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INSERTION OF A FUNCTION BELONGING TO A CERTAIN SUBCLASS OF $\mathbb{R}^X$

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Abstract. Let X be a topological space and $E(X,\mathbb{R})$ be a subset of $\mathbb{R}^X$ with the following properties: (1) Any constant function is in $E(X,\mathbb{R})$; (2) If $\alpha, \beta \in \mathbb{R}$ and $f, g \in E(X,\mathbb{R})$, then $\alpha f + \beta g \in E(X,\mathbb{R})$; (3) If (f_n) is a sequence of functions in $E(X,\mathbb{R})$ and (f_n) is uniformly convergent to f, then $f \in E(X,\mathbb{R})$; and (4) If $f \in E(X,\mathbb{R})$ and g is a constant function, then $\sup\{f,g\} \in E(X,\mathbb{R})$ and $\inf\{f,g\} \in E(X,\mathbb{R})$. Here, necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a function of $E(X,\mathbb{R})$ between two comparable real-valued functions with a certain pair of a general class of properties. The class of properties is defined by being preserved when added to a function of $E(X,\mathbb{R})$ and by being possessed by any constant function.

1. Introduction

A property P defined relative to a real-valued function on a topological space is an *E-property* provided any constant function has property P and provided the sum of a function with property P and any function in $E(X,\mathbb{R})$ also has property P. If P_1 and P_2 are E-properties, the following terminology is used: (i) A space X has the *weak* $E-$*insertion property for* (P_1, P_2) iff for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a function h in $E(X,\mathbb{R})$ such that $g \leq h \leq f$. (ii) A space X has the $E-$*insertion property for* (P_1, P_2) iff for any functions g and f on X such that

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$g < f, g$ has property P_1 and f has property P_2, then there exists a function h in $E(X, \mathbb{R})$ such that $g < h < f$. (iii) A space X has the *strong* $E-$*insertion property for* (P_1, P_2) iff for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a function h in $E(X, \mathbb{R})$ such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper for a space X with the weak $E-$insertion property for (P_1, P_2), we give a necessary and sufficient conditions in terms of lower cut sets for the space to have the $E-$insertion property for (P_1, P_2). Also for a space with the weak $E-$insertion property, we present a necessary and sufficient conditions for the space to have the strong $E-$insertion property.

1. Examples of E-property and E-insertion

In the case that $E(X, \mathbb{R})$ is the set continuous real-valued functions on X i.e. $E(X, \mathbb{R}) = C(X, \mathbb{R})$, then lower semicontinuous (lsc), upper semicontinuous (usc), continuity are examples of $E-$properties.

The following examples of $C-$insertion are known:

(a) A space$X$ has the weak $C-$insertion property for (usc, lsc) iff X is normal.

(b) A space X has the $C-$insertion property for (usc, lsc) iff X is normal and countably paracompact.

(c) A space X has the strong $C-$insertion property for (usc, lsc) iff X is perfectly normal.

Result (a) is due independently to Tong [10] and to Katětov[4]. Example (b) was proved by Katětov[4] and by Dowker [3]. Result (c) is a consequence of Theorem 3.1, Example 1.2, and Proposition 2.3 of Michael [8].

Also, we can choose $E(X, \mathbb{R}) = B_1(X, \mathbb{R})$; the set Baire-one real-valued functions on X, or $E(X, \mathbb{R}) = A(\mathbb{R}, \mathbb{R})$; the set approximately continuous real-valued functions on $\mathbb{R}$.

We now give the appropriate definitions and terminologies as follows:

DEFINITION 1.1. A real-valued function f on X is called upper (resp. lower) semiBaire-one, if for any real number t, the set $\{x \in X : f(x) < t\}$ (resp. $\{x \in X : f(x) > t\}$) is a $F_\sigma-$subset of X. We denote upper semiBaire-one by usB_1 and lower semiBaire-one by lsB_1.

In the case that $E(X,\mathbb{R}) = B_1(X,\mathbb{R})$, then ls$B_1$, us$B_1$ and B_1 are examples of E-properties.

