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ON THE INFINITE PRODUCT REPRESENTATION OF SOLUTION AND DUAL EQUATIONS OF STURM-LIOUVILLE EQUATION WITH TURNING POINT OF ORDER 4M+1

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ABSTRACT. The purpose of this paper is studying the infinite product representation of solution of boundary value problem:

$$\begin{aligned} &-y'' + q(x)y = \lambda R^2(x)y, && 0 \leq x \leq 1 \\ &y(0) = 0 = y(1), && (\mathbf{I}) \end{aligned}$$

where $\lambda = \rho^2$ is the spectral parameter and $q(x)$ is a integrable function. We also suppose that

$$R^2(x) = (x - x_1)^{4m+1} R_0(x)$$

where $0 < x_1 < 1$, $m \in N$, $R_0 > 0$ for $x \in [0, 1]$, R_0 is twice continuously differentiable on $[0, 1]$ and $R^2(x)$ has one zero in $[0, 1]$, so called turning point.

The product representation satisfies in the original equation (**I**). As a result we substituted the infinite product form in the equation (**I**) and derive the associate dual equations.

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1. **Introduction**

We consider boundary value problems L of the form

$$Ly = -y'' + q(x)y = \lambda R^2(x)y, \qquad x \in [0,1] \tag{1}$$

$$y(0) = 0 = y(1) \tag{2}$$

where $\lambda = \rho^2$ is the spectral parameter and the functions $q(x)$ and $R^2(x)$ satisfy:
(i) $R^2(x) = R_0(x)(x - x_1)^l$ is real and has one zero x_1 of order $l = 4m+1, m \in N$ in $[0,1]$ and also R_0 is positive and twice continuously differentiable .
(ii) $q(x)$ is bounded and integrable on $I = [0,1]$.
To study the infinite representation of the solution in this paper we use the asymptotic form of fundamental system of solutions(FSS) for equation (**??**) constructed in [**?**], [**?**] and we also use the asymptotic form of eigenvalues of equation (**??**) constructed in [**?**]. Note that the infinite product representation of solution with simple turning point has been studied in [**?**].

2. **Notations and preliminary results**

Let $\epsilon > 0$ be fixed and sufficiently small, and let $D_\epsilon = [0, x_1 - \epsilon] \bigcup [x_1 + \epsilon, 1]$. Further, we set $\mu = \frac{1}{2+l}$ and $\theta = 4\mu$. we also denote

$$I_+ = \{x : R^2(x) > 0\} \qquad I_- = \{x : R^2(x) < 0\}$$

$$\xi(x) = \begin{cases} 0 & \text{for } x \in I_+(x) \\ 1 & \text{for } x \in I_-(x) \end{cases}$$

$$R_+^2 = max(0, R^2(x)) \qquad R_-^2 = max(0, -R^2(x))$$

$$\gamma = 2\sin(\frac{\pi\mu}{2})$$

$$K_\pm(x) = \begin{cases} 1 & \text{for } x \in I_-(x) \\ \frac{1}{2}\csc(\frac{\pi\mu}{2})\exp(\mp i\frac{\pi}{4}) & \text{for } x \in I_+(x) \end{cases}$$

$$K_\pm^*(x) = \begin{cases} \pm i & \text{for } x \in I_-(x) \\ 2\sin(\frac{\pi\mu}{2})\exp(\pm i\frac{\pi}{4}) & \text{for } x \in I_+(x). \end{cases}$$

Let

$$S_k = \{\rho | \arg\ \rho \in [\frac{k\pi}{4}, \frac{(k+1)\pi}{4}]\}, \qquad k = \{0, 1\}.$$

