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A FRACTAL NON-CONTRACTING CLASS OF AUTOMATA GROUPS

M. J. MAMAGHANI

ABSTRACT. We present an automorphism group of the regular rooted tree of order 2 that is generated by a three state automaton and show that this group is fractal, non-contracting, weakly branch and contains a copy of the lamplighter group.

1. Introduction

The notions of automaton, fractal and branch group have received great attention of a wide range of mathematicians due to the recent works of Bartholdi-Grigorchuk [2], [3], Grigorchuk-Zuk [10],[11] [12], Grigorchuk [8] and [12] and Brunner-Sidki-Viera [5]. Automata Groups are groups generated by invertible automata and act on rooted regular trees as automorphisms. These groups have origin in 1960's and were used initially to answer Burnside problems [1] and [13]. We got acquainted with this subject in a talk given by Grigorchuk at Sharif University of Technology in Tehran in 1994. In their recent papers Grigorchuk-Zuk introducing an automaton group that is generated by a three state automaton [10, 11] have quoted "It is a question of great importance to continue the study of groups generated by automata and first of all automata with a

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small number of states." This paper has grown out to reply this question partially.
Our aim in this paper is to prove the following theorem.

Theorem 1.1. *Let G be the group generated by the automaton from figure 1, then G has the following properties*

(1) *G is fractal,*
(2) *G is not contracting,*
(3) *G is weakly branch,*
(4) *G contains a copy of the lamplighter group.*

The point is that G is an example of a weakly branch non-contracting group generated by a small automaton.

2. **Preliminaries**

The group G that we study is generated by a three state automaton. To be specific we describe this automaton by a quadruple $A = (D, Q, \varphi, \psi)$, where $D = \{0, 1\}$ is the input and output alphabet, Q is the set of states consisting of three elements, $\varphi : Q \times D \to Q$ is the transition function and $\psi : Q \times D \to D$ is the exit function. A is said to be invertible if for any $q \in Q$ the function $\psi(q,) : D \to D$ is bijective, i.e. $\psi(q, .) \in S_2$ where S_2 is the symmetric group of D, and hence $\psi(q, .)$ is either i or ϵ where

$$i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The invertible automaton A can be described simply by a directed labeled graph $\Gamma(A)$, the set of vertices of $\Gamma(A)$ is Q, an edge connects $q \in Q$ to $s \in Q$ with label t if and only if $\varphi(q, t) = s$ and a vertex q is labeled by the unique bijection $\sigma_q = \psi(q, .)$.
We observe that the automaton A is non-initial. To obtain an initial automaton A_q from A we initialize it at $q \in Q$, i.e. we choose the state q as the initial state and obtain the initial automaton $A_q = (q, D, Q, \varphi, \psi)$. Accordingly we get three initial automata

corresponding to the elements of Q. The automaton A acts on finite and infinite strings of alphabet from left to right via the initial automata A_q and changes them to the same kind of strings. In fact if we feed the string $w = tuv\dots$ to A starting from the state q then A_q will come into play and will read t the first letter of w. This means that the values $q' = \varphi(q,t)$ and $t' = \psi(q,t)$ will be calculated and then A will substitute A_q and t by $A_{q'}$ and u respectively and will go on calculations until to terminate the reading of w and producing the output $w' = t'u'v'\dots$.

We can visualize D^* the set of all strings over D as the binary rooted tree T_2. The root vertex corresponds the null string. From the above discussion we conclude that the initial automaton A_q acts on T_2 as an automorphism, i.e. A_q preserves the root and respects edge incidence. For more details refer to [10].

The initial automata $A_q, q \in Q$ corresponding to the automaton $A = (D, Q, \varphi, \psi)$ generate a group $G(A) =< A_{q_1}, A_{q_2}, A_{q_3} >$ [10] that acts on T_2 by automorphisms. This group is called the group generated by automaton A.