If a space has the strong $E-$insertion property for (P_1, P_2), then it has the weak $E-$insertion and the $E-$insertion property for (P_1, P_2). In order to see when the $E-$insertion property implies the weak $E-$ins ertion property, a technique used by Dieudonnĕ[2] is employed to prove the following result.

THEOREM 1.2. *Let P_1 and P_2 be $E-$properties and assume that X satisfies the $E-$insertion property for (P_1, P_2). If (a) the transformation $f \to f/(1 + |f|)$ preserves P_1 and P_2, and transformation $f \to f/(1 - |f|)$ preserves E for any $f \in E(X, (-1, 1))$, and*

(b) if $\inf(f, h)$ has property P_2 and $\sup(g, h)$ has property P_1 whenever f has property P_2, g has property P_1 and h is any function in $E(X,\mathbb{R})$,

then X satisfies the weak $E-$insertion propery for (P_1, P_2).

Proof. Let g and f be functions on X such that $g \le f, g$ has property P_1 and f has property P_2. If $G = g/(1+|g|)$ and $F = f/(1+|f|)$, then G has property P_1 and F has property P_2 and $-1 < G \le F < 1$. If $G_0 = G-1$ and $F_0 = F+1$, then by hypothesis there exists a function $h_0 \in E(X,\mathbb{R})$ such that $G_0 < h_0 < F_0$. Let

$$f_1 = \inf(F + 1/2, h_0 + 1/2), g_1 = \sup(G - 1/2, h_0 - 1/2).$$

Then $g_1 < f_1$ and by hypothesis, g_1 has property P_1 and f_1 has property P_2. Inductively, let

$$f_n = \inf(F + 1/2^n, h_{n-1} + 1/2^n), g_n = \sup(G - 1/2^n, h_{n-1} - 1/2^n),$$

where $h_{n-1} \in E(X,\mathbb{R})$ and $g_{n-1} < h_{n-1} < f_{n-1}$. Again by hypothesis there exists a function $h_n \in E(X,\mathbb{R})$ such that $g_n < h_n < f_n$. Since $h_n < f_n \le h_{n-1} + 1/2^n$ and $h_n > g_n \ge h_{n-1} - 1/2^n$, then $-1/2^n \le h_n - h_{n-1} \le 1/2^n$. Since $|h_n - h_{n-1}| \le 1/2^n$, the sequence (h_n) converges uniformly to $H \in E(X,\mathbb{R})$ by the Cauchy condition and the properties of $E(X, \mathbb{R})$. Since

$$G - 1/2^n \leq g_n < h_n < f_n \leq F + 1/2^n$$

and since (h_n) converges to H, it follows that $G \leq H \leq F$. Since the function $t \to t/(1-|t|)$ on $(-1, 1)$ is increasing, thus if $h = H/(1-|H|)$ then $g \leq h \leq f$ and $h \in E(X, \mathbb{R})$. Thus X satisfies the weak E-insertion property for (P_1, P_2). $\square$

2. E-insertion

If f is a real-valued function defined on a space X and if

$$\{x : f(x) < t\} \subseteq A(f, t) \subseteq \{x : f(x) \leq t\},$$

for a real number t, then $A(f, t)$ is a *lower cut set* in the domain of f at the level t. This definition is due to Brooks [1], where the terminology lower indefinite cut set is used. The main result of this section uses lower cut sets and gives a necessary and sufficient condition for a space that satisfies the weak E-insertion property to satisfy the $E-$insertion property.