In [?] it is shown that for each fixed sector $S_k (k = 0, 1)$ there exits a FSS of (??) $\{z_1(x, \rho), z_2(x, \rho)\}$, $x \in I$, $\rho \in S_k$ such that the functions $(x, \rho) \to z_s^{(j)}(x, \rho)(s = 1, 2;\ j = 0, 1)$ are continuous for $x \in I$, $\rho \in S_k$ and holomorphic for each fixed $x \in I$ with respect to $\rho \in S_k$; moreover for $|\rho| \to \infty$, $\rho \in S_k$, $x \in D_\epsilon$, $\quad j = 1, 2$

$$\begin{aligned} z_1^j(x,\rho) = \ & (\pm i\rho)^j |R(x)|^{j-\frac{1}{2}} (\exp(\mp i\tfrac{\pi}{2}\xi(x))^j \exp(\rho \textstyle\int_0^x |R_-(t)|dt) \\ & \times \exp(\pm i\rho \textstyle\int_0^x |R_+(t)|dt) K_\pm(x)\kappa(x,\rho), \end{aligned} \tag{3}$$

$$\begin{aligned} z_2^j(x,\rho) = \ & (\mp i\rho)^j |R(x)|^{j-\frac{1}{2}} (\exp(\mp i\tfrac{\pi}{2}\xi(x))^j \exp(-\rho \textstyle\int_0^x |R_-(t)|dt) \\ & \times \exp(\mp i\rho \textstyle\int_0^x |R_+(t)|dt) K_\pm^*(x)\kappa(x,\rho), \end{aligned} \tag{4}$$

$$\begin{vmatrix} z_1(x,\rho) & z_2(x,\rho) \\ z_1'(x,\rho) & z_2'(x,\rho) \end{vmatrix} = \mp(2i\rho)[1].$$

Here and in the following:
(i) The upper or lower signs in formulae correspond to the sectors S_0, S_1 respectively.
(ii) $[1] = 1 + O(\frac{1}{\rho^\theta})$ uniformly in $x \in D_\epsilon$.
(iii) $\kappa(x,\rho) = O(1)\ as\ |\rho| \to \infty, \rho \in S_k$.

3. **Representation of the solution in the form of infinite product**

Let $\varphi(x, \lambda)$ be solution of equation (??) with initial conditions

$$\varphi(0,\lambda) = 0 \qquad\qquad \frac{\partial\varphi}{\partial x}(0,\lambda) = 1. \tag{5}$$

Using the FSS $\{z_1(x,\rho), z_2(x,\rho)\}$ we obtain:

$$\varphi(x,\lambda) = \frac{1}{\omega(\lambda)}(z_1(0,\rho)z_2(x,\rho) - z_2(0,\rho)z_1(x,\rho))$$

where $\omega(\lambda) = \mp(2i\rho)[1]$. By virtue of (**??**)-(**??**), we infer that for $\rho \in S_k,\ x \in D_\epsilon,\ j = 0, 1$

$$\varphi^j(x,\lambda) = \tfrac{1}{2}(\pm i\rho)^{j-1}|R(0)|^{-\frac{1}{2}}|R(x)|^{j-\frac{1}{2}}(\exp(\mp i\tfrac{\pi}{2}\xi(x))^j \exp(\pm i\tfrac{\pi}{2}) \\ \times \exp(\rho \textstyle\int_0^x |R_-(t)|dt) \exp(\pm i\rho \int_0^x |R_+(t)|dt) K_\pm(x)\kappa(x,\rho) \qquad (6)$$

and

$$|\varphi^j(x,\lambda)| \le C|\rho|^{j-1}|\exp(\rho \int_0^x |R_-(t)|dt) \exp(\pm i\rho \int_0^x |R_+(t)|dt)|. \qquad (7)$$

It follows from (**??**) that the function $\varphi^j(x,.)$ are entire of order $\frac{1}{2}$. The function $\varphi(x,\lambda)$ has a zero set for each x, say $\{\lambda_n\}$, so that $\varphi(x,\lambda_n(x)) = 0$ which corresponds to eigenvalues of the Dirichlet problem for equation (**??**) on the closed interval $[0,x]$. Note that $\lambda_n \neq 0$ for any x by Sturm's comparison theorem since we assume that $q(x) \geq 0$. The eigenvalues of the Dirichlet problem on $[0,x]$ for (**??**) , are real and simple (see [**?**],§10.61), so we have

$$\frac{\partial\varphi}{\partial\lambda}(x,\lambda_n(x)) \neq 0.$$

We consider the Dirichlet problem corresponding to equation (**??**) on $[0,x]$ for fixed x, $x < x_1$. By result of [**?**] this problem has an infinite number of negative eigenvalues, which we denote by $\{\lambda_n\}$. By the Hadamard's theorem, the product formula is of the form