There is a close relationship between Automata Groups and wreath products. To match the needs of this paper we describe this relationship in detail. Consider the groups $G(A)$, S_2, $G(A)^D$. The latter is the group of all functions from D to $G(A)$. The function $f \in G(A)^D$ is determined by its values f_0 and f_1 at 0 and 1 respectively. Therefore we can write $f = (f_0, f_1)$. S_2 acts on D via the right action

$$\sigma x = (x, \sigma) = \sigma^{-1}x, x \in D, \sigma \in S_2.$$

Therefore S_2 also act on $G(A)^D$ via

$$(f, \sigma) = (f_{0\sigma}, f_{1\sigma}).$$

Using these data we can define the wreath product $G(A) \wr S_2$ as follows: The elements of $G(A) \wr S_2$ are the elements of the cartesian product $G(A)^D \times S_2$ and the composition of (f, σ) and (g, δ) with $f = (f_0, f_1)$ and $g = (g_0, g_1)$ is

$$(f, \sigma)(g, \delta) = (h, \sigma\delta)$$

where $h = (h_0, h_1)$ and $h_i = f_i g_{i\sigma}$. Now we can embed $G(A)$ in the wreath product $G(A) \wr S_2$ via the map

$$A_q \to (A_{q,0}, A_{q,1})\sigma_q,$$

where in fact $A_{qj}, j = 0, 1$ is the automaton to which we connect A_q with edge of label j. The expression $(A_{q,0}, A_{q,1})\sigma_q$ is called the wreath decomposition of A_q [11]. Using this embedding we will identify A_q and $(A_{q,0}, A_{q,1})\sigma_q$, will write $A_q = (A_{q,0}, A_{q,1})\sigma_q$ and will omit σ_q when $\sigma_q = i$.

The automata group G that will be studied in this paper is the group generated by the automaton of figure (1). Here we have $a \to (c, a)i$, $c \to (b, c)i$ and $b \to (c, b)\epsilon$. Therefore we write $a = (c, a)$, $c = (b, c)$ and $b = (c, b)\epsilon$.

To facilitate the study of G we now define other concepts that are necessary for this purpose. The length $|u|$ of $u \in D^*$ is the number of letters that constitute u. Let $n \geq 0$ be an integer, the set of all vertices of T_2 with length n is denoted by L_n and is called the $n-$th level of T_2.

There are four types of subgroups of G that are very useful: stabilizer of a vertex of T_2, stabilizer of a level of T_2, rigid stabilizers and stabilizer of an element of the boundary of T_2 (the so called parabolic subgroups of G[10]).

Definition 2.1. We denote the subgroup of G that stabilizes the vertex u of T_2 by $St_G(u)$, i.e.

$$St_G(u) = \{g \in G | ug = u\}.$$

Also the subgroup of G that stabilizes the level L_n of T_2 is denoted by $St_G(n)$. We have

$$St_G(n) = \{g \in G | ug = u, u \in L_n\}$$

By the x-length of the word $w \in S^*$ is the number of occurrences of x in w, we denote this by $|w|_x$.

The fact that the subgroups $St_G(n), n = 1, 2, \ldots$ are normal is proved in [4]. Considering $g \in St_G(1)$ as an automaton we observe that g corresponds to a pair $((g_0, g_1), 1) = (g_0, g_1)i = (g_0, g_1)$ in the wreath product $G(A) \wr S_2$, i.e. the label of the start state of g is i

(this is the crucial fact that g fixes 0 and 1) and we connect g to g_0 and g_1 with edges labeled 0 and 1 respectively. Consequently there is a well defined embedding

$$\psi : St_G(1) \to G \times G, \psi(g) = (g_0, g_1)$$

and hence well defined canonical projections $\phi_i : St_G(1) \to G, i = 0, 1; \phi_i(g) = g_i, i = 0, 1$ from $St_G(1)$ to the base group G.
Similarly one can define the projections $\phi_u : St_G(u) \to G$ for any vertex u.

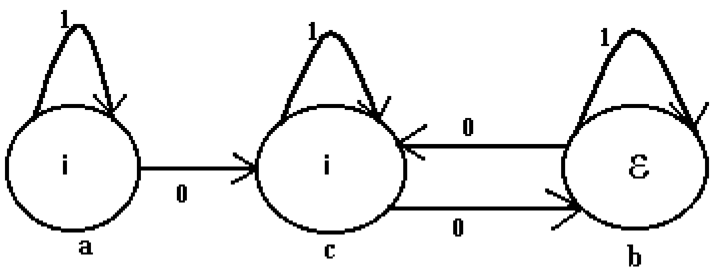

Figure 1. An automaton on Q and D

Definition 2.2. A group G that acts by automorphisms on a rooted tree T is called fractal if for every vertex u, $\phi_u(St_G(u)) = G$ after the identification of the tree with the sub-tree T_u with root at u.

Lemma 2.3. *G acts on levels of T_2 transitively*

Proof. The proof is the same as the proof of the Proposition 36 in chapter 8 of [6]. □

3. Fractal and non-contracting

In this section we prove that G is fractal but not contracting.

