Bulletin of the Iranian Mathematical Society, Vol. 28 *No. 2 (2002), pp.69-75*

TOPOLOGICALLY LEFT INVARIANT MEAN ON DUAL SEMIGROUP ALGEBRAS

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ABSTRACT. Let S be a locally compact Hausdorff semitopological semigroup, and $M(S)$ be the Banach algebra of all bounded regular Borel measures on S. In this paper, we obtain a necessary and sufficient condition for $M(S)^*$ to have a topologically left invariant mean.

1. Introduction

Let S be a locally compact Hausdorff semitopological semigroup with convolution measure algebra $M(S)$ and probability measures $M_\circ(S)$. We know that $M(S)$ is a Banach algebra with total variation norm and convolution, so we can define the first Arens product on $M(S)^{**}$, i.e. for $F, G \in M(S)^{**}$ and $f \in M(S)^*$

$$\langle FG, f\rangle = \langle F, Gf\rangle, \langle Gf, \mu\rangle = \langle G, f\mu\rangle, \langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$$

where $\mu, \nu \in M(S)$. On a Banach algebra A a functional $f \in A^*$ is called weakly almost periodic if $W(f) = \{fa; a \in A, ||a|| \leq 1\}$ is relatively weakly compact in A^* where $\langle fa, b\rangle = \langle f, ab\rangle$ for all $a, b \in A$ [8]. We denote by $wap(M(S))$ the set of all weakly almost periodic functionals on $M(S)$. Clearly $1 : M(S) \longrightarrow \mathbb{C}$ given by $\langle 1, \mu\rangle = \mu(S)$ is weakly almost periodic. A functional $M \in M(S)^{**}$ (respectively $M \in wap(M(S))^*$) is called a mean on $M(S)^*$ (respectively on $wap(M(S))$) if $||M|| = \langle M, 1\rangle = 1$, and $\langle M, f\rangle \geq 0$ where $f \in M(S)^*$ (respectively $f \in wap(M(S))$) and $f \geq 0$ ([10],

MSC(2000): Primary 22A20; Secondary 43A60.
Keywords: Semitopological semigroup, Topologically left invariant mean
Received: 17 April 2002 , Revised: 12 January 2003

[12], [13]). A mean M is said to be a topologically left invariant if $\langle M, f\mu\rangle = \langle M, f\rangle$ where $f \in M(S)^*$ (respectively $f \in wap(M(S)))$ and $\mu \in M_\circ(S)$.

Wong studied topologically left invariant mean on $M(S)^*$ and proved that $M(S)^*$ has a topologically left invariant mean if and only if there is a net (μ_α) in $M_\circ(S)$ such that $||\mu * \mu_\alpha - \mu_\alpha|| \longrightarrow 0$ $(\mu \in M_\circ(S))$[12]. Also, Day ([2], [3]) and Junghenn [5] have studied topologically left invariant mean on $M(S)^*$.

For a locally compact group G, Wong [13] has shown, there is a net (μ_α) in $M_\circ(G)$ such that $||\mu * \mu_\alpha - \mu_\alpha|| \longrightarrow 0$ for all $\mu \in M_\circ(G)$ if and only if there is a net (μ_α) in $M_\circ(G)$ such that for all compact subset K of G, $||\mu * \mu_\alpha - \mu_\alpha|| \longrightarrow 0$ uniformly over all μ in $M_\circ(G)$ which are supported in K. But for a semigroup S this matter is not known.

In this paper, among other things, we will show that if $M_\circ(S)$ has a measure ν such that the map $s \longrightarrow \delta_s * \nu$ from S into $M(S)$ is continuous, then the last statement is valid. In fact with this condition we provide an answer to a problem raised by Lau ([7], p. 322) and Day [3].