$$\varphi(x,,\lambda) = C(x)\prod_{n=1}^{\infty}(1 - \frac{\lambda}{\lambda_n}) \qquad (8)$$

where φ satisfies in initial condition (**??**) and $C(x)$ is a function of x and independent of λ, by [**?**], each function λ_n is of the form

$$\sqrt{\lambda_n} = \frac{n\pi}{p(x)}i + O(\frac{1}{n}), \qquad x < x_1$$

where

$$\lim_{x\to 0}\lambda_n(x) = -\infty, \qquad \lambda_1 > \lambda_2 > ...$$

and

$$p(x) = \int_0^x |R(t)|dt. \qquad (9)$$

In order to estimate $C(x)$ we rewrite the infinite product as

$$\begin{aligned}\varphi(x,\lambda) \quad &= C(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{\lambda_n}) \\ &= C(x)\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{\lambda_n} \\ &= C_1(x)\prod_{n=1}^{\infty}\frac{\lambda-\lambda_n}{z_n^2}\end{aligned} \tag{10}$$

with

$$C_1(x) = C(x)\prod_{n=1}^{\infty}\frac{-z_n^2}{\lambda_n(x)} \tag{11}$$

where $z_n = \frac{n\pi}{p(x)}$.

It follows from the asymptotic form of eigenvalues, $-\frac{z_n^2}{\lambda_n} = 1+O(\frac{1}{n^2})$, then the infinite product $\prod_{n=1}^{\infty}\frac{-z_n^2}{\lambda_n(x)}$ is absolutely convergent on any compact subinterval of $(0, x_1)$.

For $x \in (x_1, 1]$, fixed, the Dirichlet problem for (**??**) on $[0, x]$ has an infinite number of positive and negative eigenvalues, which we denote by $\{u_n\}$, $\{r_n\}$ respectively, it follows from results of [**?**]

$$\sqrt{u_n(x)} = \frac{n\pi - \frac{\pi}{4}}{f(x)} + (\frac{1}{n}), \qquad x_1 < x$$

where

$$f(x) = \int_{x_1}^{x} |R(t)| dt \tag{12}$$

and $r_n(x)$ is of the form

$$\sqrt{r_n(x)} = \frac{n\pi - \frac{\pi}{4}}{p(x_1)} i + (\frac{1}{n}), \qquad x_1 < x.$$

where $p(x)$ is defined in (**??**). By Hadamard's theorem, the solution on $[0, x]$ for $x > x_1$ is of the form:

$$\varphi(x,\lambda) = C(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{r_n})\prod_{n=1}^{\infty}(1-\frac{\lambda}{u_n}). \tag{13}$$

Let $\tilde{j}_m$ be the positive zeros of $J_1'(z)$, then

$$\tilde{j}_m = m^2\pi^2 - \frac{m\pi^2}{2} + O(1)$$

for more details (See [**?**],§9.5.11). Consequently we have

$$\frac{\tilde{j}_n^2}{f^2(x)u_n(x)} = 1 + O(\frac{1}{n^2}),$$
$$\frac{-\tilde{j}_n^2}{p^2(x_1)r_n(x)} = 1 + O(\frac{1}{n^2})$$

where $p(x)$ and $f(x)$ are defined in (**??**) and (**??**) respectively. Therefore the infinite products $\prod_{n=1}^{\infty} \frac{\tilde{j}_n^2}{f^2(x)u_n(x)}$ and $\prod_{n=1}^{\infty} \frac{-\tilde{j}_n^2}{p^2(x_1)r_n(x)}$ are absolutely convergent for each $x > x_1$. Then we may write

$$\varphi(x,\lambda) = C_2(x)\prod_{n=1}^{\infty}\frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{j}_n^2}\prod_{n=1}^{\infty}\frac{(u_n(x)-\lambda)f^2(x)}{\tilde{j}_n^2}$$

where

$$C_2(x) = C(x)\prod_{n=1}^{\infty}\frac{\tilde{j}_n^2}{f^2(x)u_n(x)}\prod_{n=1}^{\infty}\frac{-\tilde{j}_n^2}{p^2(x_1)r_n(x)}.$$

Now we will first approximate infinite products, then by using the asymptotic form of $\varphi(x,\lambda)$, we will determine $C_i, i = 1, 2$.

