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PERTURBATION BOUNDS FOR g-INVERSES WITH RESPECT TO THE UNITARILY INVARIANT NORM

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Abstract. Let complex matrices A and B have the same sizes. Using the singular value decomposition, we characterize the g-inverse $B^{(1)}$ of B such that the distance between a given g-inverse of A and the set of all g-inverses of the matrix B reaches minimum under the unitarily invariant norm. With this result, we derive additive and multiplicative perturbation bounds of the nearest perturbed g-inverse. These results generalize and improve the existing results published recently to some extent.
Keywords: g-inverse, additive perturbation bound, multiplicative perturbation bound, unitarily invariant norm.
MSC(2010): Primary: 15A09; Secondary: 65F20.

1. Introduction

Let $C^{m\times n}$ and $C_r^{m\times n}$ be the set of all $m \times n$ complex matrices and its subset with rank r, respectively. For a given matrix $A \in C^{m\times n}$, the symbols A^*, $A^\dagger$, $\|A\|_2$, and $\|A\|$, respectively, stand for the conjugate transpose, the Moore-Penrose inverse, the spectral norm, and the unitarily invariant norm of A. By I_m we denote the identity matrix of order m. Moreover,

$$P_A = AA^\dagger,\ P_A^\perp = I_m - AA^\dagger,\ P_{A^*} = A^\dagger A, \text{ and } P_{A^*}^\perp = I_n - A^\dagger A$$

stand for four orthogonal projectors induced by A.

Recall that the Moore-Penrose inverse $A^\dagger$ of a matrix $A \in C^{m\times n}$ is defined to be the unique solution of the four Penrose equations [2]:

$$(1.1) \qquad (1)\ AGA = A,\ (2)\ GAG = G,\ (3)\ (AG)^* = AG,\ (4)\ (GA)^* = GA.$$

If G satisfies the first equation of (1.1), then G is called a g-inverse or a $\{1\}$-inverse of A and is denoted by $A^{(1)}$. As is known, g-inverse is not unique in general. The set of all g-inverses of A is denoted by $A\{1\}$. We refer the reader to [2, 15] for basic results on the g-inverse.

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The perturbation theory of the Moore-Penrose inverse is a classical topic in matrix analysis and numerical linear algebra. The additive perturbation theory (when A is perturbed to $B = A + E$, where $A, E \in C^{m\times n}$) of the Moore-Penrose inverse was studied by many authors with respect to different norms, see [1, 2, 5, 8, 12–14, 16] and the references therein. Lately, there has been an increasing of interest in the multiplicative perturbation theory (A is perturbed to $B = D_1^* A D_2$, where D_1 and D_2 are, respectively, $m \times m$ and $n \times n$ nonsingular matrices) of the Moore-Penrose inverse due, in part, to its application to the error analysis of algorithms that solve structured least squares problem with high relative accuracy, see [3–6, 8, 9, 18].

As is known, the Moore-Penrose inverse, $\{1,3\}$-inverses, $\{1,2,3\}$-inverses, and the group inverse all belong to g-inverses. Owning to the extensive applications in matrix theory and computation [2], g-inverses receive lots of consideration. Liu et al. [7] studied the continuity properties of g-inverse under condition of rank invariant perturbations. Wei and Ling [17] obtained the additive perturbation bounds of g-inverse under the spectral and Frobenius norms. Recently, Meng et al. [10] studied the multiplicative perturbation bounds of g-inverses with respect to the spectral and Frobenius norms. The spectral norm and the Frobenius norm are special instances of unitarily invariant norms, i.e., norms $\|\cdot\|$ that satisfy $\|UXV\| = \|X\|$ for all $X \in C^{m\times n}$ and unitary matrices $U \in C^{m\times m}$ and $V \in C^{n\times n}$; see [11] for a general background of unitarily invariant norms. In this paper, we further undertake the perturbation analysis for g-inverses with respect to the unitarily invariant norm.

