Bulletin of the Iranian Mathematical Society, Vol. 28 *No. 2 (2002), pp.1-17*

ASYMPTOTIC BEHAVIOR OF MULTIVARIATE REWARD PROCESSES WITH NONLINEAR REWARD FUNCTIONS

K. KHORSHIDIAN AND A.R. SOLTANI

Abstract. In this work we study a multivariate reward process $\underline{\mathcal{Z}}(t) = (\mathcal{Z}_1(t), ..., \mathcal{Z}_p(t))$, $t \geq 0$, defined on a semi-Markov process $\{\mathcal{J}(t),\, t \geq 0\}$ with a Markov renewal process $\{(\mathcal{J}_n,\, \mathcal{T}_n),\, n = 0, 1, 2, \dots\}$ and non-linear reward functions $\rho_1, ..., \rho_p$ respectively. We follow the definitoin of Soltani 1996, for nonlinear reward processes.Usually in practice, reward functions are not of constant rate, i.e., are not linear in time, e.g., for water and electricity consumption costs. Hence we have tried to deal with general forms of reward functions, say nonlinear. Using the relation between Laplace transforms of different components of the process, the Laplace transforms of mean vector $E\underline{\mathcal{Z}}(t)$, and the covariance matrix $\Sigma(t)$, are specified. Differentiating the Laplace transforms near the point $\underline{0}$, and inverting them provides asymptotic formulas for $E\underline{\mathcal{Z}}(t)$ and $\Sigma(t)$, as $t \longrightarrow \infty$. Interestingly our results indicate that for general reward functions $\underline{\rho} = (\rho_1, ..., \rho_p)$,

$$E\underline{\mathcal{Z}}(t) = C_0 + C_1\, t + \circ(1)\ , \quad \Sigma(t) = W_0\, t + W_1\, t^2 + \circ(t)\ , \quad t \longrightarrow \infty\ ,$$

where the vectors C_0 , C_1 and the matrices W_0 , W_1 are fully specified. These results, in particular, provide asymptotic formulas for mean and variance of a univariate reward process. An example is given and followed through the paper.

MSC(2000): Primary 60K15; Secondary 60K05, 60K30

Keywords: Semi-Markov processes, Reward processes, Laplace transform

Received: 12 April 99 , Revised: 6 June 2002

© 2002 Iranian Mathematical Society.

1. Introduction

Let $\{\mathcal{J}(t),\, t \geq 0\}$ be a semi-Markov process with a Markov renewal process $\{(\mathcal{J}_n,\, \mathcal{T}_n), n = 0, 1, 2, \dots\}$. The state space of $\{\mathcal{J}_n\}$ is assumed to be $\mathcal{N} = \{0, 1, 2, \dots ,\}$. A Markov renewal process could be considered as an extension of a renewal process tied with a Markov chain. $\{\mathcal{J}_n\ ,\ n = 0, 1, 2, \dots\}$ is a Markov chain, $\{\,\mathcal{T}_n\ ,\ n = 0, 1, 2, \dots\}$ are renewal epochs of visiting successive states in underlying Markov chain, and $\mathcal{J}(t)$ is the state of the process at time $t \geq 0$. For more details, see [2], [3] and [14] .

Based on $\{\mathcal{J}(t),\, t \geq 0\}$, a multivariate reward process may be defined as $\underline{\mathcal{Z}}(t) = (\mathcal{Z}_1(t), ..., \mathcal{Z}_p(t))$, where

$$\mathcal{Z}_i(t) = \sum_{n:\, \mathcal{T}_{n+1}<t} \rho_i(\mathcal{J}_n,\, \mathcal{T}_{n+1} - \mathcal{T}_n) + \rho_i(\mathcal{J}(t)\,, X(t)),\ i = 1, ..., p\ , \tag{1.1}$$

where $X(t)$ is the age process, that is the sojourn time since the last transition until t. Each function ρ_i in (1.1) is called a reward function , and is a real function of two variables; $\rho_i : \mathcal{N} \times R \longrightarrow R$, where $\rho_i(j, \tau)$ measures the excess reward when time τ is spent in the state j , and $\mathcal{Z}_i(t)$ is the total reward gained up to time t by the i-th component of the system. If $\rho_i(j, \tau) = j\tau,\ i = 1, ..., p$ then the reward process $\underline{\mathcal{Z}}(t)$ becomes the multivariate reward process treated by Sumita and Masuda 1987, Masuda and Sumita 1991, Sumita 1993, Ball 1999, etc. In the case that $\underline{\rho}$ is of the polynomial form

$$\rho_i(k,\, x) = \sum_{n=1}^{m_i} g_{in}(k) x^n, \tag{1.2}$$

where g_{in}, $n = 1, \dots , m_i$ are given functions, an explicit formula for the mean vector $E\underline{\mathcal{Z}}(t);\ t \geq 0$, is given in Soltani 1996, which is based on $G(x, t)$ the joint distribution of $(\mathcal{J}(t)\,,\ \ X(t)\ )$. The problem with earlier results, is that in more realistic systems and situations they don't work well, because of linearity or at most polynomial nature of corresponding reward functions and the unknown distributions. Soltani and Khorshidian 1998, arrived at a formula for $E\mathcal{Z}(t)$, $t \longrightarrow \infty$, in univariate case with general reward functions using Markov Renewal Theory approach , which is difficult to extend to the multivariate case.

In the present study we will try to relax these assumptions and consider more general reward functions. Using Laplace transform techniques we determine the asymptotic behavior of $E\underline{\mathcal{Z}}(t)$, the mean vector and $\Sigma(t)$, the covariance matrix of $\underline{\mathcal{Z}}(t)$, for each $t \geq 0,\ t \longrightarrow \infty$. Interestingly our results indicate that for general reward functions $\underline{\rho}$,

$$E\underline{\mathcal{Z}}(t) = C_0 + C_1\, t + \circ(1)\ , \quad t \longrightarrow \infty,$$

and

$$\Sigma(t) = W_0\, t + W_1\, t^2 + \circ(t)\ , \quad t \longrightarrow \infty,$$

where the vectors C_0 , C_1 and the matrices W_0 , W_1 are fully specified. For more details on semi-Markov processes see [2], [3] and [14]. Concerning the asymptotic behavior of a semi-Markov process see [4] and [5].

