i,n}^{(0,q)},\ p \geq 1, \tag{4.1}$$

$$a_{mn}^{(p,q)} = \sum_{j=0}^{\infty} C_{q,n}(q+n+j) a_{m,q+n+j}^{(p,0)},\ \ q \geq 1\,, \tag{4.2}$$

and

(4.3)

$$a_{mn}^{(p,q)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{p,m}(p+m+i)\ C_{q,n}(q+n+j)\, a_{p+m+i,\,q+n+j},\ p,q \geq 1,\ m,n \geq 0,$$

where the coefficients $C_{p,m}(p+m+i)$ and $C_{q,n}(q+n+j)$ can be defined according to formula (2.13) for the case of Hahn and formula (2.14) for the cases of Meixner, Kravchuk and Charlier polynomials.

Proof. We can write equation (3.5) as

$$f^{(p,q)}(x,y) = \sum_{m=0}^{\infty} b_{m,n}^{(p,q)}(y) P_m(x),$$

where

$$b_{m,n}^{(p,q)}(y) = \sum_{n=0}^{\infty} a_{mn}^{(p,q)} P_n(y), \tag{4.4}$$

and keeping y and q fixed. In view of formula (2.10), we can deduce that

$$b_{m,n}^{(p,q)}(y) = \sum_{i=0}^{\infty} C_{p,m}(p+m+i) b_{m+i+p,n}^{(0,q)}(y). \tag{4.5}$$

Using (4.4) and (4.5) yields the formula

$$\begin{aligned}\sum_{n=0}^{\infty} a_{mn}^{(p,q)} P_n(y) &= \sum_{i=0}^{\infty} C_{p,m}(p+m+i) \left[\sum_{n=0}^{\infty} a_{m+i+p,n}^{(0,q)} P_n(y)\right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{\infty} C_{p,m}(p+m+i)\, a_{m+i+p,n}^{(0,q)}\right] P_n(y),\end{aligned}$$

which implies that

$$a_{mn}^{(p,q)} = \sum_{i=0}^{\infty} C_{p,m}(p+m+i)\, a_{p+m+i,n}^{(0,q)},\ p \geq 1,$$

and the proof of formula (4.1) is complete.

It can be also shown that formula (4.2) is true by the same procedure as outlined for (4.1), keeping x and p fixed. Formula (4.3) is obtained immediately by substituting (4.2) into (4.1). This completes the proof. $\square$

Theorem 4.2. *Assume that $f(x,y)$ and $f^{(p,q)}(x,y)$ have the expansions (3.4) and (3.5) respectively, and suppose that*

$$x^{m_1} P_i(x) = \sum_{\ell_1=0}^{2m_1} a_{m_1,\,\ell_1}(i) P_{i+m_1-\ell_1}(x), \tag{4.6}$$

$$y^{m_2} P_j(y) = \sum_{\ell_2=0}^{2m_2} a_{m_2,\,\ell_2}(j) P_{j+m_2-\ell_2}(y), \tag{4.7}$$

and

$$\begin{aligned} x^{m_1} y^{m_2} \left(\sum_{i,j=0}^{\infty} a_{i,j}^{(p,q)} P_i(x) P_j(y) \right) &= \sum_{i,j=0}^{\infty} b_{i,j}^{(p,q,m_1,m_2)} P_i(x) P_j(y) \\ &= I^{(p,q,m_1,m_2)}, \text{ say}, \end{aligned} \tag{4.8}$$

then the expansion coefficients $b_{i,j}^{(p,q,m_1,m_2)}$ are given by

$$\begin{aligned} b_{i,\,j}^{(p,q,m_1,m_2)} = \sum_{\ell_1=0}^{2m_1} \sum_{\ell_2=0}^{2m_2} & a_{m_1,\ell_1}(\ell_1+i-m_1)\, a_{m_2,\,\ell_2}(\ell_2+j-m_2) \\ & \times\, a_{\ell_1+i-m_1,\ell_2+j-m_2}^{(p,q)}, \ i,\, j \geq 0. \end{aligned} \tag{4.9}$$

Proof. In view of Corollary 2.6 and formula (4.6), we get

$$\begin{aligned} I^{(p,q,m_1,m_2)} &= y^{m_2} \sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{i,j}^{(p,q)} x^{m_1} P_i(x) \right) P_j(y) \\ &= y^{m_2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j}^{(p,q,m_1)} P_i(x)\, P_j(y), \end{aligned} \tag{4.10}$$

where

$$b_{i,j}^{(p,q,m_1)} = \sum_{\ell_1=0}^{2m_1} a_{m_1,\,\ell_1}(\ell_1+i-m_1)\, a_{\ell_1+i-m_1,\,j}^{(p,q)}. \tag{4.11}$$

In view of Corollary 2.6 and formula (4.7), Equation (4.10) takes the form

$$
\begin{aligned}
I^{(p,q,m_1,m_2)} &= \sum_{i=0}^{\infty} P_i(x) \left(\sum_{j=0}^{\infty} b_{i,j}^{(p,q,m_1)} y^{m_2} P_j(y) \right) \\
&= \sum_{i,j=0}^{\infty} b_{i,j}^{(p,q,m_1,m_2)} P_i(x) P_j(y),
\end{aligned}
$$

where

$$
b_{i,j}^{(p,q,m_1,m_2)} = \sum_{\ell_2=0}^{2m_2} a_{m_2,\,\ell_2}(\ell_2 + j - m_2)\, b_{i,\,\ell_2+j-m_2}^{(p,q,m_1)}. \tag{4.12}
$$

By substituting (4.11) into (4.12), we obtain (4.9). □

5. Construction of recurrence relations for the coefficients of expansions in series of the product of two classical discrete orthogonal polynomials

Let $f(x, y)$ have the expansion (3.4), and assume that it satisfies the linear nonhomogeneous partial difference equation

$$
\sum_{i=0}^{m} \sum_{j=0}^{n} p_{i,j}(x,y) f^{(i,j)}(x,y) = g(x,y), \tag{5.1}
$$

where $p_{i,j}(x, y)$, $i = 0, 1, ..., m,\ j = 0, 1, ..., n$, are polynomials in x and y such that $p_{m,0}, p_{0,n} \neq 0$, and the coefficients in the expansion

$$
g(x,y) = \sum_{i,j=0}^{\infty} g_{i,j} P_i(x) P_j(y),
$$

are known, then Theorems 4.1 and 4.2 enable one to construct in view of equation (5.1) the linear recurrence relation of order (d_1, d_2),

$$
\sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \alpha_{i,j}(r,k) a_{r+i,j+k} = \beta(r,k), \quad r, k \geq 0, \tag{5.2}
$$

where $\alpha_{i,j}(r, k)$, $i = 0, 1, ..., d_1$, $j = 0, 1, ..., d_2$, are polynomials in r and k such that $\alpha_{d_1,0}(r, k)$, $\alpha_{0,d_2}(r, k) \neq 0$.