Given $A, B \in C^{m\times n}$ and $A^{(1)} \in A\{1\}$, we first specify formula of the g-inverse $B^{(1)} \in B\{1\}$ singular value decomposition (SVD), such that $B^{(1)}$ is closest to $A^{(1)}$ under the unitarily invariant norm. Then we present the additive and multiplicative perturbation bounds for the nearest perturbed g-inverse with respect to the unitarily invariant norm. To our knowledge, there is no article yet discussing these problems in the literature.

Before starting our discussion, we list some lemmas which will be used in the sequel.

Lemma 1.1 ([2]). *Let $A \in C_r^{m\times n}$, then the general expressions of g-inverses of A can be written as:*

$$A\{1\} = \{A^\dagger + A^\dagger A Z(I_m - AA^\dagger) + (I_n - A^\dagger A)Z : Z \in C^{n\times m}\}.$$

Furthermore, let $A = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V^*$ be the singular value decomposition (SVD) of A, where $U \in C^{m\times m}$, $V \in C^{n\times n}$ are unitary matrices, and $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_r)$. Then the general g-inverse of A can be expressed as

$$A^{(1)} = V \begin{bmatrix} \Sigma_1^{-1} & K \\ L & M \end{bmatrix} U^*, \tag{1.2}$$

where K, L and M are arbitrary matrices of appropriate sizes. In particular, the Moore-Penrose inverse is (1.2) with $K = 0$, $L = 0$ and $M = 0$.

Lemma 1.2 ([16]). *Let $A, B = A + E \in C^{m\times n}$, then*

$$B^\dagger - A^\dagger = -B^\dagger E A^\dagger + B^\dagger (I_m - AA^\dagger) - (I_n - B^\dagger B)A^\dagger.$$

Lemma 1.3 ([10]). *Let $A, B \in C^{m\times n}$ and $B^*A = 0$ or $A^*B = 0$. Then*

$$\|A\| \le \|A + B\|.$$

2. The formula of the nearest g-inverse

Given $A, B \in C^{m\times n}$ and $A^{(1)} \in A\{1\}$, Wei and Ling [17, Theorems 3.1 and 3.2] proved that

$$B_m^{(1)} = B^\dagger + B^\dagger B A^{(1)}(I_m - BB^\dagger) + (I_n - B^\dagger B)A^{(1)}$$

is the closest g-inverse of B to $A^{(1)}$ under the Frobenius norm and the spectral norm. The following theorem shows that their results still hold when the spectral norm or Frobenius norm is generalized to any unitarily invariant norm.

Theorem 2.1. *Let $A, B \in C^{m\times n}$. For any given $A^{(1)} \in A\{1\}$, there exists a matrix $B_m^{(1)} \in B\{1\}$ of the form*

$$B_m^{(1)} = B^\dagger + B^\dagger B A^{(1)}(I_m - BB^\dagger) + (I_n - B^\dagger B)A^{(1)}$$

such that

$$\min_{B^{(1)}\in B\{1\}} \left\| B^{(1)} - A^{(1)} \right\| = \left\| B_m^{(1)} - A^{(1)} \right\| = \left\| B^\dagger - B^\dagger B A^{(1)} B B^\dagger \right\|. \tag{2.1}$$

Proof. Let $A \in C_r^{m\times n}$, $B \in C_s^{m\times n}$ respectively have the following SVDs:

$$A = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V^* \text{ and } B = \widetilde{U} \begin{bmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & 0 \end{bmatrix} \widetilde{V}^*, \tag{2.2}$$

where $U = [U_1\ U_2]$, $\widetilde{U} = [\widetilde{U}_1\ \widetilde{U}_2] \in C^{m\times m}$ and $V = [V_1\ V_2]$, $\widetilde{V} = [\widetilde{V}_1\ \widetilde{V}_2] \in C^{n\times n}$ are unitary matrices, $U_1 \in C^{m\times r}$, $\widetilde{U}_1 \in C^{m\times s}$, $V_1 \in C^{n\times r}$, $\widetilde{V}_1 \in C^{n\times s}$, $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_r)$, $\widetilde{\Sigma}_1 = diag(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_s)$, $\sigma_1 \ge \cdots \ge \sigma_r > 0$ and $\widetilde{\sigma}_1 \ge \cdots \ge \widetilde{\sigma}_s > 0$. Let $S = \widetilde{U}^* U$ and $T = \widetilde{V}^* V$ have the block form

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in C^{m\times m} \text{ and } T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in C^{n\times n} \tag{2.3}$$

with $S_{11}, T_{11} \in C^{s\times r}$, then S and T are unitary matrices.