Example 1. This is the same example as [6], with an extra vector of nonlinear reward functions entering to the multivariate case. Consider a manufacturing system which produces perfect products as well as defective products in a random manner. The system has two modes of operations: state 1 indicates that the system is under low-quality production mode in that the system produces defective with high rate while state 2 is the high-quality production mode with smaller defective production rate. There is one state corresponding to failure/setup, which is represented by state 0 . The total production rate (including both defective and perfect items) in state 1 is the same as that in state 2 , and we cannot distinguish these two modes of operation. The machine may switch its states within 0 , 1 , and 2 . Suppose that the machine produces perfect items by rate 0.1 per unit of time at state 1 , and by rate 2 at state 2 . If we define $\rho_1(0,x) = 0$, $\rho_1(1,x) = 0.1x$, $\rho_1(2,x) = 2x$, *then* $\mathcal{Z}_1(t)$ becomes the total number of perfect items produced during $(0,t]$. Let ρ_2 represent the costs of production and other services,

$$\rho_2(0,x) = 2x^3, \quad \rho_2(1,x) = 5(\mathbf{e}^{0.05x} - 1), \quad \rho_2(2,x) = 3(\mathbf{e}^{2x} - 1),$$

then $\mathcal{Z}_2(t)$ becomes the total cost during $(0,t]$. In matrix form,

$$\underline{\rho}(0,x) = \begin{pmatrix} 0 \\ 2x^3 \end{pmatrix}, \quad \underline{\rho}(1,x) = \begin{pmatrix} 0.1x \\ 5(\mathbf{e}^{0.05x} - 1) \end{pmatrix}$$

$$, \quad \underline{\rho}(2,x) = \begin{pmatrix} 2x \\ 3(\mathbf{e}^{2x}-1) \end{pmatrix}.$$

2. Notations and Preliminaries

Corresponding to a semi-Markov process $\{\mathcal{J}(t),\, t \geq 0\}$, let $A_{ij}(x)$ measures the transition probability from state i to the state j within the time interval $(0,x]$,i.e.,

$$A_{ij}(x) \;=\; P\{\,\mathcal{J}_{n+1} \,= j,\, \mathcal{T}_{n+1} \,-\, \mathcal{T}_n \,\leq\, x \;\mid\; \mathcal{J}_n \,= i\}.$$

Let $a_{ij}(x)$ denote the density of $A_{ij}(x)$ and let

$$\begin{aligned} A_i(x) &= \sum_{j\in N} A_{ij}(x), \quad \overline{A}_i(x) \;=\; 1 \,-\, A_i(x), \\ P_{ij}(t) &= P\{\,\mathcal{J}(t) \,=\, j \;\mid\; \mathcal{J}(0) \,=\, i\,\}. \end{aligned}$$

The Markov renewal kernel is denoted by $R(t) = \sum_{n=0}^{\infty} A^{n\star}$, where $A^{n\star}$ is the n-fold convolution of $A(x)$ with itself, $A(x)$ is the matrix with entries $A_{ij}(x)$. The joint distributions corresponding to the bivariate process $\{(\mathcal{J}(t), X(t)),\, t \,\geq\, 0\,\}$ and the process $\{(\mathcal{J}(t),\, X(t),\, \underline{\mathcal{Z}}(t)),\, t \geq 0\}$, respectively, are given by

$$\begin{aligned} G_{ij}(x,t) &= P\{\mathcal{J}(t) \,=\, j,\;\; X(t) \leq x \;\mid\; \mathcal{J}(0) \,=\, i\}, \\ F_{ij}(x,\underline{z},t) &= P\{\,\mathcal{J}(t) \,=\, j,\, X(t) \,\leq\, x,\, \underline{\mathcal{Z}}(t) \,\leq\, \underline{z} \;\mid\; \mathcal{J}(0) \,=\, i\,\}, \end{aligned}$$

where by $\underline{\mathcal{Z}}(t) \,\leq\, \underline{z}$ we mean $(\mathcal{Z}_1(t) \,\leq\, z_1,\, \ldots,\, \mathcal{Z}_p(t) \,\leq\, z_p)$. A vector $(w_1,...,w_p)$ in $\mathbf{R}^p$ is denoted by $\underline{w}$. The following Laplace transforms are of frequent use in subsequent sections.

$$\begin{aligned} \alpha_{ij}(s) &= \int_0^\infty e^{-sx}\, dA_{ij}(x), \\ \alpha_i(s) &= \int_0^\infty e^{-sx}\, dA_i(x), \\ \phi_{ij}(v,\underline{w},s) &= \int_0^\infty \int_{R^p} \int_0^\infty e^{-vx-\underline{w}'\underline{z}-st}\, f_{ij}(x,\underline{z},t)\, dx\, d\underline{z}\, dt, \\ \sigma_{ij}(\underline{w},s) &= \int_0^\infty \int_{R^p} e^{-\underline{w}'\underline{z}-st}\, f_{ij}(0^+,\underline{z},t)\, d\underline{z}\, dt, \\ C_{kj}(\underline{w},s) &= \int_0^\infty e^{-\underline{w}'\underline{\rho}(k,x)-sx}\, dA_{kj}(x), \\ E_j(\underline{w},s) &= \int_0^\infty e^{-\underline{w}'\underline{\rho}(j,x)-sx}\, \overline{A}_j(x) dx. \end{aligned}$$

Throughout this paper a matrix with entries y_{ij}, $i, j \in \mathcal{N}$, is denoted by $y = [\, y_{ij} \,]$ and a diagonal matrix with entries y_i, $i \in \mathcal{N}$, is denoted by $y_D = [\, \delta_{ij} y_j \,]$. The initial probability vector is denoted by $\underline{p}(0)$, and the unit vector by $\underline{e} = [\, 1 \;\; 1 \;\; \ldots \;\; 1 \,]$. The following important and informative relation between Laplace Transforms is established in [7] and [11],

$$\begin{aligned} \phi(v, \underline{w}, s) &= \sigma(\underline{w}, s) E_D(\underline{w}, s+v) \\ &= (I - C(\underline{w}, s))^{-1} E_D(\underline{w}, s+v). \end{aligned} \tag{2.1}$$

Recall from [11] that in the univariate case, when the reward function is given by

$$\rho(k,\, x) \;=\; \sum_{n=1}^{m} g_n(k) x^n,$$

it has been shown that

$$E(\mathcal{Z}_\rho(t)) \;=\; \int_0^t \underline{p}'(0) \sum_{n=1}^{m} n E^{n-1}\,(\tau) \rho_{D:n}\, \underline{e}\, d\tau, \tag{2.2}$$

where $g_n(k)$, $k \in N$, are the entries of the matrix $\rho_{D:n}$, $n = 1, \ldots, m$; and

$$E^n(t) \;=\; \int_0^\infty x^n\, G(dx, t). \tag{2.3}$$

The formula (2.2) enables one to compute the mean of the cumulative reward up to time t, whenever the reward function is a polynomial.