An example dealing with a nonhomogeneous partial difference equation of two discrete variables is considered in order to clarify application of the results obtained.

Example 5.1. Consider the nonhomogeneous partial difference equation

$$x\nabla_x u - y\nabla_y u + (x-y)u = d(x,y), \quad u(0,y)=y,\ u(x,0)=x. \tag{5.3}$$

where $d(x,y) = x^2 - y^2 + x - y$ and $u(x,y)$ are expanded as follows

$$u(x,y) = \sum_{i,j=0}^{\infty} a_{i,j}\,\tilde{C}_i^{(\alpha)}(x)\tilde{C}_j^{(\alpha)}(y), \qquad d(x,y) = \sum_{i+j\leq 2} d_{i,j}^{(\alpha)}\tilde{C}_i(x)\tilde{C}_j^{(\alpha)}(y), \tag{5.4}$$

then by virtue of formula (4.8), equation (5.3) takes the form

$$I^{(1,0,1,0)} - I^{(0,1,0,1)} + I^{(0,0,1,0)} - I^{(0,0,0,1)} = \sum_{i+j\leq 2} d_{i,j}\tilde{C}_i^{(\alpha)}(x)\tilde{C}_j^{(\alpha)}(y),$$

which in turn gives

$$b_{i,j}^{(1,0,1,0)} - b_{i,j}^{(0,1,0,1)} + b_{i,j}^{(0,0,1,0)} - b_{i,j}^{(0,0,0,1)} = d_{i,j},\ i,j = 0,1,2,\dots, \tag{5.5}$$

where

$$d_{i,j} = \begin{cases} -2(\alpha+1), & i=0,\ j=1,\\ -1, & i=0, j=2,\\ 2(\alpha+1), & i=1,\ j=0,\\ 1, & i=2,\ j=0,\\ 0, & \text{otherwise.}\end{cases}$$

By making use of formula (4.9), equation (5.5) takes the form

$$\begin{aligned}(i-j)a_{i,j} + {}& \alpha(i+1)(a_{i+1,j}^{(1,0)} + a_{i+1,j}) + (i+\alpha)a_{i,j}^{(1,0)} + (a_{i-1,j}^{(1,0)} + a_{i-1,j})\\ &-\alpha(j+1)(a_{i,j+1}^{(0,1)} + a_{i,j+1}) - (j+\alpha)a_{i,j}^{(0,1)} - (a_{i,j-1}^{(0,1)} + a_{i,j-1}) = d_{i,j}.\end{aligned} \tag{5.6}$$

Using formulae (4.1) and (4.2) with (5.6)− and after some manipulation-yield the following recurrence relation.

$$\begin{aligned}&a_{i,j} + 2(i-j+1)a_{i+1,j} + 2\alpha(i+2)a_{i+2,j} - a_{i+1,j-1} - 2\alpha(j+1)a_{i+1,j+1}\\ &= d_{i+1,j}, \quad i,j\geq 0.\end{aligned} \tag{5.7}$$

The complete solution of this example, may be obtained by solving the recurrence relation (5.7). It is worthy noting that the analytical solution for this recurrence relation is given explicitly by [see Appendix A]

$$a_{i,j} = \begin{cases} \frac{\alpha^2}{4}e^{-\alpha} + 2\alpha, & i=0,\ j=0,\\ \frac{\alpha(2-\alpha)}{8}e^{-\alpha} + 1, & i=1,\ j=0,\\ \frac{\alpha(2-\alpha)}{8}e^{-\alpha} + 1, & i=0,\ j=1,\\ \frac{(-2)^{-(i+j+2)}(2i-\alpha)(2j-\alpha)}{i!j!}e^{-\alpha}, & \text{otherwise.}\end{cases} \tag{5.8}$$

Note 1. The solution of this example can also be obtained by using the Hahn, Mexiner and Kravchuk polynomials, but details are not given here.

6. The expansion of $(x+y)^{\underline{n}}$ in series of the product of two classical discrete orthogonal polynomials

In the problem

$$(x+y)^{\underline{n}} = \sum_{i+j\le n} a_{i,j}(n) P_i(x) P_j(y), \tag{6.1}$$

$u(x,y) = (x+y)^{\underline{n}}$ satisfies the homogeneous partial difference equation

$$[x\nabla_x + y\,\nabla_y - n\,]\, u(x,y) = 0. \tag{6.2}$$

By virtue of formula (4.8), equation (6.2) takes the form

$$I^{(1,0,1,0)} + I^{(0,1,0,1)} - n\, I^{(0,0,0,0)} = 0,$$

which in turn gives

$$b_{i,\,j}^{(1,0,1,0)} + b_{i,\,j}^{(0,1,0,1)} - n\, b_{i,\,j}^{(0,0,0,0)} = 0, \;\; i,\, j = 0,1,2,\dots . \tag{6.3}$$

By making use of formula (4.9), equation (6.3) takes the form

(6.4)

$$a_{i-1,j}^{(1,0)}(n) + (\beta_i+1) a_{i,j}^{(1,0)}(n) + \gamma_{i+1} a_{i+1,j}^{(1,0)}(n) + \, a_{i,j-1}^{(0,1)}(n) + \,(\beta_j+1)\, a_{i,j}^{(0,1)}(n)$$
$$+\gamma_{j+1} a_{i,j+1}^{(0,1)}(n) - n\, a_{i,j}(n) = 0.$$

Now, using formulae (4.1) and (4.2) with (6.4)− and after some manipulation-yield the recurrence relation satisfied by $a_{i,j}(n)$.

Note 2. Repeated use of (3.6) and (3.7) to eliminate the coefficients $a_{i\pm1,j}^{(1,0)}(n)$, $a_{i,j}^{(1,0)}(n)$, $a_{i,j\pm1}^{(0,1)}(n)$ and $a_{i,j}^{(0,1)}(n)$ yields the recurrence relation satisfied by $a_{i,j}(n)$.

6.1. The link between $(x+y)^{\underline{n}}$ and Charlier-Charlier polynomials.

In the problem

$$(x+y)^{\underline{n}} = \sum_{i+j\le n} a_{i,j}(n) \tilde{C}_i^{(\alpha)}(x) \tilde{C}_j^{(\alpha)}(y), \tag{6.5}$$

the coefficients $a_{i,j}(n)$ satisfy the recurrence relation

$$\begin{aligned}(n-i-j)a_{i,j}(n) - \alpha(j+1)\, a_{i,j+1}(n) - \alpha(i+1)\, a_{i+1,j}(n) = 0,\\ i,\, j = n-1, n-2, ..., 0,\end{aligned} \tag{6.6}$$

with $a_{i,j}(n) = 0$, $i+j > n$ and $a_{n,0}(n) = \; a_{0,n}(n) = 1$. The solution of (6.6) is given by [see Appendix B]

$$a_{i,j}(n) = \begin{cases} \dfrac{(-1)^{i+j}}{i!j!}(-n)_{i+j}(2\alpha)^{n-i-j}, & i+j\le n,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, for the special case $y=0$, Eq. (6.5), after some manipulation, becomes

$$x^{\underline{n}}=\sum_{i=0}^{n}\binom{n}{i}\alpha^{n-i}\tilde{C}_i^{(\alpha)}(x),$$

which is in agreement with the result obtained by Àlvarez-Nodarse et al. [3, p. 180] and Zarzo et al. [52, p. L38].

