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SCHUR-PAIR PROPERTY AND THE STRUCTURE OF VARIETAL COVERING GROUPS

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Abstract: This paper is devoted to study the connection between the concepts of Schur-pair and the Baer-invariant of groups with respect to a given variety $\mathcal{V}$ of groups. It is shown that if a group G belongs to a certain class of groups then so does its $\mathcal{V}$-covering group G^*. Among other results, a theorem of E.W. Read in 1977 is being generalized to arbitrary varieties of groups. In addition, we consider covering groups and marginal extensions of a $\mathcal{V}$-perfect group with respect to a subvariety of abelian groups $\mathcal{V}$ and show that any $\mathcal{V}$-marginal extension of a $\mathcal{V}$-perfect group G is a homomorphic image of a $\mathcal{V}$-stem cover of G.

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1. Introduction and Preliminaries

Let F_∞ be a free group freely generated by a countable set $\{x_1, x_2, \ldots\}$. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, which is a subset of F_∞. It will be assumed that the reader is familiar with the notions of verbal subgroup, $V(G)$, and of marginal subgroup, $V^*(G)$, associated with a variety of groups $\mathcal{V}$, and a given group G. See also [23] for more information on the varieties of groups.

Let G be a group with a normal subgroup N. Then we define $[NV^*G]$ to be the subgroup of G generated by the elements of the following set:

$$\{v(g_1,\ldots,g_in,\ldots,g_r)(v(g_1,\ldots,g_r))^{-1} \mid 1 \le i \le r; v \in V; g_1,\ldots,g_r \in G; n \in N\}.$$

It is easily checked that $[NV^*G]$ is the smallest normal subgroup T of G contained in N, such that $N/T \subseteq V^*(G/T)$.

The following lemma gives basic properties of verbal and marginal subgroups of a group G with respect to the variety $\mathcal{V}$, which are useful in our investigations, see [4] for the proofs.

Lemma 1.1 *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let N be a normal subgroup of a group G. Then the following statements hold:*

(i) $V(V^*(G)) = 1$ *and* $V^*(\frac{G}{V(G)}) = \frac{G}{V(G)}$;
(ii) $V(G) = 1$ *iff* $V^*(G) = G$ *iff* $G \in \mathcal{V}$;
(iii) $[NV^*G] = 1$ *iff* $N \subseteq V^*(G)$;
(iv) $V(\frac{G}{N}) = \frac{V(G)N}{N}$ *and* $V^*(\frac{G}{N}) \supseteq \frac{V^*(G)N}{N}$;
(v) $V(N) \subseteq [NV^*G] \subseteq N \cap V(G)$. *In particular* , $V(G) = [GV^*G]$;
(vi) *If* $N \cap V(G) = 1$, *then* $N \subseteq V^*(G)$ *and* $V^*(G/N) = V^*(G)/N$;
(vii) $V^*(G) \cap V(G)$, is contained in the Frattini subgroup of G.

The following useful lemma can be proved easily. See also [4].

Lemma 1.2 *Let $\mathcal{V}$ be a variety of groups, and G be a group. If $G = HN$, where H a subgroup and N is a normal subgroup of G, then $V(G) = V(H)[NV^*G]$.*

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let G be an arbitrary group with a free presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1,$$

where F is a free group. Then the *Baer-invariant* of G with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \bigcap V(F)}{[RV^*F]}.$$

One may check that the Baer-invariant of a group G is always abelian and independent of the choice of the free presentation of G (see [7] or [8]). In particular, if $\mathcal{V}$ is the variety of abelian or nilpotent groups of class at most c ($c \geq 1$), then the Baer-invariant of the group G will be $\frac{R \cap F'}{[R, F]}$ (the so called *Schur-multiplicator* of G) or it will be $\frac{R \cap \gamma_{c+1}(F)}{[R,_c F]}$, respectively(here $\gamma_{c+1}(F)$ stands for the $(c+1)$st term of the *lower central series* of F and $[R,_c F] = [R, F, \ldots, F]$, where F is repeated c times (see also [8], [9], [10] or [12]).