2.Topologically left invariant mean

Suppose S is a locally compact Hausdorff semitopological semigroup. By the Eberlein-Smulian theorem $wap(M(S))$ is a Banach subspace of $M(S)^*$. It is easy to see that for every $f \in M(S)^*$, $\{f\mu; \mu \in M(S), ||\mu|| \leq 1\}$ is relatively weakly compact if and only if $\{\mu f; \mu \in M(S), ||\mu|| \leq 1\}$ is relatively weakly compact. So, if $f \in wap(M(S)$ then $\{\mu f; \mu \in M(S), ||\mu|| \leq 1\}$ is relatively weakly compact. Lashkarizadeh in [6] has proved $wap(S) \subseteq wap(M(S))$, where $wap(S) = \{f \in C(S); \{L_s f, s \in S\}$ is relatively weakly compact in $C(S)\}$. Also he has shown that, if S is a foundation topological semigroup with identity, then $wap(L(S)) = wap(S)$.

We recall that a semigroup S is said to be left amenable if there exists $m \in B(S)^*$ such that $m \geq 0$, $||m|| = 1$ and $\langle m, L_s f\rangle = \langle m, f\rangle$ for all $s \in S$ and all $f \in B(S)$, where $B(S)$ is the set of all bounded complex valued functions on S [1]. In the following Theorem, we give conditions on S and $M_\circ(S)$ that are sufficient to guarantee topologically left amenability of $wap(M(S))$.

Theorem 2.1. *Let S be a locally compact Hausdorff semitopological semigroup. $wap(M(S))$ has a topologically left invariant mean if any one of the following conditions holds:*

(1) *S is left amenable and there exists $\nu \in M_\circ(S)$ such that the map $s \longrightarrow \delta_s * \nu$ from S into $M(S)$ is weakly continuous and $\delta_s * \nu = \nu * \delta_s$ $(s \in S)$.*

(2) *S has an identity e and X (X is the set of all means on $wap(M(S))$ with $\sigma(X, wap(M(S)))$ topology) has a dense subset Y such that $\delta_e \in Xy$ for all $y \in Y$.*

Proof. We start by showing that for $f \in wap(M(S))$ the function $s \longrightarrow \langle f\nu, \delta_s\rangle$ is a continuous function on S. Indeed this is easy, because $\langle f\nu, \delta_s\rangle = \langle f, \nu * \delta_s\rangle = \langle f, \delta_s * \nu\rangle$. Notice too that if also $t \in S$, then $\langle f(\nu * \delta_s), \delta_t\rangle = \langle f\nu, \delta_s * \delta_t\rangle$. The iterated limit condition (or lots of other methods) now shows that $s \longrightarrow \langle f\nu, s\rangle$ is in $wap(S)$. This means in particular that if M is a left invariant mean on $B(S)$ (and in fact we only need on $wap(S)$) then $\langle M, f\nu\rangle$ is well-defined. Let M_1 be any continuous linear extension of M from $wap(S)$ to $wap(M(S))$. we claim that νM_1 is a left invariant mean on $wap(M(S))$. Indeed for $f \in wap(M(S))$,

$$\begin{aligned}\langle \delta_s(\nu M_1), f\rangle &= \langle M_1, (f\delta_s)\nu\rangle = \langle M, f(\delta_s * \nu)\rangle = \langle M, f(\nu * \delta_s)\rangle \\ &= \langle \delta_s M, f\nu\rangle = \langle M, f\nu\rangle = \langle M_1, f\nu\rangle = \langle \nu M_1, f\rangle.\end{aligned}$$

To see that this is topologically left invariant, we simply integrate, using the fact the function $s \longrightarrow \langle \nu M_1, f\delta_s\rangle = \langle M_1 f, \delta_s * \nu\rangle$ is continuous:

$$\langle \mu(\nu M_1), f\rangle = \langle \nu M_1, f\mu\rangle = \int \langle \nu M_1, f\delta_s\rangle d\mu = \int \langle \nu M_1, f\rangle d\mu = \langle \nu M_1, f\rangle.$$