6.2. The link between $(x+y)^{\underline{n}}$ and Meixner-Meixner polynomials.

In the problem

$$(x+y)^{\underline{n}}=\sum_{i+j\le n}a_{i,j}(n)\tilde{M}_i^{(\alpha,\beta)}(x)\tilde{M}_j^{(\alpha,\beta)}(y), \tag{6.7}$$

the coefficients $a_{i,j}(n)$ satisfy the recurrence relation

$$\begin{aligned}&(\beta-1)(n-i-j)a_{i,j}(n)+\beta(i+1)(i+\alpha)a_{i+1,j}(n)\\&+\beta(j+1)(j+\alpha)a_{i,j+1}(n)=0,\quad i,\,j=n-1,n-2,...,0,\end{aligned} \tag{6.8}$$

with $a_{i,j}(n)=0$, $i+j>n$ and $a_{n,0}(n)=a_{0,n}(n)=1$. The solution of (6.8) is

$$a_{i,j}(n)=\begin{cases}\dfrac{(-1)^{i+j}(-n)_{i+j}(2\alpha)_n}{i!j!(2\alpha)_{i+j}}\left(\dfrac{\beta}{1-\beta}\right)^{n-i-j}, & i+j\le n,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, for the special case $y=0$, Eq. (6.7), after some manipulation, becomes

$$x^{\underline{n}}=\sum_{i=0}^{n}\binom{n}{i}(\alpha+i)_{n-i}\left(\frac{\beta}{1-\beta}\right)^{n-i}\tilde{M}_i^{(\alpha,\beta)}(x),$$

which is in agreement with the result obtained by Àlvarez-Nodarse et al. [3, p. 180] and Zarzo et al. [52, p. L38].

6.3. The link between $(x+y)^{\underline{n}}$ and Kravchuk-Kravchuk polynomials.

In the problem

$$(x+y)^{\underline{n}}=\sum_{i+j\le n}a_{i,j}(n)\tilde{K}_i^{(s)}(x;N)\tilde{K}_j^{(s)}(y;N),\ \ n\le N, \tag{6.9}$$

the coefficients $a_{i,j}(n)$ satisfy the recurrence relation

$$\begin{aligned}&(n-i-j)a_{i,j}(n)-s(i+1)(N-i)a_{i+1,j}(n)\\&-s(j+1)(N-j)a_{i,j+1}(n)=0,\quad i,\,j=n-1,n-2,...,0,\end{aligned} \tag{6.10}$$

with $a_{i,j}(n)=0$, $i+j>n$ and $a_{n,0}(n)=a_{0,n}(n)=1$. The solution of (6.10) is

$$a_{i,j}(n)=\begin{cases}\dfrac{(-1)^{i+j}s^{n-i-j}(-n)_{i+j}(2N-i-j)!}{i!j!(2N-n)!}, & i+j\le n,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, for the special case $y=0$, Eq. (6.9), after some manipulation, becomes

$$x^{\underline{n}} = \sum_{i=0}^{n} \binom{n}{i} s^{n-i} (N-n+1)_{n-i}\, \tilde{K}_i^{(s)}(x;N),\ n \le N,$$

which is in agreement with the result obtained by Àlvarez-Nodarse et al. [3, p. 180] and Zarzo et al. [52, p. L38], .

6.4. **The link between $(x+y)^{\underline{n}}$ and Hahn-Hahn polynomials.**

The problem

$$(x+y)^{\underline{n}} = \sum_{i+j\le n} a_{i,j}(n)\, \tilde{h}_i^{(0,0)}(x;N)\, \tilde{h}_j^{(0,0)}(y;N),\ \ n \le N-1, \tag{6.11}$$

where $\tilde{h}_n^{(0,0)}(x;N) = t_n(x)$ is an important special case of the Hahn polynomials which is called the Chebyshev polynomials of a discrete variable (see [32, p. 33]). The coefficients $a_{i,j}(n)$ satisfy the recurrence relation

(6.12)

$$\begin{aligned}
&\gamma_{i,j}\ a_{i,j}(n) + \gamma_{i+1,,j-1}\ a_{i+1,j-1}(n) + \gamma_{i+1,,j}\ a_{i+1,j}(n) + \gamma_{i+1,,j+1}\ a_{i+1,j+1}(n)\\
&+ \gamma_{i+2,,j-2}\ a_{i+2,j-2}(n) + \gamma_{i+2,,j-1}\ a_{i+2,j-1}(n) + \gamma_{i+2,,j}\ a_{i+2,j}(n)\\
&+ \gamma_{i+2,,j+1}\ a_{i+2,j+1}(n) + \gamma_{i+2,,j+2}\ a_{i+2,j+2}(n) = 0,\ i,j = n-1, n-2, ..., 0,
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{i,j} &= -16(2i+3)(2i+5)(2j+3)(2j+5)(n-i-j),\\
\gamma_{i,,j+1} &= 8(j+1)(2i+3)(2i+5)(2j+3)(2j+5)(N-n+i),\\
\gamma_{i,,j+2} &= -4(j+1)_2(2i+3)(2i+5)(-N+j+2)(N+j+2)(n+j-i+3),\\
\gamma_{i+1,,j} &= 8(i+1)(2i+3)(2i+5)(2j+3)(2j+5)(N-n+j),\\
\gamma_{i+1,,j+1} &= 4(i+1)(j+1)(2i+3)(2i+5)(2j+3)(2j+5)(2N-n),\\
\gamma_{i+1,,j+2} &= -2(i+1)(j+1)_2(2i+3)(2i+5)(2j+5)\\
&\quad \times (-N+j+2)(N+j+2)(-N+n+j+3),\\
\gamma_{i+2,,j} &= -4(i+1)_2(2j+3)(2j+5)(-N+i+2)(N+i+2)(n+i-j+3),\\
\gamma_{i+2,,j+1} &= -2(i+1)_2(j+1)(2j+3)(2j+5)(-N+i+2)(N+i+2)\\
&\quad \times (n+i-N+3),\\
\gamma_{i+2,,j+2} &= -(i+1)_2(j+1)_2(-N+i+2)(N+i+2)(N+j+2)\\
&\quad \times (n+i+j+6),
\end{aligned}$$

with $a_{i,j}(n) = 0,\ i+j > n$ and $a_{n,0}(n) = a_{0,n}(n) = 1$. The solution of (6.12) is

$$\begin{aligned} a_{i,j}(n) =& \binom{n}{i}\binom{n-i}{j} \frac{(N-n+j)_{n-i-j}(i+1)_{n-i-j}}{(2i+2)_{n-i-j}} \\ & \times\ {}_4F_3\left[\begin{matrix} -n+i+j,\ j-N+1,\ \ j-(n+i+1),\ j+1 \\ N-n+j,\ \ j-n, 2j+2 \end{matrix};1\right], \\ & i+j \leq n. \end{aligned}$$