Lemma 3.1. *Let* $z_m = \frac{m\pi}{p(x)}$ *and* $\lambda_m(x), 1 \leq m$ *be a sequence of continuous functions such that for each* x

$$\lambda_m(x) = -\frac{m^2\pi^2}{p^2(x)} + O(1) \qquad 0 < x < x_1.$$

Then the infinite product

$$\prod_{m=1}^{\infty}(\frac{\lambda - \lambda_m}{z_m^2})$$

is an entire function of λ *for fixed* x *in* $(0, x_1)$ *whose roots are precisely* $\lambda_m(x), 1 \leq m$. *Moreover*

$$\prod_{m=1}^{\infty}(\frac{\lambda - \lambda_m}{z_m^2}) = \frac{sinh(\rho p(x))}{\rho p(x)}(1 + O(\frac{\log m}{m})),$$

uniformly on the circles $|\lambda| = \frac{(n+\frac{1}{2})^2\pi^2}{p^2(x)}$, *where* $\rho = \sqrt{\lambda}$ *and* $p(x)$ *is a function of* x *as defined in (**??**)*

Proof. See [**?**].

□

Lemma 3.2. *Let* $\tilde{j}_n$ *be the positive zeros of* $J_1'(z)$ *and for fixed* x *in* $(x_1, 1)$

$$u_n(x) = \frac{n^2\pi^2}{f^2(x)} - \frac{n\pi^2}{2f^2(x)} + O(1) \qquad 1 \leq n$$

be a positive sequence of continuous functions. then the infinite product

$$\prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda)f^2(x)}{\tilde{j}_n^2}$$

is an entire function of λ *for fixed* x, *whose roots are precisely* $u_n(x), 1 \leq n$. *Moreover*

$$\prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda)f^2(x)}{\tilde{j}_n^2} = 2J_1'(f(x)\rho)\{1 + O(\frac{logn}{n})\}$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{f^2(x)}$.

Proof. See [**?**] and [**?**].

□

Lemma 3.3. *Let* $\tilde{j}_n$ *be the positive zeros of* $J_1'(z)$ *and for fixed* x *in* $(x_1, 1)$

$$r_n(x) = -\frac{n^2\pi^2}{p^2(x_1)} + \frac{n\pi^2}{2p^2(x_1)} + O(1) \qquad 1 \leq n$$

be a negative sequence of continuous functions. then the infinite product

$$\prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{j}_n^2}$$

is an entire function of λ for fixed x, whose roots are precisely $r_n(x), 1 \leq n$. Moreover

$$\prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))f^2(x)}{\tilde{j}_n^2} = 2J_1'(ip(x_1)\rho)\{1 + O(\frac{logn}{n})\}$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{p^2(x_1)}$.

Proof. See [?] and [?].

□

Theorem 3.4. *Let $\varphi(x, \lambda)$ be the solution of (**??**) with the initial conditions (**??**), then for $0 \leq x < x_1$,*

$$\varphi(x, \lambda) = |R(0)R(x)|^{-\frac{1}{2}} p(x) \prod_{k=1}^{\infty} \frac{\lambda - \lambda_k(x)}{z_k^2}$$

*where p(x) is defined (**??**), $z_k = \frac{k\pi}{p(x)}$ and $\{\lambda_k(x)\}$ is the sequence of eigenvalues for the Dirichlet problem associated with (**??**) on $[0, x]$.*

Proof. For $0 < x < x_1$, $\rho \in S_0$, $|\rho| \to \infty$ by virtue of (**??**) we calculate

$$\begin{aligned} \varphi(x, \rho) &= \frac{1}{2}(i\rho)^{-1}|R(0)R(x)|^{-\frac{1}{2}} \exp(i\frac{\pi}{2}) \\ &\times \exp(\rho \int_0^x |R(t)|dt) k_+(x)\kappa(x, \rho) \end{aligned} \tag{14}$$