The problem with (2.2), is the unknown structure of $G(x, t)$ in (2.3), therefore we try to evaluate the asymptotic value of $E(\mathcal{Z}_\rho(t))$, $t \longrightarrow \infty$. What we need in theory is to obtain (2.1) and it's derivatives, but as seen in the following example, even for simple forms of $A(.)$ and $\underline{\rho}(.,.)$ it is not an easy task and therefore we proceed as next sections.

Example 1.(continued): Let $\underline{w} = (w_1, w_2)'$, and

$$a(x) = \frac{d}{dx} A(x) = \begin{bmatrix} 0 & 0.8.2e^{-2x} & 0.2.2e^{-2x} \\ 0.9.0.2e^{-0.2x} & 0 & 0.1.0.3e^{-0.3x} \\ 0.8.e^{-x} & 0.2.0.8e^{-0.8x} & 0 \end{bmatrix}.$$

W.L.O.G. it is assumed that there is no self transition (if there is a self transition, the transition functions can be modified so that the self transition function to be 0). Also

$$\overline{A}(x) = \begin{bmatrix} e^{-2x} & 0 & 0 \\ 0 & 0.9e^{-0.2x} + 0.1.e^{-0.3x} & 0 \\ 0 & 0 & 0.8.e^{-x} + 0.2.e^{-0.8x} \end{bmatrix}.$$

To arrive at (2.1), suppose we want to evaluate $C_{21}(\underline{w}, s)$ and $E_2(\underline{w}, s)$,

$$C_{21}(\underline{w}, s) \ = \ \int_0^\infty 0.16e^{-2w_1x-3w_2(\mathbf{e}^{2x}-1)-sx}\, e^{-0.8x}dx,$$

$$E_2(\underline{w}, s) \ = \ \int_0^\infty e^{-2w_1x-3w_2(\mathbf{e}^{2x}-1)-sx}(\,0.8.e^{-x} + 0.2.e^{-0.8x})dx.$$

As is seen, it is not easy to compute the above integrals, even for such uncomplicated transition probability matrix and reward functions, therefore we proceed as next sections.

3. Asymptotic Behavior Of $E\mathcal{Z}_\rho(t)$

In this section we assume that $\mathcal{Z}_\rho(t)$ is a univariate reward process corresponding to a general reward function $\rho(k, x)$. Recall from [11]that

$$\mathcal{L}_s\ E(\mathcal{Z}_\rho(t)) \ = \ -\,\underline{p}'(0)\,\frac{\partial\phi(0,w,s)}{\partial w}\,|_{w=0}\ \underline{e}\,, \tag{3.1}$$

where $\mathcal{L}_s$ stands for the Laplace transform of the given function with respect to the parameter s, and

$$\frac{\partial\phi(0,w,s)}{\partial w}\,|_{w=0}$$

$$\begin{aligned} = \ & (I-\alpha(s))^{-1}\left[-\int_0^\infty \rho(k,x)e^{-sx}A_{kj}(dx)\right](I-\alpha(s))^{-1}\frac{(I-\alpha_D(s))}{s} \\ & +(I-\alpha(s))^{-1}\left[-\delta_{kj}\int_0^\infty \rho(j,x)e^{-sx}\overline{A}_j(x)dx\right]. \end{aligned} \tag{3.2}$$

Recall from [4] that as $s \longrightarrow 0$,

$$(I-\alpha(s))^{-1} = H_0 + \frac{1}{s}H_1 + \circ(1)$$

where

$$H_1 = \frac{1}{m_1}\,\underline{e}\,\underline{\pi}'\,, \qquad m_1 = \underline{\pi}' A_1\,\underline{e}\,, \qquad A_i = \int_0^\infty x^i A(dx),$$

$\underline{\pi}$ is the unique stationary distribution, i.e., $\underline{\pi}' A(\infty) = \underline{\pi}'$,

$$H_0 = \frac{1}{m_1}\,\underline{e}\,\underline{\pi}'\{-A_1 + \frac{1}{2m_1} A_2\,\underline{e}\,\underline{\pi}'\} + \{Z_0 - \frac{1}{m_1}\,\underline{e}\,\underline{\pi}' A_1 Z_0\}\{P - \frac{1}{m_1} A_1\,\underline{e}\,\underline{\pi}'\} + I\,,$$

Z_0 is the fundamental matrix associated with the discrete time Markov chain governed by P, i.e. $Z_0 = \{I - P + \underline{e}\,\underline{\pi}'\}^{-1}$, and $\circ(1) \longrightarrow 0$, as $s \longrightarrow 0$.

Let

$$\begin{aligned} B^i_{kj} &= \int_0^\infty x^i \rho(k,x) A_{kj}(dx), \\ \theta^i_j &= \int_0^\infty \int_0^x u^i \rho(j,u) du A_j(dx), \end{aligned}$$

and $B^i = [B^i_{kj}]$, $\quad \Theta^i_D = [\delta_{kj}\theta^i_j]$. The following theorem provides the behavior of $E\mathcal{Z}_\rho(t), \ \ t \longrightarrow \infty$.