In particular, for the special case $y = 0$, equation (6.11) becomes

$$x^{\underline{n}} = \sum_{i=0}^{n} c_i^{(n)} \tilde{h}_i^{(0,0)}(x;N), \tag{6.13}$$

where

$$c_i^{(n)} = \sum_{j=0}^{n-i} a_{i,j}(n)\, \tilde{h}_j^{(0,0)}(0;N), \quad \tilde{h}_j^{(0,0)}(0;N) = \frac{(1)_j(1-N)_j}{(j+1)_j}. \tag{6.14}$$

The formula (6.14), after some manipulation, can be written in the form

$$\begin{aligned} c_{n-i}^{(n)} =& \binom{n}{i} \frac{(N-n)_i(n-i+1)_{n-i}}{(2n-2i+2)_i} \\ & \times \sum_{j=0}^{i} \binom{i}{j} \frac{(a_1)_j(a_2)_j(\alpha_1)_j}{(j+\nu)_j(b_1)_j(b_2)_j} \\ & \times_4 F_3\left[\begin{matrix} -i+j,\ j+a_1,\ \ j+a_2,\ j+\alpha_1 \\ 2j+\nu+1, j+b_1, j+b_2 \end{matrix};1\right], \end{aligned}$$

where $\nu = 1, a_1 = 1-N, a_2 = 1, b_1 = N-n, b_2 = -n,\ \alpha_1 = -2n+i-1$. Using relation (4) in Sánchez-Ruiz [40, p. 262], (see also, Luke [30], Volume II, p. 7) with suitable choices of the parameters ($p = q = 2, r = s = u = 0,\ z = 1,\ \omega = 0$), one can obtain the following identity

$$\sum_{j=0}^{i} \binom{i}{j} \frac{(a_1)_j(a_2)_j(\alpha_1)_j}{(j+\nu)_j(b_1)_j(b_2)_j}\ {}_4F_3\left[\begin{matrix} -i+j,\ j+a_1,\ \ j+a_2,\ j+\alpha_1 \\ 2j+\nu+1,\ j+b_1, j+b_1 \end{matrix};1\right] = 1,$$

then the form of $c_i^{(n)}$ becomes

$$c_i^{(n)} = \binom{n}{i} \frac{(N-n)_{n-i}(i+1)_{n-i}}{(2i+2)_{n-i}},$$

and hence Eq. (6.13) takes the form

$$x^{\underline{n}} = \sum_{i=0}^{n} \binom{n}{i} \frac{(N-n)_{n-i}(i+1)_{n-i}}{(2i+2)_{n-i}} \tilde{h}_i^{(0,0)}(x;N),$$

which is in agreement with the result obtained by Àlvarez-Nodarse et al. [3, p. 181] with $\alpha = \beta = 0$.

Note 3. It is worth mentioning here the problem

$$(x+y)^{\underline{n}} = \sum_{i+j\le n} a_{i,j}(n)\, \tilde{h}_i^{(\alpha,\beta)}(x;N)\, \tilde{h}_j^{(\alpha,\beta)}(y;N), \quad n \le N-1, \tag{6.15}$$

could be also considered in a similar way as in the above example. In fact, the corresponding recurrence relation for the expansion coefficients $a_{i,j}(n)$ can be obtained by inserting the data of monic Hahn polynomials $\tilde{h}_i^{(\alpha,\beta)}(x;N)$. However, this recurrence relation is very lengthy, but the expansion coefficients $a_{i,j}(n)$ in (6.15) has the formula:

$$\begin{aligned} a_{i,j}(n) =& \binom{n}{i}\binom{n-i}{j} \frac{(N-n+j)_{n-i-j}(i+\beta+1)_{n-i-j}}{(2i+\lambda+1)_{n-i-j}} \\ & \times\ {}_4F_3\left[\begin{matrix} -n+i+j,\ j-N+1,\ \beta+j+1,\ j-(\lambda+n+i) \\ 2j+\lambda+1,\ N-n+j,\ j-n-\beta, \end{matrix} ;1\right], \\ & i+j \le n. \end{aligned}$$

Moreover, for the special case $y=0$, by following the same manipulation and using relation (4) in [40, p. 262] with suitable choices of the parameters ($p=q=2$, $r=s=u=0$, $z=1$, $\omega=0$, $\nu=\lambda$, $a_1=1-N$, $a_2=1+\beta, b_1=N-n, b_2=-n-\beta, \alpha_1=\lambda-2n+i-1$), one can obtain

$$x^{\underline{n}} = \sum_{i=0}^{n} \binom{n}{i} \frac{(N-n)_{n-i}(i+\beta+1)_{n-i}}{(2i+\lambda+1)_{n-i}} \tilde{h}_i^{(\alpha,\beta)}(x;N),$$

which is in agreement with the result obtained by Àlvarez-Nodarse et al. [3, p. 180], and Zarzo et al. [52, p. L38], .

7. Connection problem in the sense of the product of two classical discrete orthogonal polynomials

The connection problem in this case is to determine the coefficients $a_{i,j}(n)$ in

$$Q_n(x+y) = \sum_{i+j\le n} a_{i,j}(n) P_i(x) P_j(y),$$

where $Q_n(x)$ is a classical orthogonal polynomial of a discrete variable.