Now from (**??**) and (**??**) we have

$$\begin{aligned} \varphi(x, \rho) &= C_1 \prod_{n=1}^{\infty} \frac{\lambda - \lambda_k}{z_k^2} \\ &= \frac{1}{2}(i\rho)^{-1}|R(0)R(x)|^{-\frac{1}{2}} \exp(i\frac{\pi}{2}) \\ &\times \exp(\rho \int_0^x |R(t)|dt) k_+(x)\kappa(x, \rho). \end{aligned}$$

From Lemma 3.2, uniformly on the circles $|\lambda| = \frac{(n+1/2)^2\pi^2}{p^2(x)}$, we have

$$\prod_{n=1}^{\infty} \frac{\lambda - \lambda_k}{z_k^2} = \frac{sinh(\rho p(x))}{\rho p(x)}(1 + O(\frac{logk}{k}))$$

whence on the circles $|\lambda| = \frac{(n+1/2)^2\pi^2}{p^2(x)}$ we obtain

$$C_1(x) = \frac{\varphi(x,\rho)}{\prod_{k=1}^{\infty} \frac{\lambda-\lambda_k}{z_k^2}} = |R(0)R(x)|^{-\frac{1}{2}} p(x).$$

as $|\rho| \to \infty$

□

Theorem 3.5. *Let $\varphi(t,\lambda)$ be the solution of the initial value problem (??), (??). Then for $x_1 < x$,*

$$\begin{aligned}\varphi(x,\lambda) \quad &= \frac{|R(0)R(x)|^{-\frac{1}{2}}}{2} \pi p^{\frac{1}{2}}(x_1) f^{\frac{1}{2}}(x) \csc(\frac{\pi\mu}{2}) \\ &\times \prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{j}_n^2} \prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda) f^2(x)}{\tilde{j}_n^2} \end{aligned} \tag{15}$$

Proof. For $x_1 < x < 1$, $|\rho| \to \infty$ by use of (**??**) it is obtained

$$\begin{aligned}\varphi(x,\lambda) \quad &= \frac{1}{4}(-i\rho)^{-1}|R(0)R(x)|^{-\frac{1}{2}}(-i)\exp(\rho p(x_1)) \qquad (16) \\ &\times \exp(-i\rho f(x)) \csc(\frac{\pi\mu}{2}) \exp(i\frac{\pi}{4})\kappa(x,\rho) \\ &= \frac{1}{4\rho}|R(0)R(x)|^{-\frac{1}{2}} \exp(\rho p(x_1)) \\ &\times \cos(\rho f(x) - \frac{\pi}{4}) \csc(\frac{\pi\mu}{2})(1 + O(\frac{1}{\rho^\theta}))\end{aligned}$$

By Lemma 3.2 and 3.3, on the circles $|\lambda| = \min\{\frac{n^2\pi^2}{p^2(x_1)}, \frac{n^2\pi^2}{f^2(x)}\}$ we have

$$\begin{aligned}&\varphi(x,\lambda) = C_2(x) \prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{j}_n^2} \prod_{n=1}^{\infty} \frac{(u_n(x) - \lambda) f^2(x)}{\tilde{j}_n^2} \\ &= 4J_1'(f(x)\rho)J_1'(ip(x_1)\rho)\{1 + O(\frac{logn}{n})\} \qquad (17) \\ &= \frac{4\exp(\rho p(x_1))}{\pi p^{\frac{1}{2}}(x_1) f^{\frac{1}{2}}(x)\rho}(\cos(f(x)\rho - \frac{\pi}{4}) + O(\frac{1}{\rho})).\end{aligned}$$

Now by virtue of (??), (??) and let of $|\rho| \to \infty$ it is calculated

$$\begin{aligned} C_2(x) &= \frac{\varphi(x,\rho)}{\prod_{n=1}^{\infty} \frac{(\lambda - r_n(x))p^2(x_1)}{\tilde{j}_n^2} \prod_{n=1}^{\infty} \frac{(u_n(x)-\lambda)f^2(x)}{\tilde{j}_n^2}} \\ &= \frac{1}{16}\pi|R(0)R(x)|^{-\frac{1}{2}} \csc(\frac{\pi\mu}{2})p^{\frac{1}{2}}(x_1)f^{\frac{1}{2}}(x) \end{aligned}$$