Theorem 3.3. *Let $\mathcal{Z}_\rho(t)$ be a reward process given by (1.1) with reward function $\rho(k,x)$ and suppose that B^i, and Θ^i_D, $\ i = 0,1,2$ exists, then as $t \longrightarrow \infty$,*

$$E\mathcal{Z}_\rho(t) = \underline{p}'(0)\left(H_0 B^0 - H_1 B^1 + H_1 \Theta^0_D + \ H_1 B^0\, t\right)\underline{e} + \circ(1).$$

Proof. By applying the Taylor's Theorem to the term e^{-sx} in the integrals in (3.2) we obtain that

$$\begin{aligned} &\textstyle\int_0^\infty \rho(k,x) e^{-sx} A_{kj}(dx) \\ &\quad = \int_0^\infty \rho(k,x)(1 - sx + \frac{s^2x^2}{2} e^{-s^* x}) A_{kj}(dx) \\ &\quad = \int_0^\infty \rho(k,x) A_{kj}(dx) - s\int_0^\infty x\rho(k,x) A_{kj}(dx) \\ &\qquad\qquad + \frac{s^2}{2}\int_0^\infty x^2 \rho(k,x) e^{-s^* x} A_{kj}(dx) \\ &\quad = B^0_{kj} - \ s\,B^1_{kj} \ + \circ_{kj}(s), \qquad\qquad s \longrightarrow 0 \end{aligned}$$

where $0 < s^* < s$ and

$$\begin{aligned} \circ_{kj}(s) = \frac{s^2}{2}\int_0^\infty x^2 \rho(k,x) e^{-s^* x} A_{kj}(dx) \ &< \ \frac{s^2}{2}\int_0^\infty x^2 \rho(k,x) A_{kj}(dx) \\ &= \ \frac{s^2}{2} B^2_{kj}. \end{aligned}$$

In matrix form

$$\left[\int_0^\infty \rho(k,x)e^{-sx}A_{kj}(dx)\right] = B^0 - s\,B^1 + \circ(s), \tag{3.4}$$

where $\circ(s) = [\circ_{kj}(s)]$. Also

$$\begin{aligned}
&\textstyle\int_0^\infty \rho(j,x)e^{-sx}\overline{A}_j(x)dx \\
&= \int_0^\infty A_j(dx)\int_0^x \rho(j,u)e^{-su}du \\
&= \int_0^\infty \int_0^x e^{-su}\rho(j,u)duA_j(dx) \\
&= \int_0^\infty \int_0^x (1 - su + \frac{s^2u^2}{2}e^{-s^*u})\rho(j,u)duA_j(dx) \quad 0 < s^* < s \\
&= \int_0^\infty \int_0^x \rho(j,u)duA_j(dx) - s\int_0^\infty \int_0^x u\rho(j,u)duA_j(dx) \\
&\qquad + \frac{s^2}{2}\int_0^\infty \int_0^x u^2e^{-s^*u}\rho(j,u)duA_j(dx) \\
&= \theta_j^0 - s\,\theta_j^1 + \circ_j(s), \qquad s \longrightarrow 0
\end{aligned}$$

where

$$\begin{aligned}
\circ_j(s) &= \frac{s^2}{2}\int_0^\infty \int_0^x u^2e^{-s^*u}\rho(j,u)duA_j(dx) \\
&< \frac{s^2}{2}\int_0^\infty \int_0^x u^2\rho(j,u)duA_j(dx) \\
&= \frac{s^2}{2}\theta_j^2.
\end{aligned}$$

In matrix form, with $\circ(s) = [\delta_{kj}\circ_j(s)]$,

$$\left[\delta_{kj}\int_0^\infty \rho(j,x)e^{-sx}\overline{A}_j(x)dx\right] = \Theta_D^0 - s\,\Theta_D^1 + \circ(s). \tag{3.5}$$

The fact $(I - \alpha(s))\,\underline{e} = (I - \alpha_D(s))\,\underline{e}$, provides that

$$(I - \alpha(s))^{-1}(I - \alpha_D(s))\,\underline{e} = \underline{e}\,.$$

Substitute (3.4) and (3.5) in (3.2) and use Keilson's result for the term $(I-\alpha(s))^{-1}$ to obtain that

$$
\begin{aligned}
- s\,\frac{\partial\phi(0,w,s)}{\partial w}\,|_{w=0}\,\underline{e} &= (H_0+\frac{1}{s}H_1+\circ(1))(B^0-\ s\,B^1\ +\circ(s))\,\underline{e} \\
&+\ \ s\,(H_0+\frac{1}{s}H_1+\circ(1))(\Theta^0_D-\ s\,\Theta^1_D+\circ(s))\,\underline{e} \\
&= (H_0B^0-H_1B^1+H_1\Theta^0_D+\frac{1}{s}H_1B^0)\,\underline{e}+\circ(1).
\end{aligned}
$$

The proof is complete by (3.1).

Example 1.*(continued)*: In this section we compute $E\mathcal{Z}_i(t)\quad i=1,2$ as discussed in section 1,

$$E\mathcal{Z}_i(t)=\underline{p}'(0)\left(H_0B^0_i-H_1B^1_i+H_1\Theta^0_{D:i}+\ H_1B^0_i\,t\right)\,\underline{e}+\circ(1),\ \ i=1,2.$$

At first we compute the needed matrices and vectors,

$$P=A(\infty)=\left[\begin{array}{ccc} 0 & 0.8 & 0.2 \\ 0.9 & 0 & 0.1 \\ 0.8 & 0.2 & 0 \end{array}\right],$$

$$\pi'P=\pi'\Longrightarrow\pi'=(0.4\overline{6},0.4,0.1\overline{3}).$$

$$A_1=\left[\begin{array}{ccc} 0 & 0.4 & 0.1 \\ 4.5 & 0 & 1/3 \\ 0.8 & 0.25 & 0 \end{array}\right], A_2=\left[\begin{array}{ccc} 0 & 0.4 & 0.1 \\ 45 & 0 & 20/9 \\ 1.6 & 5/8 & 0 \end{array}\right],$$

giving that

$$m_1=\pi'A_1\underline{e}=(0.4\overline{6},0.4,0.1\overline{3})\left[\begin{array}{ccc} 0 & 0.4 & 0.1 \\ 4.5 & 0 & 1/3 \\ 0.8 & 0.25 & 0 \end{array}\right]\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]=2.30\overline{6},$$

$$H_1=\frac{\underline{e}\pi'}{m_1}=\left[\begin{array}{ccc} 0.2023 & 0.1734 & 0.0578 \\ 0.2023 & 0.1734 & 0.0578 \\ 0.2023 & 0.1734 & 0.0578 \end{array}\right],$$

$$H_0=\left[\begin{array}{ccc} 0.9805 & 0.0051 & 0.0033 \\ 0.1873 & 4.6027 & 0.9311 \\ -.0613 & 0.1519 & 1.0546 \end{array}\right].$$