For the given $A^{(1)}$, from Lemma 1.1 there exist matrices $K \in C^{\,r\times(m-r)}$, $L \in C^{(n-r)\times r}$ and $M \in C^{(n-r)\times(m-r)}$ such that $A^{(1)} = V \begin{bmatrix} \Sigma_1^{-1} & K \\ L & M \end{bmatrix} U^*$. Since any $B^{(1)} \in B\{1\}$ has the form $B^{(1)} = \widetilde{V} \begin{bmatrix} \widetilde{\Sigma}_1^{-1} & \widetilde{K} \\ \widetilde{L} & \widetilde{M} \end{bmatrix} \widetilde{U}^*$, where $\widetilde{K} \in$

$C^{\,s\times(m-s)}$, $\widetilde{L}\in C^{(n-s)\times s}$ and $\widetilde{M}\in C^{(n-s)\times(m-s)}$ are arbitrary matrices, we have

$$\begin{aligned}\|B^{(1)}-A^{(1)}\| &= \left\|\begin{bmatrix}\widetilde{\Sigma}_1^{-1} & \widetilde{K}\\ \widetilde{L} & \widetilde{M}\end{bmatrix}\widetilde{U}^*U-\widetilde{V}^*V\begin{bmatrix}\Sigma_1^{-1} & K\\ L & M\end{bmatrix}\right\|\\ &= \left\|\begin{bmatrix}\widetilde{\Sigma}_1^{-1}S_{11}+\widetilde{K}S_{21}-H_1 & \widetilde{\Sigma}_1^{-1}S_{12}+\widetilde{K}S_{22}-H_2\\ 0 & 0\end{bmatrix}+\right.\\ &\qquad \left.+\begin{bmatrix}0 & 0\\ \widetilde{L}S_{11}+\widetilde{M}S_{21}-H_3 & \widetilde{L}S_{12}+\widetilde{M}S_{22}-H_4\end{bmatrix}\right\|, \end{aligned} \tag{2.4}$$

where S_{ij} and T_{ij} are defined by (2.2) and

$$H_1=T_{11}\Sigma_1^{-1}+T_{12}L,\ \ H_2=T_{11}K+T_{12}M,$$

$$H_3=T_{21}\Sigma_1^{-1}+T_{22}L,\ \ H_4=T_{21}K+T_{22}M.$$

Applying Lemma 1.3 to (2.4), we obtain

$$\begin{aligned}&\left\|\begin{bmatrix}\widetilde{\Sigma}_1^{-1}S_{11}+\widetilde{K}S_{21}-H_1 & \widetilde{\Sigma}_1^{-1}S_{12}+\widetilde{K}S_{22}-H_2\\ 0 & 0\end{bmatrix}\right\|\\ \leq\ &\left\|B^{(1)}-A^{(1)}\right\|.\end{aligned} \tag{2.5}$$

Choosing

$$\widetilde{L}_m=H_3S_{11}^*+H_4S_{12}^*\ \text{and}\ \widetilde{M}_m=H_3S_{21}^*+H_4S_{22}^*,$$

it is easy to check that

$$\widetilde{L}_mS_{11}+\widetilde{M}_mS_{21}-H_3=0,\ \ \widetilde{L}_mS_{12}+\widetilde{M}_mS_{22}-H_4=0. \tag{2.6}$$

Combining (2.4)–(2.6), we get

$$\begin{aligned}&\min_{B^{(1)}\in B\{1\}}\left\|B^{(1)}-A^{(1)}\right\|\\ =\ &\min_{\widetilde{K}\in C^{s\times(m-s)}}\left\|\begin{bmatrix}\widetilde{\Sigma}_1^{-1}S_{11}+\widetilde{K}S_{21}-H_1 & \widetilde{\Sigma}_1^{-1}S_{12}+\widetilde{K}S_{22}-H_2\end{bmatrix}\right\|.\end{aligned}$$