2) Let (M_α) be a net in X, $M \in X$ and $M_\alpha \longrightarrow M$ in the $\sigma(X, wap(M(S)))$ topology. If $M_1 \in X$, $f \in wap(M(S))$, since $\{Mf; M \in X\}$ is relatively weakly compact (of course we can define the first Arens product on $wap(M(S))^*$ and so Mf is well defined), for every subnet (M_β) of (M_α) there is a subnet (M_γ) of (M_β) such that $M_\gamma f \longrightarrow Mf$ in the weak topology. Hence $\langle M_1 M_\gamma, f\rangle \longrightarrow \langle M_1 M, f\rangle$. Consequently $\langle M_1 M_\alpha, f\rangle \longrightarrow \langle M_1 M, f\rangle$, i.e. X is a semitopological semigroup (in the $\sigma(X, wap(M(S)))$ topology). Now, let E, E_1 be two idempotents lying in the same minimal left ideal. Since $EE_1 = E$, by assumption and an argument similar to the proof in ([1], Theorem

1.5.9), we have $E = E_1$. So the minimal left ideals of X are groups. On the other hand the minimal left ideals of X are affine. By Corollary 1.3.23 in [1] every minimal idempotent is right zero. Therefore if E is a minimal idempotent, it is clear that E is a topologically left invariant mean on $wap(M(S))$.

Theorem 2.2. *Let $M_\circ(S)$ contains a measure ν such that the map $s \longrightarrow \delta_s * \nu$ from S into $M(S)$ is continuous when $M(S)$ has the norm topology. Then the following statements are equivalent:*

(1) *$M(S)^*$ has a topologically left invariant mean.*
(2) *There is a net (μ_α) in $M_\circ(S)$ such that for every compact subset K of S, $||\mu * \mu_\alpha - \mu_\alpha|| \longrightarrow 0$ uniformly over all μ in $M_\circ(S)$ which are supported in K.*
(3) *For every finite subset Ω of $M_\circ(S)$, $\epsilon > 0$, there exists a measure $\mu \in M_\circ(S)$ such that $||\nu * \mu - \mu|| < \epsilon$ for all $\nu \in \Omega$.*

Proof. Reader could point out that the hypothesis about the continuity of $s \longrightarrow \delta_s * \nu$ is needed only for (1) $\longrightarrow$ (2). Let $M(S)^*$ has a topologically left invariant mean. Then by ([12], Theorem 3.1), there is a net (ν_α) in $M_\circ(S)$ such that $||\mu * \nu_\alpha - \nu_\alpha|| \longrightarrow 0$ for all $\mu \in M_\circ(S)$. Now, let $\nu \in M_\circ(S)$ and the map $s \longrightarrow \delta_s * \nu$ be continuous. For all α, we take $\mu_\alpha = \nu * \nu_\alpha$. It is easy to see that $||\mu * \mu_\alpha - \mu_\alpha|| \longrightarrow 0$ $(\mu \in M_\circ(S))$ and $s \longrightarrow \delta_s * \mu_\alpha$ is continuous. So, if K is a compact subset of S and $\epsilon > 0$, then for $s \in K$ there is a neighbourhood U_s of s such that for every α and $t \in U_s$

$$||\delta_s * \mu_\alpha - \delta_t * \mu_\alpha|| < \epsilon/2.$$

But K is a compact subset of S, hence we can choose a finite subset $\{s_1, s_2, ..., s_n\}$ of K which $K \subseteq \bigcup_{i=1}^n U_{s_i}$. Also, we can find an $\alpha_\circ$ such that for every $\alpha \geq \alpha_\circ$ and $1 \leq i \leq n$,

$$||\delta_{s_i} * \mu_\alpha - \mu_\alpha|| < \epsilon/2.$$

Let $A_1 = U_{s_1}$, $A_i = U_{s_i} \setminus \bigcup_{j=1}^{i-1} U_{s_j}$, $2 \leq i \leq n$, and $\mu \in M_\circ(S)$. Since for every α the map $s \longrightarrow \delta_s * \mu_\alpha$ is continuous, therefore by ([11], Chapter 3), $\int_K \delta_s * \mu_\alpha d\mu(s) \in M(S)$ and $\int_K \delta_s * \mu_\alpha d\mu(s) = \mu\chi_K * \mu_\alpha$. Consequently for $\mu \in M_\circ(S)$ with $supp\, \mu \subseteq K$ and $f \in M(S)^*$, we can write