We know that the classical orthogonal polynomials of a discrete variable satisfy the second order difference equation of the form (2.2), then we get

$$\left[\sigma(x+y-1)\nabla_x^2 + [\tau(x+y-1)-\lambda_n]\nabla_x + \lambda_n\right] Q_n(x+y) = 0.$$

7.1. **The Charlier− Charlier-Charlier connection problem.**

In the problem

$$\tilde{C}_n^{(\gamma)}(x+y) = \sum_{i+j\le n} a_{i,j}(n)\tilde{C}_i^{(\alpha)}(x)\tilde{C}_j^{(\alpha)}(y), \tag{7.1}$$

where $\tilde{C}_n^{(\gamma)}(x+y)$ satisfies the difference equation

$$[(x+y-1)\nabla_x^2 + (\gamma - x - y - n + 1)\nabla_x + n]\tilde{C}_n^{(\gamma)}(x+y) = 0,$$

the coefficients $a_{i,j}(n)$ satisfy the partial difference equation

(7.2)
$$\begin{aligned}&(n-i)a_{i,j}(n) - (i+1)a_{i+1,j-1}(n) - (i+1)(1+i+j-n+2\alpha-\gamma)a_{i+1,j}(n)\\ &-\alpha(i+1)(j+1)a_{i+1,j+1}(n) - (\alpha-\gamma)(i+1)(i+2)a_{i+2,j}(n) = 0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = n-1, n-2, ..., 0,\end{aligned}$$

with $a_{i,j}(n) = 0$ if $i+j > n$, $a_{-1,j}(n) = a_{i,-1}(n) = 0$ and $a_{n,0}(n) = a_{0,n}(n) = 1$.

The solution of (7.2) is [see Appendix C]

$$a_{i,j}(n) = \begin{cases} \dfrac{(-1)^{i+j}}{i!j!}(-n)_{i+j}\,(2\alpha-\gamma)^{n-i-j}, & i+j\le n,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, for the special case $y = 0$, Eq. (7.1), after some manipulation, becomes

$$\tilde{C}_n^{(\gamma)}(x) = \sum_{i=0}^{n} \binom{n}{i} (\alpha-\gamma)^{n-i}\ \tilde{C}_i^{(\alpha)}(x),$$

which is in agreement with the result obtained by Àlvarez-Nodarse et al. [3, p. 187], and Area et al. [8, p. 316].
For the case $\gamma = 2\alpha$, we get

$$\tilde{C}_n^{(2\alpha)}(x+y) = \sum_{i=0}^{n} \binom{n}{i} \tilde{C}_i^{(\alpha)}(x)\tilde{C}_{n-i}^{(\alpha)}(y).$$

7.2. **The Meixner− Meixner-Meixner connection problem.**

In the problem

$$\tilde{M}_n^{(\alpha,\beta)}(x+y) = \sum_{i+j\le n} a_{i,j}(n)\tilde{M}_i^{(\gamma,\beta)}(x)\tilde{M}_j^{(\gamma,\beta)}(y), \tag{7.3}$$

where $\tilde{M}_n^{(\alpha,\beta)}(x+y)$ satisfies the difference equation

$$[(x+y-1)\nabla_x^2 + [\alpha\beta + (\beta-1)(x+y+n-2)]\nabla_x + n(1-\beta)]\tilde{M}_n^{(\alpha,\beta)}(x+y) = 0,$$

the coefficients $a_{i,j}(n)$ satisfy the recurrence relation

(7.4)
$$\begin{aligned}
&(\beta-1)^2(n-i)a_{i,j}(n)-(\beta-1)^2(i+1)a_{i+1,j-1}(n)\\
&-\beta(i+1)(j+1)(j+\gamma)a_{i+1,j+1}(n)-(\beta-1)(i+1)(n-2\beta-(\beta+1)(i+j)\\
&+\beta(n+\alpha-2\gamma))a_{i+1,j}(n)-(i+1)_2(\beta(i-n-\alpha+\gamma+3)-1)a_{i+2,j}(n)=0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad j=n-1,n-2,...,0,
\end{aligned}$$

with $a_{i,j}(n)=0$, $i+j>n$, $a_{-1,\,j}(n)=a_{i,-1}(n)=0$ and $a_{n,0}(n)=a_{0,n}(n)=1$. The solution of (7.4) is given by

$$a_{i,j}(n)=\begin{cases}\left(\dfrac{\beta}{\beta-1}\right)^{n-i-j}\dfrac{(-1)^{i+j}(\alpha)_n(-n)_{i+j}(\alpha-2\gamma)_{n-i-j}}{i!j!(\alpha)_{i+j}(\alpha+i+j)_{n-i-j}}, & i+j\leq n,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, for the special case $y=0$, equation (7.3), after some manipulation, becomes

$$\tilde{M}_n^{(\alpha,\beta)}(x)=\sum_{i=0}^{n}\binom{n}{i}(\alpha-\gamma)_{n-i}\left(\frac{\beta}{\beta-1}\right)^{n-i}\tilde{M}_i^{(\gamma,\beta)}(x),$$

which agrees with the result obtained by Àlvarez-Nodarse et al. [3, p. 188], and Area et al. [8, p. 318].

For the case $\alpha=2\gamma$, we get

$$\tilde{M}_n^{(2\gamma,\beta)}(x+y)=\sum_{i=0}^{n}\binom{n}{i}\tilde{M}_i^{(\gamma,\beta)}(x)\tilde{M}_{n-i}^{(\gamma,\beta)}(y).$$

7.3. **The Kravchuk− Kravchuk-Kravchuk connection problem.**
In the problem

$$\tilde{K}_n^{(s)}(x+y;N)=\sum_{i+j\leq n}a_{i,j}(n)\tilde{K}_i^{(s)}(x;N)\tilde{K}_j^{(s)}(y;N),\ n\leq N, \tag{7.5}$$

where $\tilde{K}_n^{(s)}(x+y;N)$ satisfies the difference equation

$$[(1-s)(x+y-1)\nabla_x^2+[N\,s-(x+y+n-1)]\nabla_x+n]\tilde{K}_n^{(s)}(x+y;N)=0,$$

the coefficients $a_{i,j}(n)$ satisfy the recurrence relation

(7.6)
$$\begin{aligned}
&(n-i)a_{i,j}(n)-(i+1)a_{i+1,j-1}(n)+s(s-1)(i+1)(j+1)(N-j)a_{i+1,j+1}(n)\\
&-(i+1)(1+(1-2s)(i+j)-n+s(N+2n-2))a_{i+1,j}(n)\\
&+s(s-1)(i+1)_2(n-i-2)a_{i+2,j}(n)=0,\ i,\ j=n-1,n-2,...,0,
\end{aligned}$$

with $a_{i,j}(n)=0$, $i+j>n$, $a_{-1,j}(n)=a_{i,-1}(n)=0$ and $a_{n,0}(n)=a_{0,n}(n)=1$. The solution of (7.6) is given by

$$a_{i,j}(n)=\begin{cases} s^{n-i-j}\dfrac{(-1)^{i+j}(-N)_n(-n)_{i+j}(N)_{n-i-j}}{i!j!(-N)_{i+j}(-N+i+j)_{n-i-j}}, & i+j\le n,\\ 0, & \text{otherwise.}\end{cases}$$