Proposition 3.1. *G is a fractal group.*

Proof. As indicated in the last section we define the homomorphism $\psi : St_G(1) \to G \times G$ by $\psi = (\phi_0, \phi_1)$ where

$$\psi(a) = (\phi_0(a), \phi_1(a)) = (c, a), \psi(b^2) = (\phi_0(b^2), \phi_1(b^2)) = (cb, bc)$$
$$\psi(c) = (\phi_0(c), \phi_1(c)) = (b, c),$$
$$\psi(b^{-1}ab) = (\phi_0(b^{-1}ab), \phi_1(b^{-1}ab)) = (b^{-1}ab, c)$$
$$\psi(b^{-1}cb) = (\phi_0(b^{-1}cb), \phi_1(b^{-1}cb)) = (b^{-1}cb, c^{-1}bc)$$

we observe that

$$a^{-1}bab^{-1}a = [a, b^{-1}]a \in G$$
$$\phi_0[a, b^{-1}]a = a$$

and also

$$\phi_0[bcb^{-1}] = c$$
$$\phi_1[bcb^{-1}] = b$$

Therefore each of the projections of $St_G(1)$ in G is G itself, i.e. G is fractal.□

Definition 3.2. A group G is called contracting if there is $\lambda < 1$ and $C, L \in \mathbb{N}$ such that for any vertex u of level $l > L$ we have

$$|g_u| < \lambda|g| + C$$

For information on g_u refer to [11].

Proposition 3.3. *$b^n = 1$ if and only if $n = 0$.*

This proposition will be proved through the following lemmas. But before proceeding we introduce some notations.
For a fixed positive integer n define the elements 0_n, 1_n and w_n of D^* as follows.

$$0_n = \overbrace{00\ldots0}^{n}$$
$$1_n = \overbrace{11\ldots1}^{n}$$

and

$$w_1 = 0$$

$$w_n = 0\overbrace{1\ldots1}^{n-1} = o1_{n-1}, ...n > 1.$$

We note that each of these words is of length n except for w_1 which is of length 1 .

Lemma 3.4. *For any positive odd integer n we have $b^n \neq 1$.*

Proof. This is obvious from the definition of b. □

Lemma 3.5. *Let n be a positive even integer. For any positive integer m we have*

$$c^{n-1}b(1_{m+1}) = 0b^n(1_m) \tag{3.1}$$

Proof. We calculate

$$c^{n-1}b(1_{m+1}) = c^{n-1}(0)b(1_m) = ob^n(1_m)$$

□

Lemma 3.6. *If for some $n = 2^k$ and for some word $x \in D^*$ the relations*

$$b^n(w_n x) = w_{n-1}c^{n-2}bc(1x) = w_{n-1}ob^{n-1}c(x) \tag{3.2}$$

hold then we have

$$(bc)^{\frac{n}{2}}(1_{n-1}x) = 1_{n-2}c^{n-2}bc(1x) \tag{3.3}$$

Proof. using the relations $b(0x) = 1c(x)$ and $b(1x) = 0b(x)$ repeatedly we have

$$b^n(w_n x) = b^n(01_{n-1}x) = o(bc)^{\frac{n}{2}}(1_{n-1}x)$$

comparing the words in (4.2) by the right hand side of this relation we obtain

$$o(bc)^{\frac{n}{2}}(1_{n-1}x) = 01_{n-2}c^{n-2}bc(1x).$$

Therefore using the notion of equality of words we have

$$(bc)^{\frac{n}{2}}(1_{n-1}x) = 1_{n-2}c^{n-2}bc(1x).$$

The proof is complete. □

We remark that under the conditions of Lemma 3.6 we have

$$(bc)^{\frac{n}{2}}(1_{n-1}x) = 1_{n-2}ob^{n-1}c(x) \tag{3.4}$$

Lemma 3.7. *For $n = 2^k, k = 1, 2, \ldots$ and for any $x \in D^*$ we have*

$$b^n(w_{n-1}1x) = w_{n-1}c^{n-2}bc(1x) \tag{3.5}$$

Proof. For $n = 2$ we have

$$b^2(w_1 1x) = b^2(01x) = b(1c(1x)) = w_1 c^{2-2}bc(1x).$$

Therefore (3.5) is true for $k = 1$. Let (3.5) be true for k. For $k + 1$ we write $n = 2^k$ and

$$m = 2^{k+1} = 2.2^k = 2n$$

so that assuming the truth of (3.5) we have to prove the truth of

$$b^m(w_{m-1}1x) = w_{m-1}c^{m-2}bc(1x)$$

or the truth of

$$b^{2n}(w_{2n-1}1x) = w_{2n-1}c^{2n-2}bc(1x)$$

Using the hypothesis of induction we write

$$b^{2n}(w_{2n-1}1x) = b^n b^n(w_{n-1}1_n 1x) = b^n(w_{n-1}c^{n-2}bc(1_n 1x)) \tag{3.6}$$