Similarly, using the following equality

$$\begin{aligned}&\begin{bmatrix}\widetilde{\Sigma}_1^{-1}S_{11}+\widetilde{K}S_{21}-H_1 & \widetilde{\Sigma}_1^{-1}S_{12}+\widetilde{K}S_{22}-H_2\end{bmatrix}\\ =\ &\begin{bmatrix}\widetilde{\Sigma}_1^{-1}S_{11}-H_1+\widetilde{K}_mS_{21} & \widetilde{\Sigma}_1^{-1}S_{12}-H_2+\widetilde{K}_mS_{22}\end{bmatrix}\\ &+\begin{bmatrix}(\widetilde{K}-\widetilde{K}_m)S_{21} & (\widetilde{K}-\widetilde{K}_m)S_{22}\end{bmatrix}\end{aligned}$$

with $\widetilde{K}_m = H_1 S_{21}^* + H_2 S_{22}^*$, we get

$$\begin{aligned}
&\min_{B^{(1)}\in B\{1\}} \left\|B^{(1)} - A^{(1)}\right\| = \left\|B_m^{(1)} - A^{(1)}\right\| \\
= &\left\|\begin{bmatrix} \widetilde{\Sigma}_1^{-1}S_{11} - H_1 + \widetilde{K}_m S_{21} & \widetilde{\Sigma}_1^{-1}S_{12} - H_2 + \widetilde{K}_m S_{22} \end{bmatrix}\right\| \\
= &\left\|\begin{bmatrix} \widetilde{\Sigma}_1^{-1}S_{11} & \widetilde{\Sigma}_1^{-1}S_{12} \end{bmatrix} - (H_1 S_{11}^* + H_2 S_{12}^*)\begin{bmatrix} S_{11}^* & S_{12}^* \end{bmatrix}\right\| \\
= &\left\|\begin{bmatrix} \widetilde{\Sigma}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\widetilde{U}^*U - \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}\widetilde{V}^*V\begin{bmatrix} \Sigma_1^{-1} & K \\ L & M \end{bmatrix}U^*\widetilde{U}\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}\widetilde{U}^*U\right\| \\
= &\left\|B^\dagger - B^\dagger B A^{(1)} B B^\dagger\right\|,
\end{aligned}$$

where

$$\begin{aligned}
B_m^{(1)} &= \widetilde{V}\begin{bmatrix} \widetilde{\Sigma}_1^{-1} & \widetilde{K}_m \\ \widetilde{L}_m & \widetilde{M}_m \end{bmatrix}\widetilde{U}^* \\
&= \widetilde{V}\begin{bmatrix} \widetilde{\Sigma}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\widetilde{U}^* + \widetilde{V}\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}\widetilde{V}^*V\begin{bmatrix} \Sigma_1^{-1} & K \\ L & M \end{bmatrix}U^*\widetilde{U} \times \\
&\quad \begin{bmatrix} 0 & 0 \\ 0 & I_{m-s} \end{bmatrix}\widetilde{U}^* + \widetilde{V}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-s} \end{bmatrix}\widetilde{V}^*V\begin{bmatrix} \Sigma_1^{-1} & K \\ L & M \end{bmatrix}U^*\widetilde{U}\widetilde{U}^* \\
&= B^\dagger + B^\dagger B A^{(1)}(I_m - BB^\dagger) + (I_n - B^\dagger B)A^{(1)}.
\end{aligned}$$

Here the last equality follows from the facts

$$B^\dagger = \widetilde{V}\begin{bmatrix} \widetilde{\Sigma}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\widetilde{U}^*,\ B^\dagger B = \widetilde{V}\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}\widetilde{V}^*,$$

$$I_m - BB^\dagger = \widetilde{U}\begin{bmatrix} 0 & 0 \\ 0 & I_{m-s} \end{bmatrix}\widetilde{U}^* \text{ and } I_n - B^\dagger B = \widetilde{V}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-s} \end{bmatrix}\widetilde{V}^*.$$

The proof is completed. □

Remark 2.2. It is worthy to point out that, for any given $A^{(1)} \in A\{1\}$, the nearest g-inverse is unique under the Frobenius norm. However, the nearest g-inverse may not be unique under the other unitarily invariant norms [17].