□

4. Dual equations

By the implicit function theorem $\lambda_n(x), u_n(x)$ and $r_n(x)$ are twice continuously differentiable functions. For $x < x_1$, the condition

$$\varphi(x, \lambda_n(x)) = 0$$

gives , as usual ,

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial\lambda}.\lambda_n' = 0 \tag{18}$$

and differentiating again

$$\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial^2\varphi}{\partial x\partial\lambda}.\lambda_n' + \frac{\partial^2\varphi}{\partial\lambda^2}.(\lambda_n')^2 + \frac{\partial\varphi}{\partial\lambda}.\lambda_n'' = 0 \tag{19}$$

The first term in (??) is zero at $(x, \lambda_n(x))$ by virtue of (??) . Thus

$$2\frac{\partial^2\varphi}{\partial x\partial\lambda}.\lambda_n' + \frac{\partial^2\varphi}{\partial\lambda^2}.(\lambda_n')^2 + \frac{\partial\varphi}{\partial\lambda}.\lambda_n'' = 0. \tag{20}$$

Similarly for $x_1 < x$, the conditions

$$\varphi(x, u_n(x)) = 0$$
$$\varphi(x, r_n(x)) = 0$$

give the equations

$$2\frac{\partial^2\varphi}{\partial x\partial\lambda}.u_n' + \frac{\partial^2\varphi}{\partial\lambda^2}.(u_n')^2 + \frac{\partial\varphi}{\partial\lambda}.u_n'' = 0$$

$$2\frac{\partial^2\varphi}{\partial x\partial\lambda}.r_n' + \frac{\partial^2\varphi}{\partial\lambda^2}.(r_n')^2 + \frac{\partial\varphi}{\partial\lambda}.r_n'' = 0 \tag{21}$$

If we make use of the infinite product form of $\varphi(x, \lambda)$, substitute this in (??), in the case $x < x_1$ and in (??) for $x > x_1$ it will be

obtained the *dual* of the equation (**??**). Indeed we need the various derivatives of $\varphi(x,\lambda)$ at the points $(x,\lambda_n(x))$ for $x < x_1$ and at the points $(x,u_n(x))$ and $(x,r_n(x))$ for $x > x_1$.
Now , we first calculate the various derivatives of $\varphi(x,\lambda)$ for $x < x_1$. In this case, from (**??**), it can be written

$$\varphi(x,\lambda) = C(x)\prod_{k=1}^{\infty}(1-\frac{\lambda}{\lambda_k(x)}) \tag{22}$$

where C is a function independent of λ; by using (**??**) it is obtained

$$C_1 = |R(0)R(x)|^{-\frac{1}{2}}p(x) = C\prod_{k=1}^{\infty}\frac{-z_k^2}{\lambda_k}$$

where $z_k = \frac{k\pi}{p(x)}$ and $p(x)$ is determined in (**??**). Therefore

$$C(x) = |R(0)R(x)|^{-\frac{1}{2}}p(x)\prod_{k=1}^{\infty}\frac{-\lambda_k}{z_k^2}\ . \tag{23}$$

We calculate $\frac{\partial\varphi}{\partial\lambda}$, $\frac{\partial^2\varphi}{\partial\lambda^2}$ and $\frac{\partial^2\varphi}{\partial x\partial\lambda}$ at the points $(x,\lambda_n(x))$ by using (**??**). In determining of $\frac{\partial^2\varphi}{\partial\lambda\partial x}$, the interchange of summation and differentiation in

$$\frac{d}{dx}\sum_{k=1}^{\infty}\log(1-\frac{\lambda}{\lambda_k(x)})$$

is valid, because by results of [**?**], the differentiated series

$$\sum_{k\neq n}\frac{-\lambda_n\lambda_k'(x)}{(\lambda_k(x)-\lambda_n)\lambda_k(x)}$$

is uniformly convergent. We define F_n by

$$F_n = F_n(x,\lambda_n(x)) = \prod_{k\neq n,1\leq k}(1-\frac{\lambda_n(x)}{\lambda_k(x)}). \tag{24}$$