Now using again the hypothesis of induction (or Lemma 3.6)we have

$$c^{n-2}bc(1_n 1x) = c^{n-2}bc(11_n x)) = 0b^{n-1}c(1_n x) \tag{3.7}$$

We put from (3.7) in (3.6) and obtain

$$b^{2n}(w_{2n-1}1x) = b^n(w_{n-1}0b^{n-1}c(1_n x)) \tag{3.8}$$

Now again in (3.7) we apply induction hypothesis and obtain

$$b^{2n}(w_{2n-1}1x) = w_{n-1}c^{n-2}bc(0b^{n-1}c(1_n x)) \tag{3.9}$$

We use the relations $n = 2^k$, $c(0) = 0b$, $c(1) = 1c$, $b(0) = 1c$ and $b(1) = 1b$ and write(3.9) as follows

$$b^{2n}(w_{2n-1}1x) = w_n c^{n-1}b^n c(1_n x) = w_n c^{n-1}b^n c(11_{n-1}x) =$$

$$w_n c^{n-1} b^n(1) c(1_{n-1}x) = w_{n+1} c^{n-1} (cb)^{\frac{n}{2}} c(1_{n-1}x) =$$
$$w_{n+1} c^n (cb)^{\frac{n}{2}} (1_{n+1}x)$$

The proof of the Lemma (3.7) is complete. □

We remark here that for $n = 2k$ and for the word $w = 0_{n-1}110$ one can prove $a^n(w) \neq w$ and hence $a^n \neq 1$. Therefore by the following lemma $b^n \neq 1$ for $n = 1, 2, \ldots$.

Lemma 3.8. *For the integer n, the relations $b^n = 1$, $c^n = 1$ and $a^n = 1$ are equivalent .*

Proof. Let $b^n = 1$. From $c = (b, c)$ we obtain that

$$c^n = (b^n, c^n) = (1, c^n)$$

Therefore by definition of c we have $c^n = 1$
Conversely let $c^n = 1$. Then by definition of c we have

$$1 = (b^n, 1)$$

which together with the definition of b imply $b^n = 1$. We observe that $a^n = 1$ is equivalent to $c^n = 1$. The proof is complete. □

Proposition 3.9. *G is not contracting*

Proof. For $x = x_\theta$ the empty word and for $n = 2^k, k = 1, 2, \ldots$ from Lemma 3.7 we obtain

$$b^n(w_n) = w_{n-1}0$$

and hence $b^{2^k} \neq 1$ for $k = 1, 2, \ldots$. This together with the definition of b imply that

$$b^{2^k} = ((cb)^{2^{k-1}}, (bc)^{2^{k-1}})$$

This implies that

$$|b^{2^k}| = 2^k < 2^{k+1} = |(cb)^{2^{k-1}}| + |(bc)^{2^{k-1}}| \qquad (3.10)$$

Now let k be any positive integer and consider the element a^{2^k} whose leftmost coordinate at any level of T_2 is c^{2^k} and the left coordinate of this last element is b^{2^k}. This together with (3.10) implies that G is non-contracting. The proof of the Proposition 3.3 is complete. □

4. G is weakly branch

In this section we show that G is weakly branch.

Definition 4.1. A group H that acts spherically transitively on tree T_d is called weakly branch if none of its rigid stabilizers $Rist_H(u)$ is trivial.
Recall that by $Rist_H(u)$ we mean a subgroup of H that acts trivially out of T_u in T_2. Of course here we deal with T_2 and $H = G$.