Now consider the bivariate reward process $\underline{\rho}(k,x)$ of Example 1,

$$\underline{\rho}(0,x)=\left(\begin{array}{c} 0 \\ 2x^3 \end{array}\right),\quad \underline{\rho}(1,x)=\left(\begin{array}{c} 0.1x \\ 5(\mathbf{e}^{0.05x}-1) \end{array}\right)$$

$$, \quad \underline{\rho}(2,x) = \begin{pmatrix} 2x \\ 3(\mathbf{e}^{2x}-1) \end{pmatrix}.$$

Then

$$B_1^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0.45 & 0 & 0.0\overline{3} \\ 1.6 & 0.5 & 0 \end{bmatrix}, B_1^1 = \begin{bmatrix} 0 & 0 & 0 \\ 4.5 & 0 & 0.2\overline{2} \\ 3.2 & 1.25 & 0 \end{bmatrix},$$

$$\Theta_1^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.36\overline{1} & 0 \\ 0 & 0 & 14.1 \end{bmatrix}, B_2^0 = \begin{bmatrix} 0 & 1.2 & 44.\overline{4} \\ 1.5 & 0 & 0.1 \\ 0.6 & 0.2 & 0 \end{bmatrix},$$

$$B_2^1 = \begin{bmatrix} 0 & 2.4 & 0.6 \\ 63.\overline{8} & 0 & 3.4\overline{3} \\ 1.35 & .58\overline{3} & 0 \end{bmatrix}, \Theta_2^0 = \begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 27.8\overline{3} & 0 \\ 0 & 0 & .28\overline{3} \end{bmatrix}.$$

Assume that the initial probabilities are $p'(0) = (0.2\ ,\ 0.3\ ,\ 0.5)$. It follows from Theorem 3.3, that as $t \longrightarrow \infty$,

$$EZ_\rho(t) = \begin{bmatrix} 2.548 \\ 5.697 \end{bmatrix} + \begin{bmatrix} 0.205 \\ 9.558 \end{bmatrix} t \ + \ \circ(1)\,.$$

The next section is devoted to the asymptotic behavior of $\Sigma(t)$, the covariance matrix of multivariate process $\underline{\mathcal{Z}}(t) = (\mathcal{Z}_1(t), \mathcal{Z}_2(t), ..., \mathcal{Z}_p(t))$.

4. Asymptotic Behavior Of The Covariance Matrix

In this section we consider a multivariate reward process $\underline{\mathcal{Z}}(t) = (\mathcal{Z}_1(t), \mathcal{Z}_2(t),$ $..., \mathcal{Z}_p(t))$, with a multidimensional reward function $\underline{\rho} = (\rho_1, ..., \rho_p)$, and obtain an asymptotic formula for $\Sigma(t)\,,\quad t \longrightarrow \infty\,,$ under mild conditions on $\underline{\rho}$. First note that from the relation

$$\int_0^\infty e^{-st} E\{e^{-\underline{w}'\underline{z}(t)}\}\, dt = \underline{p}'(0)\phi(0,\underline{w},s)\,\underline{e},$$

it follows that

$$\int_0^\infty e^{-st} E\{\mathcal{Z}_i(t)\mathcal{Z}_j(t)\}\, dt = \underline{p}'(0)\frac{\partial^2\phi(0,\underline{w},s)}{\partial w_i \partial w_j}\Big|_{\underline{w}=\underline{0}}\ \underline{e}\,. \tag{4.1}$$

Also it follows from (2.1) that

$$\phi(0,\underline{w},s) = \sigma(\underline{w},s)E_D(\underline{w},s).$$

Theorem 4.2. *Suppose that $\rho_r(k,x)$, r=1,2,...,p, satisfy the following conditions*

$$\begin{aligned}
B^i_{r:kj} &= \int_0^\infty x^i \rho_r(k,x) A_{kj}(dx) < \infty \qquad i = 0,1,2,\\
B^i_{rs:kj} &= \int_0^\infty x^i \rho_r(k,x)\rho_s(k,x) A_{kj}(dx) < \infty \qquad i = 0,1,2,\\
\theta^i_{j:r} &= \int_0^\infty \int_0^x u^i \rho_r(j,u) du A_j(dx) < \infty \qquad i = 0,1,2,\\
\theta^i_{j:rs} &= \int_0^\infty \int_0^x u^i \rho_r(j,u)\rho_s(j,u) du A_j(dx) < \infty \quad i = 0,1,2.
\end{aligned}$$

and denoting $B^i_r = [B^i_{r:kj}]$, $B^i_{rs} = [B^i_{rs:kj}]$, $\Theta^i_{D:r} = [\delta_{kj}\theta^i_{j:r}]$, $\Theta^i_{D:rs} = [\delta_{kj}\theta^i_{j:rs}]$. *Then as* $t \longrightarrow \infty$,

$$E\,\mathcal{Z}_r(t)\mathcal{Z}_s(t) = \underline{p}'(0)\{Y_1 t^2 + Y_0 t\}\,\underline{e} + \circ(t)\,,$$

where

$$Y_{1:rs} = H_1(B^0_r H_1 B^0_s + B^0_s H_1 B^0_r)$$

and

$$\begin{aligned}
Y_{0:rs} = \; & H_0(B^0_r H_1 B^0_s + B^0_s H_1 B^0_r)\\
& + H_1(B^0_r H_0 B^0_s + B^0_s H_0 B^0_r - B^0_r H_1 B^1_s - B^0_s H_1 B^1_r\\
& \quad - B^1_r H_1 B^0_s - B^1_s H_1 B^0_r + B^0_r H_1 \Theta^0_{D:s} + B^0_s H_1 \Theta^0_{D:r} + B^0_{rs}).
\end{aligned}$$