7.4. **The Hahn− Hahn-Hahn connection problem.**

In the problem

(7.7)
$$\tilde{h}_n^{(\gamma,\delta)}(x+y;M)=\sum_{i+j\le n}a_{i,j}(n)\tilde{h}_i^{(0,0)}(x;N)\tilde{h}_j^{(0,0)}(y;N),\ n\le\min\{N-1,M-1\},$$

where $\tilde{h}_n^{(\gamma,\delta)}(x+y;M)$ satisfies the difference equation

$$\begin{aligned}&[(x+y-1)(M+\gamma-x-y+1)\nabla_x^2+[(\delta+1)(M-1)-(\mu+1)(x+y-1)]\nabla_x\\&+n(\mu+n)]\tilde{h}_n^{(\gamma,\delta)}(x+y;M)=0,\ \mu=\gamma+\delta+1,\end{aligned}$$

the coefficients $a_{i,j}(n)$ satisfy the recurrence relation

(7.8)
$$\begin{aligned}&\gamma_{i,j}\ a_{i,j}(n)+\gamma_{i+1,,j-1}\ a_{i+1,j-1}(n)+\gamma_{i+1,,j}\ a_{i+1,j}(n)+\gamma_{i+1,,j+1}\ a_{i+1,j+1}(n)\\&+\gamma_{i+2,,j-2}\ a_{i+2,j-2}(n)+\gamma_{i+2,,j-1}\ a_{i+2,j-1}(n)+\gamma_{i+2,,j}\ a_{i+2,j}(n)\\&+\gamma_{i+2,,j+1}\ a_{i+2,j+1}(n)+\gamma_{i+2,,j+2}\ a_{i+2,j+2}(n)=0,\ i,j=n-1,n-2,...,0,\end{aligned}$$

where

$$\begin{aligned}\gamma_{i,j}&=(i+3)_2(\mu+n+i)(n-i),\\ \gamma_{i+1,j-1}&=-(i+1)(i+3)_2(2i+\mu+1),\\ \gamma_{i+2,j-2}&=-(i+1)_4,\\ \gamma_{i+2,j-1}&=\frac{1}{2}(i+1)_4(5+2M-N(N+2)+\gamma-\delta),\\ \gamma_{i+1,j}&=\frac{1}{4}(i+1)(i+4)\,[i^2(4M+4(\gamma-1)-N(\mu+5))+2((1-6N)(\mu+1)\\&\quad+6n(n+\mu))+i(12M-N(19+7\mu)+2(-8+5\gamma-\delta+2n(n+\mu)))],\\ \gamma_{i+1,j+1}&=(i+1)(i+3)_2[-2i+[(1+j)^2(j-N+1)(j+N+1)(\mu+1)]\\&\quad\times[4(3+4j(j+2))]^{-1}],\\ \gamma_{i+2,j}&=\frac{1}{16}(i+1)_4\,[-16(M+1)-8(N-1)^2+16\,n\,(n+\mu)\\&\quad+4(\mu+n(n+\mu)-3)-16\gamma+8(N-1)(M-\delta)\\&\quad+4(N-1)(2M-2N+\gamma-\delta)+4(\gamma-3\delta+4M+N(\mu-3)\end{aligned}$$

$$
\begin{aligned}
& - 4n(n+\mu) + 6) + \frac{8\,i\,(N+i+2)(-N+i+2)}{(2i+3)(2i+5)} \\
& + \frac{8(i+1)(N+i+3)(-N+i+3)}{(2i+5)(2i+7)} - \frac{4\,j^2(N^2-j^2)}{4j^2-1} \\
& + \frac{4(j+1)^2(N+j+1)(-N+j+1)}{3+4\,j\,(j+2)} - \frac{16(\mu+1)(i+2)}{(3+i)_2} \\
& - \frac{(i+1)_2(i+3)(N+j+1)_2(-N+j+1)_2(\mu+1)}{(2i+1)(2i+3)^2(2i+5)} \\
& + \frac{4n(n+\mu)(N+i+2)(-N+i+2)}{(2i+3)(2i+5)} \\
& + \frac{4n(n+\mu)(N+i+3)(-N+i+3)}{(2i+5)(2i+7)} \\
& + \frac{16(3+\gamma+2\delta+M(\delta+1)-n(n+\mu))}{(i+3)} \\
& + \frac{4(81+272\,i+316\,i^2+168\,i^3+42\,i^4+4i^5}{(2i+3)(2i+5)(2i+7)} \\
& - \frac{4(9+2i(16+i(2i+11)))N^2)}{(2i+3)(2i+5)(2i+7)}], \\
\gamma_{i+2,j+1} = {} & \frac{1}{4}(i+1)_4[2(1-N^2) \\
& + [2(2j+1)(2j+3)]^{-1} \times \\
& \quad [(j+1)^2(N+j+1)(-N+j+1)(\delta-\gamma+2N-2M-4)], \\
\gamma_{i+2,j+2} = {} & -\frac{(i+1)_4(j+1)^2(j+2)^2(-N+j+1)_2(N+j+1)_2}{16(2j+1)(2j+3)^2(2j+5)},
\end{aligned}
$$

with $a_{i,j}(n) = 0$, $i+j > n$, $a_{-1,\,j}(n) = a_{i,-1}(n) = 0$ and $a_{n,0}(n) = a_{0,n}(n) = 1$. The solution of (7.8) is given by

$$
a_{i,j}(n) = \begin{cases}
\dfrac{(-1)^{i+j}(1-M)_n(1+\delta)_n(-n)_{i+j}\ (n+\mu)_{i+j}}{i!\,j!(1-M)_{i+j}(1+\delta)_{i+j}(n+\mu)_n} & \\[2ex]
\times \displaystyle\sum_{\ell=0}^{n-i-j} \frac{1}{\ell!} \frac{(-n+i+j)_\ell\ (n+\mu+i+j)_\ell\ (j+1)_\ell\ (-N+j+1)_\ell}{(-M+i+j+1)_\ell(2j+2)_\ell(\delta+i+j+1)_\ell} & \\[2ex]
\times\ {}_4F_3\left[\begin{matrix} -\ell, -N+i+1, i+1, -2j-\ell-2 \\ N-\ell-j, -j-\ell, 2i+2 \end{matrix};1\right], & i+j \le n, \\
0, & \text{otherwise.}
\end{cases}
$$