Lemma 4.2. *$Rist_G(0)$ and $Rist_G(1)$ are nontrivial.*

Proof. Let

$$bc^{-1} = (c, b)\epsilon(b^{-1}, c^{-1}) = (1, 1)\epsilon$$

and

$$t = a^{-1}b = (c^{-1}, a^{-1})(c, b)\epsilon = (1, a^{-1}b)\epsilon = (1, t)\epsilon.$$

Observe that $t\epsilon = (1, t) \in Rist_G(1)$ and $(t, 1) = \epsilon(t\epsilon)\epsilon \in Rist_G(0)$. This proves the lemma. □

Lemma 4.3. *The subgroups $Rist_G(11)$, $Rist_G(10)$ and $Rist_G(01)$ and $Rist_G(00)$ are nontrivial.*

Proof. We write ϵ for $(1, 1)\epsilon$. Observe that $[b, c^{-1}] = (b^{-1}c, c^{-1}b) \in G$ and therefore $u = (\epsilon, \epsilon) \in G$. We have $(1, t)u = (\epsilon, t\epsilon) = ((1, 1)\epsilon, (1, t))$. Therefore $((1, t)u)^2 = (1, 1, 1, t^2) \in Rist_G(11)$.
From $(t^{-1}\epsilon)u = (t^{-1}, 1, (1, 1)\epsilon)$ we conclude that $(t^{-2}, 1, 1, 1) \in Rist_G(00)$.
The relation $\epsilon(1, t)\epsilon u = (t\epsilon, \epsilon)$ implies that $(1, t^2, 1, 1) \in Rist_G(01)$.
Finally we have $(\epsilon t^{-1}u)^2 = (1, 1, t^{-2}, 1) \in Rist_G(10)$. Since $t^2 = (t, t) \neq 1$ the proof is complete. □

Proposition 4.4. *For $n \in \mathbb{N}$ and $u = u_1u_2 \cdots u_n \in D^*$, $Rist_G(u)$ is nontrivial.*

Proof. The proof is easily carried out by induction taking into account the proof of the above lemma.
For example from $W = (1,1,1,t^2) \in G$, $t^2 = (t,t)$ and $(t\epsilon)^2 = (1,t^2)$ we obtain $W^2 = (1,1,1,1,1,1,t^2,t^2)$ and $W' = (1,1,1,1,1,(1,t^2), t^2) = (1,1,1,1,1,(t\epsilon)^2,t,t)$. Therefore $(W')^2W^{-2} = (1,1,1,1,1,(t\epsilon)^2, 1,1)$ together with $t\epsilon \neq 1$ prove that $Rist_G(101)$ is nontrivial. The proof is complete. □

5. G and lamplighter group

In this section we prove that G contains a copy of the lamplighter group. The lamplighter group L is an automaton group that is generated by the automaton from figure 2 [10].

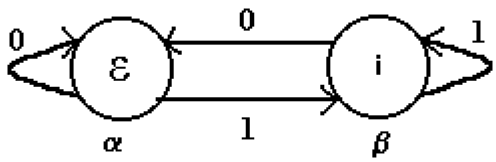

Figure 2.

According to [10] p 223, L has a presentation of the form

$$L =< \alpha, \gamma | \gamma^2 = 1, [\gamma^{\alpha^i}, \gamma^{\alpha^j}], i, j \in \mathbb{Z} >$$

Where $\gamma = \alpha^{-1}\beta$. Let H be the subgroup of G that is generated by elements b and c.

Lemma 5.1. *H and L are isomorphic.*

Proof. Comparing the automata generating H and L we define the function $\varphi : L \to H$ by defining it in generators as $\varphi(\alpha) = b^{-1}$ and $\varphi(\beta) = c^{-1}$ and extending it linearly to obtain a homomorphism . Now we prove φ is in fact an isomorphism. To this end we show that if R is a relator of L then $\varphi(R) = 1$. For $R = \gamma^2$ we have

$$\varphi(\gamma^2) = \varphi(\alpha^{-1}\beta\alpha^{-1}\beta) = bc^{-1}bc^{-1} = 1$$

Also if $R = R_{i,j} = [\gamma^{\alpha^i}, \gamma^{\alpha^j}]$ we compute

$$\varphi(R) = [(bc^{-1})^{b^{-i}}, (bc^{-1})^{b^{-j}}]$$

Let $\lambda = bc^{-1}$ we have $\lambda^{b^{-i}} = b^i\gamma b^{-i}$.
For $i = 2k$ and $j = 2l$ even and nonnegative we have $b^i = ((cb)^k, (bc)^k)$ and therefore we observe that

$$\lambda^{b^{-i}} = ((cb)^k, (bc)^k)(1,1)\epsilon((cb)^{-k}, (bc)^{-k}) =$$
$$((cb)^k(bc)^{-k}, (bc)^k(cb)^{-k})\epsilon = (\delta_k, {\delta_k}^{-1})\epsilon$$

where $\delta_k = (cb)^k(bc)^{-k}$. Therefore we have

$$\varphi(R) = (\delta_k, {\delta_k}^{-1})\epsilon(\delta_l, {\delta_l}^{-1})\epsilon(\delta_k, {\delta_k}^{-1})\epsilon(\delta_l, {\delta_l}^{-1})\epsilon =$$
$$(\delta_k{\delta_l}^{-1}, {\delta_k}^{-1}\delta_l)(\delta_k{\delta_l}^{-1}, {\delta_k}^{-1}\delta_l) =$$
$$(\delta_k{\delta_l}^{-1}\delta_k{\delta_l}^{-1}, {\delta_k}^{-1}\delta_l{\delta_k}^{-1}\delta_l)$$