Since

$$\frac{\partial\varphi}{\partial\lambda} = C\sum_{i=1}^{\infty}\frac{-1}{\lambda_i(x)}\prod_{k\neq i,1\leq k}(1-\frac{\lambda}{\lambda_k(x)}),$$

we have

$$\frac{\partial\varphi}{\partial\lambda}(x,\lambda_n) = \frac{-CF_n}{\lambda_n(x)},$$

$$\frac{\partial^2\varphi}{\partial\lambda^2}(x,\lambda_n(x)) = \frac{2CF_n}{\lambda_n(x)}\sum_{i\neq n,1\leq i}\frac{1}{\lambda_i}(1-\frac{\lambda_n(x)}{\lambda_i(x)})^{-1},$$

$$\frac{\partial^2\varphi}{\partial\lambda\partial x} = \frac{-C'(x)F_n}{\lambda_n(x)} + \frac{C(x)\lambda_n' F_n}{\lambda_n^2} - C(x)F_n\sum_{i\neq n,1\leq i}\frac{\lambda_i'}{\lambda_i^2}(1-\frac{\lambda_n(x)}{\lambda_i(x)})^{-1}$$

$$-\frac{C(x)F_n\lambda_n'}{\lambda_n}\sum_{i\neq n,1\leq i}\frac{1}{\lambda_i}(1-\frac{\lambda_n(x)}{\lambda_i(x)})^{-1}.$$

Placing these terms into (**??**) we obtain

$$\lambda_n'' + \frac{2C'\lambda_n'}{C} + 2\lambda_n\lambda_n'\sum_{i\neq n,1\leq i}\frac{\lambda_i'}{\lambda_i^2}(1-\frac{\lambda_n(x)}{\lambda_i(x)})^{-1} - 2\frac{(\lambda_n')^2}{\lambda_n} = 0. \tag{25}$$

Dividing the above equation by λ_n' and integrating from a fixed number $\alpha \neq 0$ up to x , we obtain

$$\lambda_n'(x) = \frac{\lambda_n^2(x)\lambda_n'(\alpha)C^2(\alpha)}{\lambda_n^2(\alpha)C^2(x)}e^{-2S_n(x,\lambda_n)} \tag{26}$$

where

$$S_n(x,\lambda_n) = \sum_{i\neq n}\int_\alpha^x\frac{\lambda_i'\lambda_n}{\lambda_i}(\lambda_i-\lambda_n)^{-1} \tag{27}$$

and $C(x)$ is determined in (**??**) .
Similarly , for the case $x > x_1$ from (**??**) and theorem 2 , we have

$$\varphi(x,\lambda) = a(x)\prod_{k=1}^{\infty}(1-\frac{\lambda}{r_k(x)})\prod_{k=1}^{\infty}(1-\frac{\lambda}{u_k(x)}) \tag{28}$$

with

$$\begin{aligned} a(x) \quad &= \frac{1}{16}\pi|R(0)R(x)|^{-\frac{1}{2}}\csc(\frac{\pi\mu}{2})p(x_1)^{\frac{1}{2}}f^{\frac{1}{2}}(x) \\ &\times\prod_{k=1}^{\infty}\frac{f^2(x)u_k(x)}{\tilde{j}_k^2}\prod_{k=1}^{\infty}\frac{p^2(x_1)r_k(x)}{\tilde{j}_k^2} \end{aligned} \tag{29}$$

where $f(x)$,$p^2(x)$ are defined in (**??**), (**??**) and $\tilde{j}_k$, k = 1 ,2 , ... are the positive zeros of $J_1'(z)$.

as before , we calculate the various derivatives of $\varphi(x,\lambda)$ and evaluate these at the fixed points $(x,u_n(x)),(x,r_n(x))$. Since , by results of [?], the series

$$\sum_{k\neq n}\frac{-u_n u_k'(x)}{(u_k(x)-u_n)u_k(x)}$$

is uniformly convergent, we obtain $\frac{\partial^2\varphi}{\partial\lambda\partial x}$ from (??) in terms of u_n and r_n .
Suppose

$$G_n(x,\lambda)=\prod_{k\neq n,1\leq k}(1-\frac{\lambda}{u_k(x)})$$

$$H_n(x,\lambda)=\prod_{1\leq k}(1-\frac{\lambda}{r_k(x)}).$$

Then ,

$$G_n=G_n(x,u_n(x))=\prod_{k\neq n,1\leq k}(1-\frac{u_n(x)}{u_k(x)}) \tag{30}$$

$$H_n=H_n(x,u_n(x))=\prod_{1\leq k}(1-\frac{u_n(x)}{r_k(x)}) \tag{31}$$

so that

$$\prod_{k\neq i,1\leq k}(1-\frac{u_n(x)}{r_k(x)})=H_n(1-\frac{u_n}{r_i})^{-1}. \tag{32}$$