THEOREM 2.1. *Let P_1 and P_2 be E-property and X be a space that satisfies the weak E-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the E-insertion property for (P_1, P_2) iff there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence (D_n) of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by functions in $E(X, \mathbb{R})$.*

Proof. Assume that X has the weak E-insertion property for (P_1, P_2). Let g and f be functions such that $g < f, g$ has property P_1 and f has property P_2. By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by functions in $E(X, \mathbb{R})$. Let k_n be a function in $E(X, [0, 1])$ such that

$k_n = 0$ on $A(f-g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the properties of $E(X, \mathbb{R})$, the function k is in $E(X, \mathbb{R})$. Since $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f-g, 3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If x is any point in X, then $x \notin A(f-g, 1)$ or for some n,

$$x \in A(f-g, 3^{-n+1}) - A(f-g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are E-properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak E-insertion property for (P_1, P_2), then there exists a function $h \in E(X, \mathbb{R})$ such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that X satisfies the E-insertion property for (P_1, P_2). (The technique of this proof is by Katětov[4]).

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a function $h \in E(X, \mathbb{R})$ such that $g < h < f$. We follow an idea contained in Lane [6]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the E-insertion property for (P_1, P_2), then there exists a function $k \in E(X, \mathbb{R})$ such that $0 < k < f - h$. Let $A(f-g, 3^{-n+1})$ be any lower cut set for $f - g$ and let $D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \varnothing$. Since

$$\begin{aligned} A(f-g, 3^{-n+1}) &\subseteq \{x \in X : (f-g)(x) \leq 3^{-n+1}\} \\ &\subseteq \{x \in X : k(x) \leq 3^{-n+1}\} \end{aligned}$$

and since $\{x \in X : k(x) \leq 3^{-n+1}\}$ and $\{x \in X : k(x) \geq 3^{-n+2}\}$ $= X \setminus D_n$ are completely separated by $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$ $\in E(X, \mathbb{R})$, it follows that for each $n, A(f-g, 3^{-n+1})$ and $X \setminus D_n$ are completely separated by functions in $E(X, \mathbb{R})$. $\square$

3. Strong E-insertion

The main result of this section uses lower cut sets and gives a necessary and sufficient condition for a space that satisfies the weak $E-$insertion property to satisfy the strong $E-$insertion property.

THEOREM 3.1. *Let P_1 and P_2 be E-property and X be a space that satisfies the weak E-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g \leq f, g$ has property P_1 and f has property P_2. The space X has the strong E-insertion property for (P_1, P_2) iff there exists a sequence $(A(f-g, 2^{-n}))$ of lower cut sets for $f-g$ and there exists a sequence (F_n) of subsets of X such that*

(i) $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$, and

(ii) for each n, the sets $A(f-g, 2^{-n})$ and F_n are completely separated by functions in $E(X, \mathbb{R})$.

Proof. Suppose that there is a sequence $(A(f-g, 2^{-n}))$ of lower cut sets for $f-g$ and suppose that there is a sequence (F_n) of subsets of X such that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n, there exists a function $k_n \in E(X, [0, 2^{-n}])$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f-g, 2^{-n})$. The function k from X into $[0, 1/4]$ which is defined by

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x)$$

is in $E(X, \mathbb{R})$ by the Cauchy condition and the properties of $E(X, \mathbb{R})$, (1) $k^{-1}(0) = \{x \in X : (f-g)(x) = 0\}$ and (2) if $(f-g)(x) > 0$ then $k(x) < (f-g)(x)$: In order to verify (1), observe that if $(f-g)(x) = 0$, then $x \in A(f-g, 2^{-n})$ for each n and hence $k_n(x) = 0$ for each n. Thus $k(x) = 0$. Conversely, if $(f-g)(x) > 0$, then there exists an n such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f-g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f-g, 2^{-n})$$

and that $(A(f-g, 2^{-n}))$ is a decreasing sequence. Thus if $(f-g)(x) > 0$ then either $x \notin A(f-g, 1/2)$ or there exists a smallest n such that $x \notin A(f-g, 2^{-n})$ and $x \in A(f-g, 2^{-j})$ for $j = 1, \ldots, n-1$.
In the former case,

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f-g)(x),$$

and in the latter,

$$k(x) = 1/4\sum_{j=n}^{\infty} k_j(x) \leq 1/4\sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f-g)(x).$$

Thus $0 \leq k \leq f-g$ and if $(f-g)(x) > 0$ then $(f-g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$ then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are E-properties, then g_1 has property P_1 and f_1 has property P_2. Since by hypothesis X has the weak E-insertion property for (P_1, P_2) , then there exists a function $h \in E(X, \mathbb{R})$ such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Therefore X has the strong E-insertion property for (P_1, P_2). (The technique of this proof is by Lane [6].)