Proof. Without loss of generality, we evaluate $E\mathcal{Z}_1(t)\mathcal{Z}_2(t)$. Differentiating of (2.1) gives that

$$\frac{\partial\phi(0,\underline{w},s)}{\partial w_1} = \frac{\partial\sigma(\underline{w},s)}{\partial w_1}E_D(\underline{w},s) + \sigma(\underline{w},s)\frac{\partial E_D(\underline{w},s)}{\partial w_1},$$

and

$$\begin{aligned}
\frac{\partial^2\phi(0,\underline{w},s)}{\partial w_1\partial w_2} = \; & \frac{\partial^2\sigma(\underline{w},s)}{\partial w_1\partial w_2}E_D(\underline{w},s) + \frac{\partial\sigma(\underline{w},s)}{\partial w_1}\frac{\partial E_D(\underline{w},s)}{\partial w_2}\\
& + \frac{\partial\sigma(\underline{w},s)}{\partial w_2}\frac{\partial E_D(\underline{w},s)}{\partial w_1} + \sigma(\underline{w},s)\frac{\partial^2 E_D(\underline{w},s)}{\partial w_1\partial w_2}.
\end{aligned} \tag{4.3}$$

Also from (2.1),

$$\sigma(\underline{w},s) = \sigma(\underline{w},s)C(\underline{w},s) + I,$$

which implies that,

$$\frac{\partial \sigma(\underline{w},s)}{\partial w_1} = \frac{\partial \sigma(\underline{w},s)}{\partial w_1} C(\underline{w},s) + \sigma(\underline{w},s) \frac{\partial C(\underline{w},s)}{\partial w_1},$$

or

$$\frac{\partial \sigma(\underline{w},s)}{\partial w_1} = (I - C(\underline{w},s))^{-1} \frac{\partial C(\underline{w},s)}{\partial w_1} (I - C(\underline{w},s))^{-1} \tag{4.4}$$

similarly

$$\frac{\partial \sigma(\underline{w},s)}{\partial w_2} = (I - C(\underline{w},s))^{-1} \frac{\partial C(\underline{w},s)}{\partial w_2} (I - C(\underline{w},s))^{-1}.$$

By using a similar method and formula (4.4) we obtain that

$$\begin{aligned} &\frac{\partial^2 \sigma(\underline{w},s)}{\partial w_1 \partial w_2} \\ = \; & (I - C(\underline{w},s))^{-1} \{ \frac{\partial C(\underline{w},s)}{\partial w_1} (I - C(\underline{w},s))^{-1} \frac{\partial C(\underline{w},s)}{\partial w_2} \\ + \; & \frac{\partial C(\underline{w},s)}{\partial w_2} (I - C(\underline{w},s))^{-1} \frac{\partial C(\underline{w},s)}{\partial w_1} + \frac{\partial^2 C(\underline{w},s)}{\partial w_1 \partial w_2} \} (I - C(\underline{w},s))^{-1}. \end{aligned} \tag{4.5}$$

Now note that

$$C_{kj}(\underline{w},s) \; = \; \int_0^\infty e^{-\sum_{i=1}^p w_i \rho_i(k,x) - sx} \, dA_{kj}(x),$$

and therefore $C_{kj}(\underline{0},s) \; = \; \int_0^\infty e^{-sx} \, dA_{kj}(x)$, or $C(\underline{0},s) = \alpha(s)$. Also

$$\frac{\partial C(\underline{w},s)}{\partial w_1} \Big|_{\underline{w}=\underline{0}} = \left[- \int_0^\infty \rho_1(k,x) \, e^{-sx} \, dA_{kj}(x) \right], \tag{4.6}$$

and

$$\frac{\partial^2 C(\underline{w},s)}{\partial w_1 \partial w_2} \Big|_{\underline{w}=\underline{0}} = \left[\int_0^\infty \rho_1(k,x) \rho_2(k,x) \, e^{-sx} \, dA_{kj}(x) \right]. \tag{4.7}$$

Moreover

$$E_j(\underline{w},s) \; = \; \int_0^\infty e^{-\sum_{i=1}^p w_i \rho_i(k,x) - sx} \, \overline{A}_j(x) dx,$$

giving that $E_j(\underline{0},s) \; = \; \dfrac{1 - \alpha_j(s)}{s}$ or $E_D(\underline{0},s) = \dfrac{I - \alpha_D(s)}{s}$.

Also

$$\frac{\partial E_j(\underline{w},s)}{\partial w_1}\Big|_{\underline{w}=\underline{0}} = -\int_0^\infty \rho_1(k,x)\, e^{-sx}\, \overline{A}_j(x)dx, \tag{4.8}$$

and

$$\frac{\partial^2 E_j(\underline{w},s)}{\partial w_1 \partial w_2}\Big|_{\underline{w}=\underline{0}} = \int_0^\infty \rho_1(k,x)\, \rho_2(k,x)\, e^{-sx}\, \overline{A}_j(x)dx. \tag{4.9}$$

Similar to one dimensional case, it follows from (4.6)-(4.9) that

$$\frac{\partial C(\underline{w},s)}{\partial w_1}\Big|_{\underline{w}=\underline{0}} = -B_1^0 + s\, B_1^1 + \circ(s) \tag{4.10}$$

$$\frac{\partial^2 C(\underline{w},s)}{\partial w_1 \partial w_2}\Big|_{\underline{w}=\underline{0}} = B_{12}^0 - s\, B_{12}^1 + \circ(s) \tag{4.11}$$

$$\frac{\partial E_D(\underline{w},s)}{\partial w_1}\Big|_{\underline{w}=\underline{0}} = -\Theta_{D:1}^0 + s\, \Theta_{D:1}^1 + \circ(s) \tag{4.12}$$

$$\frac{\partial^2 E_D(\underline{w},s)}{\partial w_1 \partial w_2}\Big|_{\underline{w}=\underline{0}} = \Theta_{D:12}^0 - s\, \Theta_{D:12}^1 + \circ(s) \tag{4.13}$$

Therefore (4.4) together with (4.10) and (4.11) implies that

$$\begin{aligned}\frac{\partial^2 \sigma(\underline{w},s)}{\partial w_1 \partial w_2} E_D(\underline{w},s)\Big|_{\underline{w}=\underline{0}} &= (I-\alpha(s))^{-1}\{ \\ &(-B_1^0 + s\, B_1^1 + \circ(s))(I-\alpha(s))^{-1}(-B_2^0 + s\, B_2^1 + \circ(s)) \\ &+(-B_2^0 + s\, B_2^1 + \circ(s))(I-\alpha(s))^{-1}(-B_1^0 + s\, B_1^1 + \circ(s)) \\ &+(B_{12}^0 - s\, B_{12}^1 + \circ(s))\ \}\ (I-\alpha(s))^{-1}\frac{(I-\alpha_D(s))}{s}.\end{aligned} \tag{4.14}$$