$$|\langle f,\mu*\mu_\alpha\rangle-\langle f,\mu_\alpha\rangle|=|\sum_{i=1}^{n}\int_{A_i}\langle f,\delta_s*\mu_\alpha\rangle-\langle f,\mu_\alpha\rangle d\mu(s)|$$

$$\leq\sum_{i=1}^{n}\int_{A_i}|\langle f,\delta_s*\mu_\alpha\rangle-\langle f,\delta_{s_i}*\mu_\alpha\rangle|d\mu(s)$$

$$+\sum_{i=1}^{n}|\langle f,\delta_{s_i}*\mu_\alpha\rangle-\langle f,\mu_\alpha\rangle|\mu(A_i)\leq$$

$$\sum_{i=1}^{n}\int_{A_i}||f||||\delta_s*\mu_\alpha-\delta_{s_i}*\mu_\alpha||d\mu(s)$$

$$+\sum_{i=1}^{n}||f||||\delta_{s_i}*\mu_\alpha-\mu_\alpha||\mu(A_i)<||f||\epsilon$$

where $\alpha\geq\alpha_\circ$. So (1) implies (2).

(2) implies (3) is easy. Now, assume that (3) holds. For every finite subset Ω of $M_\circ(S)$ and $\epsilon>0$, we associate the nonvoid subset $A_{\Omega,\epsilon}=\{\eta\in M_\circ(S);||\mu*\eta-\eta||<\epsilon$ for all $\mu\in\Omega\}$. Since the family $\{A_{\Omega,\epsilon};\Omega$ is a finite subset of $M_\circ(S)$ and $\epsilon>0\}$ has the finite intersection property, so there exists $M\in M(S)^{**}$ such that $M\in\bigcap_{\Omega,\epsilon}$weak*-closure $A_{\Omega,\epsilon}$. Now let $f\in M(S)^*$ with $||f||=1$ and $\mu\in M_\circ(S)$. For $\epsilon>0$, there exists $\eta\in A_{\{\mu\},\epsilon/3}$ such that $|\langle\eta,f\rangle-\langle M,f\rangle|<\epsilon/3$ and $|\langle\eta,f\mu\rangle-\langle M,f\mu\rangle|<\epsilon/3$. So

$$|\langle M,f\rangle-\langle M,f\mu\rangle|\leq|\langle M,f\rangle-\langle\eta,f\rangle|+|\langle\eta,f\rangle-\langle\mu*\eta,f\rangle|$$

$$+|\langle\mu*\eta,f\rangle-\langle M,f\mu\rangle|<\epsilon.$$

Therefore $\langle M,f\rangle=\langle M,f\mu\rangle$. It is trivial that M is a mean, and so (1) holds. This completes our proof.

Let S be a topological semigroup with identity. We define $L(S)=\{\mu\in M(S);s\longrightarrow\delta_s*|\mu|$ and $s\longrightarrow|\mu|*\delta_s$ are weakly continuous$\}$. In the following Theorem, we may assume that S is a locally compact Hausdorff foundation topological semigroup, i.e. $\bigcup\{supp\,\mu;\mu\in L(S)\}$ is dense in S. It is well known that $L(S)$ is an ideal in $M(S)$ and has an approximate identity [6]. We also note that for $\mu\in L(S)$ both mapping $x\longrightarrow|\mu|*\delta_x$ and $x\longrightarrow\delta_x*|\mu|$ from S into $M(S)$ are norm continuous [4].