Conversely, assume that X satisfies the strong E-insertion for (P_1, P_2). Let g and f be functions on X satisfying P_1 and P_2 respectively such that $g \leq f$. Thus there exists $h \in E(X, \mathbb{R})$ such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. We follow an idea contained in Powderly [9]. Now consider the functions 0 and $f-h.0$ satisfies property P_1 and $f-h$ satisfies property P_2. Thus there exists function $h_1 \in E(X, \mathbb{R})$ such that $0 \leq h_1 \leq f-h$ and if $0 < (f-h)(x)$ for any x in X, then $0 < h_1(x) < (f-h)(x)$. We next show that

$$\{x \in X : (f-g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.$$

If x is such that $(f-g)(x) > 0$, then $g(x) < f(x)$. Therefore $g(x) < h(x) < f(x)$. Thus $f(x) - h(x) > 0$ or $(f-h)(x) > 0$. Hence $h_1(x) > 0$. On the other hand, if $h_1(x) > 0$, then since $(f-h) \geq h_1$ and $f-g \geq f-h$, therefore $(f-g)(x) > 0$. For each n, let

$$A(f-g, 2^{-n}) = \{x \in X : (f-g)(x) \leq 2^{-n}\}$$
$$, F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}$$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Since $\{x \in X : (f-g)(x) > 0\} = \{x \in X : h_1(x) > 0\}$, it follows that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is in $E(X,[0,2^{-n}])$ which completely separates F_n and $A(f-g,2^{-n})$. From its definition and by the properties of $E(X,\mathbb{R})$, it is clear that k_n is in $E(X,[0,2^{-n}])$. Let $x \in F_n$. Then, from the definition of $k_n, k_n(x) = 2^{-n}$. If $x \in A(f-g,2^{-n})$, then since $h_1 \leq f-h \leq f-g, h_1(x) \leq 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of k_n. Hence k_n completely separates F_n and $A(f-g,2^{-n})$. $\square$

THEOREM 3.2. *Let P_1 and P_2 be E-properties and assume that the space X satisfied the weak E-insertion property for (P_1, P_2). The space X satisfies the strong E-insertion property for (P_1, P_2) iff X satisfies the strong E-insertion property for (P_1, E) and for (E, P_2).*

Proof. Assume that X satisfies the strong E-insertion property for (P_1, E) and for (E, P_2). If g and f are functions on X such that $g \leq f, g$ satisfies property P_1, and f satisfies property P_2, then since X satisfies the weak E-insertion property for (P_1, P_2) there is a function $k \in E(X,\mathbb{R})$ such that $g \leq k \leq f$. Also, by hypothesis there exist functions h_1 and h_2 in $E(X,\mathbb{R})$ such that $g \leq h_1 \leq k$ and if $g(x) < k(x)$ then $g(x) < h_1(x) < k(x)$ and such that $k \leq h_2 \leq f$ and if $k(x) < f(x)$ then $k(x) < h_2(x) < f(x)$. If a function h is defined by $h(x) = (h_2(x) + h_1(x))/2$, then h is in $E(X,\mathbb{R}), g \leq h \leq f$, and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Hence X satisfies the strong E-insertion property for (P_1, P_2). The converse is obvious since any function in $E(X,\mathbb{R})$ must satisfy both properties P_1 and P_2. $\square$
(The technique of this proof is by Lane [7].)

Remark. In conclusion, let us mention that, in the case that $E(X,\mathbb{R}) = C(X,\mathbb{R})$, Theorems 1.2, 2.1, 3.1 of Lane [6] and Proposition 2.1 of Lane [7] are respectively consequences of our Theorems 1.1, 2.1, 3.1 and 3.2.

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