From (4.4), (4.10) and (4.12) it follows that,

$$\begin{aligned}\frac{\partial \sigma(\underline{w},s)}{\partial w_1}\frac{\partial E_D(\underline{w},s)}{\partial w_2}\Big|_{\underline{w}=\underline{0}} &= (I-\alpha(s))^{-1}(-B_1^0 + s\, B_1^1 + \circ(s)) \\ &\qquad (I-\alpha(s))^{-1}(-\Theta_{D:2}^0 + s\, \Theta_{D:2}^1 + \circ(s)),\end{aligned} \tag{4.15}$$

and by (2.1) and (4.13),

$$\sigma(\underline{w},s)\frac{\partial^2 E_D(\underline{w},s)}{\partial w_1 \partial w_2}\Big|_{\underline{w}=\underline{0}} = (I-\alpha(s))^{-1}(\Theta_{D:12}^0 - s\, \Theta_{D:12}^1 + \circ(s)). \tag{4.16}$$

Using the Keilson's approximation of $(I-\alpha(s))^{-1}$, and substituting (4.14)-(4.16) in (4.3) imply that as $s \longrightarrow 0$,

$$\frac{\partial^2 \phi(0,\underline{w},s)}{\partial w_1 \partial w_2}\Big|_{\underline{w}=\underline{0}}\ \underline{e}$$

$$\begin{aligned}
= \ \{ & \frac{1}{s^3} H_1(B_1^0 H_1 B_2^0 + B_2^0 H_1 B_1^0) + \frac{1}{s^2} H_0(B_1^0 H_1 B_2^0 + B_2^0 H_1 B_1^0) \\
& + \frac{1}{s^2} H_1(B_1^0 H_0 B_2^0 + B_2^0 H_0 B_1^0 - B_1^0 H_1 B_2^1 - B_2^0 H_1 B_1^1 - B_1^1 H_1 B_2^0 \\
& - B_2^1 H_1 B_1^0 + B_1^0 H_1 \Theta_{D:2}^0 + B_2^0 H_1 \Theta_{D:1}^0 + B_{12}^0) \ \} \ \underline{e} + \circ(\frac{1}{s^2}).
\end{aligned}$$

giving the result.

Corollary 4.17. *Let* $\underline{\rho} = (\rho_1, ..., \rho_p)$, *then as* $t \longrightarrow \infty$, *the asymptotic covariance matrix of* $\underline{\mathcal{Z}}(t) = (\mathcal{Z}_1(t), ..., \mathcal{Z}_p(t))$ *is given by*

$$\Sigma(t) = W_0\, t + W_1\, t^2 \ + \circ(t),$$

where

$$W_{1:rs} = \underline{p}'(0) \ \{ \ H_1(B_r^0 H_1 B_s^0 + B_s^0 H_1 B_r^0) - H_1 B_r^0 \mathbf{P}(0) H_1 B_s^0 \ \} \underline{e} \ ,$$

$$\begin{aligned}
W_{0:rs} \ = \ & \underline{p}'(0) \ [\ H_0(B_r^0 H_1 B_s^0 + B_s^0 H_1 B_r^0) \\
& + H_1(B_r^0 H_0 B_s^0 + B_s^0 H_0 B_r^0 - B_r^0 H_1 B_s^1 - B_s^0 H_1 B_r^1 \\
& \quad - B_r^1 H_1 B_s^0 - B_s^1 H_1 B_r^0 + B_r^0 H_1 \Theta_{D:s}^0 + B_s^0 H_1 \Theta_{D:r}^0 + B_{rs}^0) \\
& - H_1 B_r^0 \mathbf{P}(0)(H_0 B_s^0 - H_1 B_s^1 + H_1 \Theta_{D:s}^0) \\
& - (H_0 B_r^0 - H_1 B_r^1 + H_1 \Theta_{D:r}^0) \mathbf{P}(0) H_1 B_s^0 \ ] \underline{e} \, ,
\end{aligned}$$

and $\mathbf{P}(0) = \ \underline{e}\, \underline{p}'(0)$.

Corollary 4.18. *Let* $\mathcal{Z}_\rho(t)$ *be a one-dimensional reward process corresponding to a reward function* $\rho(k,x)$, *then*

$$Var(\mathcal{Z}_\rho(t)\,) = \underline{p}'(0) \Big(U_0\, t + U_1\, t^2 \Big) \ \underline{e} + \circ(t)\,, \qquad t \longrightarrow \infty,$$

where

$$U_1 = 2 H_1 B^0 H_1 B^0 - H_1 B^0 \mathbf{P}(0) H_1 B^0 \,,$$

$$\begin{aligned} U_0 &= 2H_0B^0H_1B^0 \\ &\quad +2H_1\left(B^0H_0B^0 - B^0H_1B^1 - B^1H_1B^0 + B^0H_1\Theta_D^0 \right) \\ &\quad +H_1B^* - H_1B^0\mathbf{P}(0)(H_0B^0 - H_1B^1 + H_1\Theta_D^0) \\ &\quad -(H_0B^0 - H_1B^1 + H_1\Theta_D^0)\mathbf{P}(0)H_1B^0 , \end{aligned}$$

and

$$B^* = \left[\int_0^\infty \rho^2(k,x)A_{kj}(dx) \right].$$

Example 1.*(continued):* Here we show that the analyses performed in preceding sections can be implemented and are useful to exploit the partial observations of stochastic systems. Calculations give

$$B_{11}^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0.45 & 0 & 0.0\overline{2} \\ 6.4 & 2.5 & 0 \end{bmatrix}, B_{12}^0 = \begin{bmatrix} 0 & 0 & 0 \\ 4.5 & 0 & 0.31\overline{3} \\ 2.7 & 7.6 & 0 \end{bmatrix},$$

$$B_{22}^0 = \begin{bmatrix} 0 & 0.176 & 0.044 \\ 4.5 & 0 & 0.5 \\ 9 & 2.25 & 0 \end{bmatrix}$$

and then corollary 4.17, gives that as $t \longrightarrow \infty$,

$$\Sigma(t) = \begin{bmatrix} 0.0421 & 1.9613 \\ 1.9613 & 91.358 \end{bmatrix} t^2 + \begin{bmatrix} 0.519 & 15.093 \\ 15.093 & -358.3 \end{bmatrix} t \; + \; + \circ(t).$$