3. Perturbation bounds for the nearest g-inverse

By using the result obtained in the previous section, we can derive the additive and multiplicative perturbation bounds for the nearest g-inverse under the unitarily invariant norm. Notice that in the analysis of the previous section, we do not enforce the condition $\operatorname{rank}(B) = \operatorname{rank}(A)$. Liu et al. [7] proved that, for stable perturbations the condition $\operatorname{rank}(B) = \operatorname{rank}(A)$ is necessary. Hence, in this section, we study the additive perturbation bounds for the nearest g-inverse under the condition $\operatorname{rank}(B) = \operatorname{rank}(A)$.

Theorem 3.1. *Let* $A, B = A + E \in C^{m\times n}$ *with* $rank(B) = rank(A)$ *and* $\|A^\dagger\|_2\|E\|_2 < 1$. *For any given* $A^{(1)} \in A\{1\}$ *of the form*

$$A^{(1)} = A^\dagger + A^\dagger AZ(I_m - AA^\dagger) + (I_n - A^\dagger A)Z,\ \ Z \in C^{n\times m},$$

let $B_m^{(1)}$ *be as in Theorem 2.1. Then*

$$\begin{aligned} \|B_m^{(1)} - A^{(1)}\| &\leq \|A^\dagger\|_2\big(\|P_B E\|_2\|A^\dagger + (I_n - A^\dagger AZ)\| + \\ &\quad \|P_A^\perp E P_{B^*}\|_2\|A^\dagger AZ(I_m - AA^\dagger)\|\big) + \mathcal{O}(\|E\|_2\|E\|). \end{aligned} \tag{3.1}$$

Proof. It follows from Lemma 1.2, (2.1), $B^\dagger = B^\dagger BB^\dagger$ and $B = A + E$, that

$$\begin{aligned} &\left\|B_m^{(1)} - A^{(1)}\right\| = \left\|B^\dagger - B^\dagger BA^{(1)}BB^\dagger\right\| = \left\|B^\dagger B\left(B^\dagger - A^{(1)}\right)BB^\dagger\right\| \\ = &\ \Big\|B^\dagger B\Big(- B^\dagger EA^\dagger + B^\dagger(I_m - AA^\dagger) - A^\dagger AZ(I_m - AA^\dagger) \\ &-(I_n - A^\dagger A)Z\Big)BB^\dagger\Big\| \\ \leq &\ \|B^\dagger EA^\dagger + B^\dagger E(I_n - A^\dagger A)Z\| + \|B^\dagger(I_m - AA^\dagger)EB^\dagger\| \\ &+\|A^\dagger AZ(I_m - AA^\dagger)EB^\dagger\| \\ \leq &\ \|B^\dagger\|_2\Big(\|P_B E\|_2\|A^\dagger + (I_n - A^\dagger A)Z\| + \|P_A^\perp EP_{B^*}\|_2\|A^\dagger AZ \times \\ &(I_m - AA^\dagger)\|\Big) + \|B^\dagger\|_2^3\|E\|_2\|E\|. \end{aligned}$$

Also from the conditions of this theorem, we observe

$$\|B^\dagger\|_2 \leq \frac{\|A^\dagger\|_2}{1 - \|A^\dagger\|_2\|E\|_2},$$

from which the inequality of (3.1) follows. □

Remark 3.2. The bound (3.1) in Theorem 3.1 is reduced to the bound (4.2) in [17] when the applied unitarily invariant norm is the spectral norm or Frobenius norm.