An exact sequence $1 \longrightarrow A \longrightarrow G^* \longrightarrow G \longrightarrow 1$ is said to be a *$\mathcal{V}$-stem cover* of G, if (i) $A \subseteq V(G^*) \cap V^*(G^*)$, and (ii) $A \cong \mathcal{V}M(G)$. In this case G^* is called a *$\mathcal{V}$-covering group* of G. Note that if $\mathcal{V}$ is taken to be the variety of abelian groups, then we have the usual definition of *covering group.*

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let G and H be groups. Then (α, β) is said to be a *$\mathcal{V}$-isologism* between G and H, if there exist isomorphisms $\alpha : \frac{G}{V^*(G)} \longrightarrow \frac{H}{V^*(H)}$ *and* $\beta : V(G) \longrightarrow V(H)$, such that for all $v(x_1, \ldots, x_r) \in V$ and all $g_1, \ldots, g_r \in G$, we have

$$\beta(v(g_1, \ldots, g_r)) = v(h_1, \ldots, h_r),$$

whenever $h_i \in \alpha(g_i V^*(G))$, $i = 1, \ldots, r$. In this case we write $G \sim_{\mathcal{V}} H$, and say that G is $\mathcal{V}$*-isologic* to H. In particular, if $\mathcal{V}$ is the variety of abelian groups we obtain the notion of *isoclinism* due to P. Hall [3], (see also [17] and [18]).

The following lemma of H.N. Hekster [4] is needed, in our investigation.

Lemma 1.3 *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let G be a group with a subgroup H and a normal subgroup N. Then the following statements hold.*

(i) $H \sim_{\mathcal{V}} HV^*(G)$. In particular, if $G = HV^*(G)$ then $G \sim_{\mathcal{V}} H$. Conversely, if $\frac{G}{V^*(G)}$ satisfies the descending chain condition on subgroups and $G \sim_{\mathcal{V}} H$, then $G = HV^*(G)$.

(ii) $\frac{G}{N \cap V(G)} \sim_{\mathcal{V}} \frac{G}{N}$. In particular, if $N \cap V(G) = \langle 1 \rangle$ then $G \sim_{\mathcal{V}} \frac{G}{N}$. Conversely, if $V(G)$ satisfies the descending chain condition on normal subgroups and $G \sim_{\mathcal{V}} \frac{G}{N}$, then $N \cap V(G) = \langle 1 \rangle$.

In section 2, we deal with the connection between the Schur pair property and the Baer-invariant of groups. In fact, it will be shown that if $(\mathcal{V}, \mathcal{X})$ is a Schur-pair, and G^* is a $\mathcal{V}$-covering group of a group G then $G \in \mathcal{X}$ if and only if $G^* \in \mathcal{X}$ (see Corollary 2.3).

In section 3, we study the varietal covering groups and among the other results, a theorem of E.W. Read [24] is being generalized, extensively. Section 4 is devoted to study the $\mathcal{V}$-covering groups and $\mathcal{V}$-marginal extensions of a $\mathcal{V}$-perfect group, when $\mathcal{V}$ is taken to be a subvariety of abelian groups.

2. Schur-pair and the Baer-invariant of groups

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and let $\mathcal{X}$ be a class of groups. Then $(\mathcal{V}, \mathcal{X})$ is said to be a *Schur-pair*, when G is any group with $\frac{G}{V^*(G)} \in \mathcal{X}$ it implies that $V(G) \in \mathcal{X}$.

In particular, if $\mathcal{X}$ is the class of all finite groups then the above property is known as a Hall's first conjecture (see [3]).

In this section we give some equivalent conditions that $(\mathcal{V}, \mathcal{X})$ has Schur-pair property if and only if, when G is in $\mathcal{X}$ then its Baer-invariant $\mathcal{V}M(G)$ is also in $\mathcal{X}$. For the class of finite groups, we have the remarkable theorem of C.R. Leedham-Green and S. McKay [7], which reads as follows:

Theorem 2.1([7; Theorem 1.17]) *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let $\mathcal{X}$ be the class of finite groups. Then the following conditions are equivalent:*

(a) $(\mathcal{V}, \mathcal{X})$ is a Schur-pair;

(b) For any finite group G, the order of the Baer-invariant of G, $|\mathcal{V}M(G)|$, divides a power of $|G|$.

Let $\mathcal{X}$ be an arbitrary class of groups, which is extension, quotient, and normal subgroup closed, i.e. $\mathcal{X} = PQS_n\mathcal{X}$. Then we are able to prove a result similar to Theorem 2.1 for the class $\mathcal{X}$, which is much larger than the class of finite groups.

Theorem 2.2 *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and let$\mathcal{X}$ be a class of groups with $\mathcal{X} = PQS_n\mathcal{X}$. Then the following conditions are equivalent:*

(a) $(\mathcal{V}, \mathcal{X})$ is a Schur-pair;

(b) If G is any group in $\mathcal{X}$, then so is $\mathcal{V}M(G)$.