5. A Renewal Theory Approach

In this section we use the renewal theory to obtain $\Sigma(t)$ for a more general function $\rho_r(k,x)$, rather than analytic functions. By conditioning on the first renewal epoch, in the univariate case, we obtain that

$$\begin{aligned} E_i\mathcal{Z}(t) &= (1 - A_i(t))\rho(i,t) + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)\{\rho(i,x) + E_j\mathcal{Z}(t-x)\} \\ &= (1 - A_i(t))\rho(i,t) + \int_0^t A_i(dx)\rho(i,x) \\ &\quad + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)E_j\mathcal{Z}(t-x) \\ &= g(i,t) + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)E_j\mathcal{Z}(t-x), \end{aligned}$$

where E_i is the conditional expectation given $\mathcal{J}(0) = i$. The above equation has the form $f = g + A \star f$, with

$$g(i,t) = (1 - A_i(t))\rho(i,t) + \int_0^t A_i(dx)\rho(i,x)\,,$$

and has the solution

$$E_i\mathcal{Z}(t) = \sum_{j\in\mathcal{N}} \int_0^t R_{ij}(dx)g(j,t-x)\,, \tag{5.1}$$

which provides a formula for $E_i\mathcal{Z}(t)$, the behavior of $E_i\mathcal{Z}(t)$, $t \longrightarrow \infty$ is completely specified in [12]. In the multivariate case, by conditioning on the first renewal epoch one obtains that

$$\begin{aligned}
&E_i\mathcal{Z}_r(t)\mathcal{Z}_s(t)\\
&= (1 - A_i(t))\rho_r(i,t)\rho_s(i,t)\\
&\quad + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)E_j\{\,\rho_r(i,x) + \mathcal{Z}_r(t-x)\,\}\{\,\rho_s(i,x) + \mathcal{Z}_s(t-x)\,\}\\
&= (1 - A_i(t))\rho_r(i,t)\rho_s(i,t) + \int_0^t A_i(dx)\rho_r(i,x)\,\rho_s(i,x)\\
&\quad + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)\{\,\rho_r(i,x)\,E_j\mathcal{Z}_s(t-x) + \rho_s(i,x)\,E_j\mathcal{Z}_r(t-x)\,\}\\
&\quad + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)E_j\mathcal{Z}_r(t-x)\mathcal{Z}_s(t-x)\\
&= g_{rs}(i,t) + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)E_j\mathcal{Z}_r(t-x)\mathcal{Z}_s(t-x).
\end{aligned}$$

The equation given above is a Markov renewal equation with

$$\begin{aligned}
g_{rs}(i,t) &= (1 - A_i(t))\rho_r(i,t)\rho_s(i,t) + \int_0^t A_i(dx)\rho_r(i,x)\rho_s(i,x)\\
&\quad + \sum_{j\in\mathcal{N}} \int_0^t A_{ij}(dx)\{\,\rho_r(i,x)\,E_j\mathcal{Z}_s(t-x) + \rho_s(i,x)\,E_j\mathcal{Z}_r(t-x)\,\}
\end{aligned}$$

and has the solution

$$E_i\mathcal{Z}_r(t)\mathcal{Z}_s(t) = \sum_{j\in\mathcal{N}} \int_0^t R_{ij}(dx)g_{rs}(j,t-x)\,. \tag{5.2}$$

The $\Sigma(t)$ and its asymptotic behavior may be specified by using (5.1), (5.2) and the Markov Renewal Limit Theorems(due to Cinlar). We expect the exact analysis to be hard and interesting and can be the basis of a further study.

References

[1] F. Ball, Central Limit Theorems For Multivariate Semi-Markov Sequences and Processes, with Applications. *J. Appl. Prob.* 36,(1999)415-432.

[2] E. Cinlar, Markov Renewal Theory. *Adv. Appl. Prob.* 1,(1969)123-187.

[3] E. Cinlar, Markov Renewal Theory: a Survey. *Management Sci.* 21,(1975)727-752.

[4] J. Keilson, On the Matrix Renewal Function for Markov Renewal Processes. *Ann. Math. Statist.* 40,(1969)1901-1907.

[5] J. Keilson, A Process with Chain Dependent Growth Rate. Part II: The Ruin and Ergodic Problems. *Adv. Appl. Prob.* 3,(1971) 315-338.

[6] Y. Masuda, Partially Observable Semi-Markov Reward Processes. *J. Appl. Prob.* 30,(1993) 548-560.

[7] Y. Masuda, and U. Sumita, A Multivariate Reward Processes Defined on a Semi-Markov Process and its First Passage Time Distributions. *J. Appl. Prob.* 28,(1991) 360-373.

[8] R. A. Mclean, and M. F. Neuts, The Integral of a Step Function Defined on a Semi Markov Process. *SIAM J. Appl. Math.* 15,(1967) 726-737.

[9] G. A. Parham, and A. R. Soltani, First Passage Time for Reward Processes with Nonlinear Reward Functions: Asymptotic Behavior. *Pak. J. Statist.* 14(1), (1998) 65-80.

[10] H. L. Royden, *Real Analysis*. Collier Macmillan Limited, London.,(1968).

[11] A. R. Soltani, Reward Processes with nonlinear Reward Functions. *J. Appl. Prob.* 33, (1996) 1011-1017.

[12] A. R. Soltani, and K. Khorshidian, Reward Processes for Semi-Markov Processes: Asymptotic Behavior. *J. Appl. Prob.* 35,(1998).

[13] U. Sumita, and Y. Masuda, An Alternative Approach to the Analysis of Finite Semi-Markov and Related Processes. *Commun. Statist.-Stochastic Models* 3(1), (1987) 67-87.

[14] J. L. Teugels, A Bibliography on Semi-Markov Processes. *J. Comput. Appl. Math.* 2, (1976) 125-144.

Math. and Stat. Dept.,
Persian Gulf Univ.,
Bushehr, Iran.
e-mail:K_khorshidian@yahoo.com

Stat. Dept.,
Shiraz Univ.,
Shiraz, Iran
e-mail:Soltani@kuc01.kuniv.edu.kw