Note 4. The problem

(7.9)
$$\tilde{h}_n^{(\gamma,\delta)}(x+y;M)=\sum_{i+j\le n}a_{i,j}(n)\tilde{h}_i^{(\alpha,\beta)}(x;N)\tilde{h}_j^{(\alpha,\beta)}(y;N),\ n\le\min\{N,M\},$$

could be also considered in a similar way as in the above example. In fact, the corresponding recurrence relation for the expansion coefficients $a_{i,j}(n)$ can be obtained by inserting the data of monic Hahn polynomials $\tilde{h}_i^{(\alpha,\beta)}(x;N)$. However, this recurrence relation is very lengthy, but the expansion coefficients $a_{i,\,j}(n)$ in (7.9) has the formula:

$$\begin{aligned}a_{i,\,j}(n)&=\frac{(-1)^{i+j}(1-M)_n(1+\delta)_n(-n)_{i+j}\ (n+\mu)_{i+j}}{i!\,j!(1-M)_{i+j}(1+\delta)_{i+j}(n+\mu)_n}\\&\times\sum_{\ell=0}^{n-i-j}\frac{1}{\ell!}\frac{(-n+i+j)_\ell\ (n+\mu+i+j)_\ell\ (j+\beta+1)_\ell\ (-N+j+1)_\ell}{(-M+i+j+1)_\ell(2j+\lambda+1)_\ell(\delta+i+j+1)_\ell}\\&\times{}_4F_3\left[\begin{matrix}-\ell,-N+i+1,\beta+i+1,-2j-\ell-\lambda-1\\N-\ell-j,-j-\beta-\ell,2i+\lambda+1\end{matrix};1\right].\end{aligned}$$

Remark 7.1. Up to now, and to the best of the author's knowledge that analytical solutions of the recurrence relations of variable coefficients with two indices may be very difficult in general (see [47–50]), but the authors in these papers present the formulae of general solutions for some definite classes of homogeneous and non-homogeneous recurrences of variable coefficients with two indices. Hence, the given analytical solutions of the recurrences (5.7), (6.6), (6.8), (6.10), (6.12), (7.2), (7.4), (7.6) and (7.8) may be obtained by alternative method. This method depends on using some algebraic manipulation, the aid of the symbolical expression of Newton's formula for a function $f(x)$,

$$f(x)=\sum_{k=0}^{\infty}\binom{x}{k}\Delta^k f(0), \tag{7.10}$$

and the known representations for the falling factorials in terms of the classical discrete orthogonal polynomials (see [25, pp.85-6]), to find the expansions (5.4), (6.5), (6.7), (6.9), (6.11), (7.1), (7.3), (7.5) and (7.7) [see Appendices A, B and C]. Then, the obtained expansions coefficients will be actually the analytical solutions of these recurrences.

Remark 7.2. It should be mentioned that one of our aims here is to emphasize the systematic character and simplicity of our algorithm to build linear recurrence relations of the form (5.2), which allows one to implement it in any computer algebra (here Mathematica Version 8) symbolic language has been used.

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Appendices

Appendix A

The analytical solution of (5.3) is $u(x,y) = x\,y\,2^{-x}2^{-y}+x+y$. The analytical solution (5.8) can be obtained as follows.

First, we show that

$$f(x) = x\ 2^{-x} = \sum_{i=0}^{\infty} f_i\,\tilde{C}_i^{(\alpha)}(x), \tag{A.1}$$

where

$$f_i = \frac{(-2)^{i-1}}{i!}(2i-\alpha)\,e^{-\alpha/2}.$$

It is easy to see that $\Delta^k f(x) = (-1)^k 2^{-x-k}(x-k)$, then $\left[\Delta^k f(x)\right]_{x=0} = (-1)^{k+1}2^{-k}\,k$. Using (7.10) and the equality

$$\binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(x-k+1)\,k!} = \frac{x^{\underline{k}}}{k!},$$

yield the formula

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{x^{\underline{k}}}{k!}(-2)^{-k}. \tag{A.2}$$

Substituting the formual [3, p. 180]

$$x^{\underline{k}} = \sum_{i=0}^{k}\binom{k}{i}\alpha^{k-i}\tilde{C}_i^{(\alpha)}(x),$$

in (A.2), expanding and collecting similar terms, lead to

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty}\left(\frac{(-2)^{-(i+1)}}{i!}(2i-\alpha)\sum_{k=0}^{\infty}\frac{(-\alpha/2)^k}{k!}\right)\tilde{C}_i^{(\alpha)}(x) \\ &= \sum_{i=0}^{\infty}\left(\frac{(-2)^{-(i+1)}}{i!}(2i-\alpha)\,e^{-\alpha/2}\right)\tilde{C}_i^{(\alpha)}(x)\,, \end{aligned}$$

then (A.1) is obtained. Now, in view of (A.1), one can see that

$$\begin{aligned} u(x,y) = &\sum_{i,j=0}^{\infty}\tfrac{(-2)^{-(i+j+2)}(2i-\alpha)(2j-\alpha)}{i!j!}\ \tilde{C}_i^{(\alpha)}(x)\tilde{C}_j^{(\alpha)}(y) \\ &+2\alpha\tilde{C}_0^{(\alpha)}(x)\tilde{C}_0^{(\alpha)}(y)+\tilde{C}_1^{(\alpha)}(x)\tilde{C}_0^{(\alpha)}(y)+\tilde{C}_0^{(\alpha)}(x)\tilde{C}_1^{(\alpha)}(y). \end{aligned} \tag{A.3}$$

The expansion (A.3) can be written in the form

$$u(x,y)=\sum_{i,j=0}^{\infty} a_{i,j}\ \tilde{C}_i^{(\alpha)}(x)\tilde{C}_j^{(\alpha)}(y),$$

where the coefficients $a_{i,j}$ are given in (5.8). By using Mathematica, it can be checked that the formula (5.8) satisfies the recurrence relation (5.7).

Appendix B

In the expansion

$$(x+y)^{\underline{n}}=\sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}(n)P_i(x)P_j(y),$$

the coefficients $a_{i,j}(n)$ have the form

$$a_{i,j}(n)=\sum_{r=0}^{n-i-j}\binom{n}{j+r} d_i^{(n-j-r)} d_j^{(j+r)}, \tag{B.1}$$

where $d_i^{(n)}(i=0,1,...,n)$ are the coefficients in the expansion

$$x^{\underline{n}}=\sum_{i=0}^{n} d_i^{(n)}\ P_i(x). \tag{B.2}$$

The formula (B.1) can be proved as follows:

By using (7.10), it is not difficult to show that

$$(x+y)^{\underline{n}}=\sum_{k=0}^{n} b_k^{(n)}(y)\,x^{\underline{k}},\ \ b_k^{(n)}(y)=\binom{n}{k}\,y^{\underline{n-k}}. \tag{B.3}$$

Substituting (B.2) in (B.3), expanding and collecting similar terms give

$$(x+y)^{\underline{n}}=\sum_{i=0}^{n} c_i^{(n)}(y)P_i(x), \tag{B.4}$$

where

$$c_i^{(n)}(y)=\sum_{j=0}^{n-i} b_{i+j}^{(n)}(y)\,d_i^{(i+j)}.$$

The coefficients $c_i^{(n)}(y)$ can be written in the form

$$c_i^{(n)}(y)=\sum_{j=0}^{n-i}\lambda_j^{(i)}(n)\,y^{\underline{j}},\quad \lambda_j^{(i)}(n)=\binom{n}{j}\,d_i^{(n-j)}.$$