We have

$$\frac{\partial\varphi}{\partial\lambda}(x,u_n)=\frac{-aH_nG_n}{u_n(x)}$$

$$\begin{aligned}\frac{\partial^2\varphi}{\partial\lambda^2}(x,u_n(x)) &= \frac{2aH_nG_n}{u_n(x)}\sum_{1\leq i}\frac{1}{r_i(x)-u_n(x)}\\ &+ \frac{2aH_nG_n}{u_n(x)}\sum_{1\leq i,i\neq n}\frac{1}{u_i(x)-u_n(x)}\\ \frac{\partial^2\varphi}{\partial\lambda\partial x}(x,u_n(x)) &= \frac{-a'(x)H_nG_n}{u_n(x)}+\frac{a(x)H_nG_nu_n'}{u_n^2}\end{aligned}$$

$$
\begin{aligned}
&- \frac{a(x)H_nG_nu_n'}{u_n}\sum_{1\le i}\frac{1}{r_i(x)-u_n(x)}\\
&- a(x)H_nG_n\sum_{1\le i}\frac{r_i'}{r_i}(r_i(x)-u_n(x))^{-1}\\
&- a(x)H_nG_n\sum_{1\le i, i\ne n}\frac{u_i'}{u_i}(u_i(x)-u_n(x))^{-1}\\
&- \frac{a(x)H_nG_nu_n'}{u_n}\sum_{1\le i, i\ne n}\frac{1}{u_i(x)-u_n(x)}
\end{aligned}
$$

Placing these terms into (**??**) we obtain

$$
\begin{aligned}
u_n'' \quad &+ \quad \frac{2a'u_n'}{a} + 2u_nu_n'\{\sum_{i\ne n, 1\le i}\frac{u_i'}{u_i}(u_i(x)-u_n(x))^{-1}\\
&+ \sum_{1\le i}\frac{r_i'}{r_i}(r_i(x)-u_n(x))^{-1}\} - 2\frac{(u_n')^2}{u_n} = 0. \qquad (33)
\end{aligned}
$$

Similarly for negative eigenvalue $r_n(x)$ we get

$$
\begin{aligned}
r_n'' \quad &+ \quad \frac{2a'r_n'}{a} + 2r_nr_n'\{\sum_{i\ne n, 1\le i}\frac{r_i'}{r_i}(r_i(x)-r_n(x))^{-1}\\
&+ \sum_{1\le i}\frac{u_i'}{u_i}(u_i(x)-r_n(x))^{-1}\} - 2\frac{(r_n')^2}{r_n} = 0. \qquad (34)
\end{aligned}
$$

Dividing the equation (**??**) by u_n' , the equation (**??**) by r_n' and integrating from x up to 1 , we obtain

$$
u_n'(x) = \frac{u_n^2(x)u_n'(1)a^2(1)}{u_n^2(1)a^2(x)}e^{2T_n(x,u_n,r_n)} \qquad (35)
$$

$$
r_n'(x) = \frac{r_n^2(x)r_n'(1)a^2(1)}{r_n^2(1)a^2(x)}e^{2T_n(x,r_n,u_n)} \qquad (36)
$$

where

$$
T_n(x,u_n,r_n) = \sum_{i\ne n}\int_x^1 \frac{u_i'u_n}{u_i}(u_i-u_n)^{-1}dv + \sum_i\int_x^1\frac{r_i'u_n}{r_i}(r_i-u_n)^{-1}dv, \qquad (37)
$$

and $a(x)$ is determined in (**??**).

The system of equations (**??**), (**??**) and (**??**) are dual to the original equation (**??**) and involves only the functions $\lambda_n(x)$, $u_n(x)$ and $r_n(x)$.

Note that the proof of existence and uniqueness solution for dual equation (**??**), (**??**) and (**??**) in one simple turning point case is given by authors in submitted paper [**?**].

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