Theorem 2.3. *Let S be a foundation topological semigroup with identity. Then the following are equivalent:*

1. *$M(S)^*$ has a topologically left invariant mean.*
2. *There is a net (ν_β) in $M_\circ(S)$ with finite support such that for all $\mu \in M_\circ(S)$ and $\nu \in L(S)$, $||\mu * \nu_\beta * \nu - \nu_\beta * \nu|| \longrightarrow 0$.*
3. *There is a net (ν_β) in $M_\circ(S)$ with finite support such that for all compact subset K of S and $\nu \in L(S)$, $||\mu*\nu_\beta*\nu-\nu_\beta*\nu|| \longrightarrow 0$ uniformly over all μ in $M_\circ(S)$ which are supported in K.*

Proof. Let $M(S)^*$ has a topologically left invariant mean. Then there is a net (γ_α) in $M_\circ(S)$ such that $||\mu * \gamma_\alpha - \gamma_\alpha|| \longrightarrow 0$ ([12], Theorem 3.1). Now, if $\eta \in M_\circ(S) \bigcap L(S)$, we take $\mu_\alpha = \gamma_\alpha * \eta$ (for all α). Since $L(S)$ is an ideal in $M(S)$, so $\mu_\alpha \in L(S)$. Let $\epsilon > 0$ be given. For all α, we choose $\eta_\alpha \in M_\circ(S) \bigcap L(S)$ with compact support and $||\eta_\alpha - \mu_\alpha|| < \epsilon/4$. On the other hand, $L(S)$ has a positive approximate identity of norm one ([6], Lemma 3.4). So there is a $\xi_\alpha \in M_\circ(S) \bigcap L(S)$ such that

$$||(\eta_\alpha - \mu_\alpha) * \xi_\alpha|| < \epsilon/2.$$

For $s \in S$, there exists a neighbourhood U_s of s such that $||\delta_s * \xi_\alpha - \delta_t * \xi_\alpha|| < \epsilon/2$ ($t \in U_s$). But $supp\, \eta_\alpha$ is compact, hence we can find a finite subset $\{s_1, s_2, ..., s_n\}$ of S with $supp\, \eta_\alpha \subseteq \bigcup_{i=1}^{n} U_{s_i}$. If $A_1 = U_{s_1}$ and $A_i = U_{s_i} \setminus \bigcup_{j=1}^{i-1} U_{s_j}$, $2 \le i \le n$, we define $\nu_\alpha = \sum_{i=1}^{n} \eta_\alpha(A_i)\delta_{s_i}$. So for $f \in L(S)^*$, $||f|| \le 1$, we have

$$|\langle f, \eta_\alpha * \xi_\alpha\rangle - \langle f, \nu_\alpha * \xi_\alpha\rangle| = |\int \langle f, \delta_s * \xi_\alpha\rangle - \langle f, \nu_\alpha * \xi_\alpha\rangle d\eta_\alpha(s)|$$

$$\le \sum_{i=1}^{n} \int_{A_i} |\langle f, \delta_s * \xi_\alpha\rangle - \langle f, \delta_{s_i} * \xi_\alpha\rangle| d\eta_\alpha(s) < \epsilon/2.$$

Therefore $||\eta_\alpha * \xi_\alpha - \nu_\alpha * \xi_\alpha|| < \epsilon/2$ and $||\mu_\alpha * \xi_\alpha - \nu_\alpha * \xi_\alpha|| < \epsilon$. Consequently, since for all $\mu \in M_\circ(S)$, $||\mu*\mu_\alpha - \mu_\alpha|| \longrightarrow 0$, we may therefore determine a net (ν_β) in $M_\circ(S)$ with finite support and a net (ξ_β) as an approximate identity in $L(S)$ such that $||\mu * \nu_\beta * \xi_\beta - \nu_\beta * \xi_\beta|| \longrightarrow 0$ for all $\mu \in M_\circ(S)$. Hence it is easy to see that $||\mu * \nu_\beta * \nu - \nu_\beta * \nu|| \longrightarrow 0$ for all $\nu \in L(S)$ and $\mu \in M_\circ(S)$. So (1) implies (2).
If (2) holds, an argument similar to the proof of Theorem 2.2 implies (3).
By ([12], Theorem 3.1), (3) implies (1).

Acknowledgment

This work has been done while the author was visiting The Department of Mathematical Science of Alberta University. The author would like to thank both Professor A. T. Lau and University of Alberta for their hospitality.

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