Theorem 3.3. *Let* $A \in C^{m\times n}$ *and* $B = D_1^* AD_2$, *where* D_1 *and* D_2 *are respectively* $m \times m$ *and* $n \times n$ *nonsingular matrices. For any given* $A^{(1)} \in A\{1\}$ *of the form*

$$A^{(1)} = A^\dagger + A^\dagger AZ(I_m - AA^\dagger) + (I_n - A^\dagger A)Z,\ \ Z \in C^{n\times m},$$

let $B_m^{(1)}$ *be as in Theorem 2.1. If* $\max\{\|I_m - D_1\|_2, \|I_n - D_2\|_2\} < 1$, *then we have*

$$\begin{aligned} \|B_m^{(1)} - A^{(1)}\| &\leq \|I_n - s_1 D_2^{-1}\|_2\|A^\dagger + (I_n - A^\dagger A)Z\| \\ &\quad +\|I_m - \overline{s_3}D_1^{-1}\|_2\|A^\dagger AZ(I_m - AA^\dagger)\| \\ &\quad +\frac{\|I_m - \overline{s_1}D_1\|_2 + \|I_m - \overline{s_2}D_1^{-1}\|_2}{\Phi(D_1, D_2)}\|A^\dagger\|, \end{aligned}$$

where $\Phi(D_1, D_2) = (1-\|I_m - D_1\|_2)(1-\|I_n - D_2\|_2)$ *and* s_1, s_2, s_3 *are arbitrary complex numbers.*

Proof. Since $B = D_1^* A D_2$, we have

$$BD_2^{-1} = D_1^* A \text{ and } D_1^{-*}B = AD_2, \tag{3.2}$$

where D_1^{-*} denotes the inverse of the conjugate transpose of D_1. It follows from Lemma 1.2 and (3.2) that for any complex numbers s_1, s_2 and s_3

$$\begin{aligned}
& B^\dagger - B^\dagger B A^{(1)} B B^\dagger \\
= \; & B^\dagger B\Big(B^\dagger A A^\dagger - B^\dagger B A^\dagger + B^\dagger(I_m - AA^\dagger) - A^\dagger A Z(I_m - AA^\dagger) \\
& -(I_n - A^\dagger A)Z\Big)BB^\dagger \\
= \; & B^\dagger B\Big(B^\dagger(I_m - s_1 D_1^*)AA^\dagger - B^\dagger B(I_n - s_1 D_2^{-1})A^\dagger + B^\dagger(I_m - AA^\dagger) \times \\
& (I_m - s_2 D_1^{-*}) - A^\dagger A Z(I_m - AA^\dagger)(I_m - s_3 D_1^{-*}) - (I_n - s_1 D_2^{-1}) \times \\
& (I_n - A^\dagger A)Z\Big)BB^\dagger.
\end{aligned}$$

Now, combining the above equality with (2.1), we get

$$\begin{aligned}
& \|B_m^{(1)} - A^{(1)}\| = \|B^\dagger - B^\dagger B A^{(1)} B B^\dagger\| \\
\leq \; & \big(\|I_m - s_1 D_1^*\|_2 + \|I_m - s_2 D_1^{-*}\|_2\big)\, \|B^\dagger\| \\
& + \|I_n - s_1 D_2^{-1}\|_2 \|A^\dagger + (I - A^\dagger A)Z\| \\
& + \|I_m - s_3 D_1^{-*}\|_2 \|A^\dagger A Z(I - AA^\dagger)\|.
\end{aligned} \tag{3.3}$$

Using (3.2), it is easy to verify that $B^\dagger = B^\dagger B D_2^{-1} A^\dagger D_1^{-*} B B^\dagger$. If $\max\{\|I_m - D_1\|_2, \|I_n - D_2\|_2\} < 1$, then

$$\begin{aligned}
\|B^\dagger\| \; & \leq \; \|D_2^{-1}\|_2 \|D_1^{-1}\|_2 \|A^\dagger\| \\
& = \; \|\left(I_n - (I_n - D_2)\right)^{-1}\|_2 \|[I_m - (I_m - D_1)]^{-1}\|_2 \|A^\dagger\| \\
& \leq \; \frac{\|A^\dagger\|}{\Phi(D_1, D_2)},
\end{aligned}$$

which together with (3.3) complete the proof of this theorem. □

Remark 3.4. Choosing $s_1 = s_2 = s_3 = 1$, Theorem 3.3 is reduced to [10, Theorem 5.1] when the applied unitarily invariant norm is the spectral norm or Frobenius norm.

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