Using (B.2), expanding and collecting similar terms, yields the formula (B.5)

$$c_i^{(n)}(y) = \sum_{j=0}^{n-i} \left(\sum_{r=0}^{j} d_r^{(j)}\, P_r(y) \right) \lambda_j^{(i)}(n) = \sum_{j=0}^{n-i} \left(\sum_{r=0}^{n-i-j} \lambda_{r+j}^{(i)}(n)\, d_j^{(r+j)} \right) P_j(y).$$

Substituting (B.5) in (B.4) leads to

$$(x+y)^{\underline{n}} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \left(\sum_{r=0}^{n-i-j} \lambda_{j+r}^{(i)}(n)\, d_j^{(r+j)} \right) P_i(x) P_j(y), \tag{B.6}$$

then (B.6) takes the form

$$(x+y)^{\underline{n}} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{i,j}(n) P_i(x) P_j(y),$$

where

$$a_{i,j}(n) = \sum_{r=0}^{n-i-j} \lambda_{j+r}^{(i)}(n)\, d_j^{(r+j)} = \sum_{r=0}^{n-i-j} \binom{n}{j+r} d_i^{(n-j-r)} d_j^{(r+j)}.$$

The solution of recurrence relation (6.6)**:** In this case $d_i^{(n)} = \binom{n}{i}\alpha^{n-i}$, then

$$a_{i,j}(n) = \sum_{r=0}^{n-i-j} \binom{n}{j+r} \binom{n-j-r}{i} \binom{j+r}{j} \alpha^{n-i-j}. \tag{B.7}$$

The formula (B.7), after some manipulation, can be written in the form

$$\begin{aligned} a_{i,j}(n) &= (-1)^{i+j} \frac{(-n)_{i+j}}{i!j!} \alpha^{n-i-j} \sum_{r=0}^{n-i-j} \binom{n-i-j}{r} \\ &= (-1)^{i+j} \frac{(-n)_{i+j}}{i!j!} (2\alpha)^{n-i-j}. \end{aligned} \tag{B.8}$$

By using Mathematica, one can see that the formula (B.8) satisfies the recurrence relation (6.6). Similarly, the analytical solutions of recurrence relations (6.8), (6.10) and (6.12) can be obtained.

Appendix C

In the expansion

$$Q_n(x+y) = \sum_{i+j \le n} A_{i,j}(n) P_i(x) P_j(y),$$

the coefficients $a_{i,j}(n)$ have the form

$$A_{i,j}(n) = \sum_{r=0}^{n-i-j} \sum_{s=0}^{r} \binom{r+i+j}{j+s} d_i^{(r+i-s)} d_j^{(j+s)} a_{r+i+j}(n), \tag{C.1}$$

where $a_k(n)$, $k = 0, 1, ..., n$, are the coefficients in the expansion

$$Q_n(x) = \sum_{k=0}^{n} a_k(n)\, x^{\underline{k}}. \tag{C.2}$$

The formula (C.1) can be proved as follows:

By using (C.2), one can see that

$$Q_n(x+y) = \sum_{k=0}^{n} a_k(n)\, (x+y)^{\underline{k}}. \tag{C.3}$$

Substituting (B.1) in (C.3) yields,

$$Q_n(x+y) = \sum_{k=0}^{n} a_k(n) \left(\sum_{i=0}^{k} \sum_{j=0}^{k-i} a_{i,j}(k) P_i(x) P_j(y) \right),$$

then expanding and collecting similar terms lead to

$$Q_n(x+y) = \sum_{i=0}^{n} B_i^{(n)}(y)\, P_i(x), \tag{C.4}$$

where

$$B_i^{(n)}(y) = \sum_{r=i}^{n} \left(\sum_{j=0}^{r-i} a_{i,\,j}(r)\, P_j(y) \right) a_r. \tag{C.5}$$

Again, by expanding and collecting similar terms, the formula (C.5) takes the form

$$B_i^{(n)}(y) = \sum_{j=0}^{n-i} \left(\sum_{r=0}^{n-i-j} a_{i,\,j}(r+i+j)\, a_{r+i+j} \right) P_j(y). \tag{C.6}$$

Substituting (C.6) in (C.4) gives

$$Q_n(x+y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \left(\sum_{r=0}^{n-i-j} a_{i,\,j}(r+i+j)\, a_{r+i+j} \right) P_i(x)\, P_j(y).$$

Using (B.1) yields the formula

$$Q_n(x+y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} A_{i,j}(n)\, P_i(x)\, P_j(y),$$

where

$$A_{i,j}(n) = \sum_{r=0}^{n-i-j} \sum_{s=0}^{r} \binom{r+i+j}{j+s} d_i^{(r+i-s)} d_j^{(j+s)} a_{r+i+j}.$$

The solution of recurrence relation (7.2): In this case $d_i^{(n)} = \binom{n}{i}\alpha^{n-i}$ and $a_k(n) = \frac{(-1)^k(-n)_k}{k!}(-\gamma)^{n-k}$ then

$$A_{i,j}(n) = \sum_{r=0}^{n-i-j} \left(\sum_{s=0}^{r} \binom{r+i+j}{j+s} \binom{r+i-s}{i} \binom{j+s}{j} \right) \times \alpha^r (-\gamma)^{n-r-i-j} (-1)^{r+i+j} \frac{(-n)_{r+i+j}}{(r+i+j)!}.$$

By using Mathematica, one can obtain

$$\sum_{s=0}^{r} \binom{r+i+j}{j+s} \binom{r+i-s}{i} \binom{j+s}{j} = 2^r \binom{r+i}{i} \binom{i+j+r}{j},$$

then

$$A_{i,j}(n) = \sum_{r=0}^{n-i-j} \binom{r+i}{i} \binom{i+j+r}{j} (2\alpha)^r (-\gamma)^{n-r-i-j} (-1)^{r+i+j} \frac{(-n)_{r+i+j}}{(r+i+j)!}.$$

Using the equality $(-n)_k = (-1)^k k! \binom{n}{k}$ leads to the form

$$A_{i,j}(n) = \sum_{r=0}^{n-i-j} \binom{r+i}{i} \binom{i+j+r}{j} \binom{n}{r+i+j} (2\alpha)^r (-\gamma)^{n-r-i-j}.$$

Using Mathematica, after some manipulation, yields the formula

$$A_{i,j}(n) = \frac{(-1)^{i+j}}{i!j!} (-n)_{i+j} (2\alpha - \gamma)^{n-i-j}. \tag{C.7}$$

By using Mathematica, it can be seen that the formula (C.7) satisfies the recurrence relation (7.2). Similarly, the analytical solutions of recurrence relations (7.4), (7.6) and (7.8) can be obtained.

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