Now by induction on k we prove $\varphi(R) = 1$ for any fixed l. For $k = 0$ we have $\delta_0 = 1$ and so

$$\varphi(R) = ({\delta_l}^{-2}, {\delta_l}^2)$$

Now we compute ${\delta_l}^2$. We have

$${\delta_l}^2 = (cb)^l(bc)^{-l}(cb)^l(bc)^{-l}$$

Since $cb = (bc, cb)\epsilon, bc^{-1} = (b^{-2}, c^{-2})\epsilon, b^{-1}c = \epsilon, \epsilon b = (b, c) = c$, we have

$$(bc)^{-2}(cb)^2 = c^{-1}b^{-1}c^{-1}\epsilon bcb = c^{-1}b^{-1}c^{-1}ccb = c^{-1}b^{-1}cb = c\epsilon b$$
$$= c^{-1}c = 1$$

Thus for $l = 2h$ even We have proved $(bc)^{-l}(cb)^l = 1$. Therefore ${\delta_l}^2$ reduces to

$${\delta_l}^2 = (cb)^l(bc)^{-l}$$

Therefore we have

$${\delta_l}^2 = (bc)^l(bc)^{-l}(cb)^l(bc)^{-l} = (bc)^l(bc)^{-l} = 1$$

Thus when l is even and $k = 0$ we have $\varphi(R) = 1$.
Now let $l = 2h + 1$ be odd and $k = 0$. We have

$$\varphi(R) = ({\delta_{2h+1}}^{-2}, {\delta_{2h+1}}^2)$$

and

$$\delta_{2h+1}{}^2 = (cb)^{2h+1}(bc)^{-l}[(bc)^{-2h}(cb)^{2h}](cb)(bc)^{-2h-1}$$

And this regarding the case l even and $cbc^{-1}b^{-1}cbc^{-1}b^{-1} = 1$ easily reduces to 1. Therefore we have proved the induction step.
Now assume that for a fixed but arbitrary l and for $k \leq n$ we have $\varphi(R) = 1$. We prove

$$\varphi(R) = (\delta_{n+1}\delta_l{}^{-1}\delta_{n+1}\delta_l{}^{-1}, \delta_{n+1}{}^{-1}\delta_l\delta_{n+1}{}^{-1}\delta_l) = 1$$

We write

$$\delta_{n+1}\delta_l{}^{-1}\delta_{n+1}\delta_l{}^{-1} = \delta_1\delta_n\delta_l{}^{-1}\delta_n\delta_l^{-1}\delta_l\delta_1\delta_l{}^{-1} = \delta_1\delta_l\delta_1\delta_l{}^{-1}$$

By inversions and conjugations and using the induction hypothesis we observe that

$$\delta_1\delta_l\delta_1\delta_l{}^{-1} \to \delta_l\delta_1{}^{-1}\delta_l{}^{-1}\delta_1{}^{-1} \to \delta_1{}^{-1}\delta_l\delta_1{}^{-1}\delta_l{}^{-1}$$
$$\to \delta_1{}^{-1}\delta_l\delta_1{}^{-1}\delta_l\delta_l{}^{-2} \to \delta_l{}^{-2}.$$

Since $\delta_l{}^{-2} = 1$ the proof is complete in this case .
The proof in other two cases is quite similar. The proof of the lemma is complete. □
Being isomorphic to an automata group L acts on T_2 by automorphisms. Therefore as a corollary for Lemma 4.1 we have:

Corollary 5.2. *The lamplighter as an automata group is non-contracting*

We note that this corollary implies that any automata group that contains a copy of L is non-contracting.

Corollary 5.3. *Any automata group containing an isomorphic copy of the lamplighter group is non-contracting*

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M. J. Mamaghani
Dept. of Mathematics and Statistics
Univ. of Allameh Tabatabaii
Tehran
Iran
e-mail:j_mamaghan@atu.ac.ir