Proof. Let $1 \to R \to F \to G \to 1$ be a free presentation of the group G. Then

$$1 \longrightarrow \frac{R}{[RV^*F]} \longrightarrow \frac{F}{[RV^*F]} \longrightarrow G \longrightarrow 1,$$

is a $\mathcal{V}$-marginal extension of G. Now if $(\mathcal{V}, \mathcal{X})$ is a Schur-pair and $G \in \mathcal{X}$, then using the property $\mathcal{X} = Q\mathcal{X}$ and $\frac{R}{[RV^*F]} \subseteq V^*(\frac{F}{[RV^*F]})$ we have

$$\frac{\frac{F}{[RV^*F]}}{V^*(\frac{F}{[RV^*F]})} \in \mathcal{X}.$$

Hence $\frac{V(F)}{[RV^*F]} \in \mathcal{X}$, and so $\mathcal{V}M(G) \in \mathcal{X}$.

Conversely, with the same notation, let E be a group with marginal factor group $\frac{E}{V^*(E)} \cong G$. By the assumption $G \in \mathcal{X}$ and hence

$$1 \longrightarrow \mathcal{V}M(G) \longrightarrow \frac{V(F)}{[RV^*F]} \longrightarrow V(G) \longrightarrow 1,$$

is a $\mathcal{V}$-marginal extension of $V(G) \in \mathcal{X}$, then $\frac{V(F)}{[RV^*F]}$ is also in $\mathcal{X}$. It is easily checked that $V(E)$ is a homomorphic image of $\frac{V(F)}{[RV^*F]}$. Therefore $V(E) \in \mathcal{X}$, i.e. $(\mathcal{V}, \mathcal{X})$ is a Schur-pair. ■

The following interesting corollary states that a group G in the above mentioned class of groups $\mathcal{X}$ has the same structure as its covering group, and its proof follows from the above theorem.

Corollary 2.3 *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and let $\mathcal{X}$ be a class of groups with $\mathcal{X} = PQS_n\mathcal{X}$. Let $(\mathcal{V}, \mathcal{X})$ be a Schur-pair and G^* be a $\mathcal{V}$ -covering group of G. Then $G \in \mathcal{X}$ if and only if $G^* \in \mathcal{X}$.*

Remark. J.A. Hulse and J.C. Lennox in [5] did introduce a generalized version of the Schur-pair property as follows: $(\mathcal{V}, \mathcal{X})$ is said to be an *ultra Schur-pair*, if for any group G with a normal subgroup N such that $\frac{N}{N \cap V^*(G)} \in \mathcal{X}$, it holds that $[NV^*G] \in \mathcal{X}$, see also [13] and [20].

Now, considering this notion we have been able to prove a result similar to Theorem 1.3.

3. Varietal Covering groups

This section is devoted to study the covering groups of a group G, with respect to a given variety of groups $\mathcal{V}$. One should note that, in general, groups might not possess $\mathcal{V}$-covering groups. In [19], we presented a class of groups lacking $\mathcal{V}$-covering groups, with respect to a certain given variety $\mathcal{V}$, (see also [7]).

However, I.Schur in [25] showed the existence of covering groups for finite groups and then M.R. Jones (see [26]) generalized it to every group in the variety of abelian groups. In [14] we have also shown that every group has a $\mathcal{V}$-covering group with respect to the variety of abelian groups of exponent m, when m is a square-free positive integer. Now, assuming the existence of a covering group of a given group G with respect to a variety $\mathcal{V}$, we are able to give the structure of such covering groups.

Theorem 3.1 *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and let G be a group with a free presentation $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$. Then*

(i) If S is a normal subgroup of F such that

$$\frac{R}{[RV^*F]} = \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]},$$

then $G^* = F/S$ is a $\mathcal{V}$-covering group of G.

(ii) Every $\mathcal{V}$-covering group of G is a homomorphic image of $\frac{F}{[RV^*F]}$.

(iii) For any $\mathcal{V}$-covering group G^* of the group G with a $\mathcal{V}$-stem cover

$$1 \longrightarrow A \longrightarrow G^* \longrightarrow G \longrightarrow 1,$$

there exists a normal subgroup S of F as in part (i), satisfying also $F/S \cong G^*$ and $R/S \cong A$.

Proof. (i) Put $A = R/S$. Then $G^*/A \cong F/R \cong G$ and $A \cong \mathcal{V}M(G)$. From the assumption we have $R \subseteq V(F)S$. Clearly

$$A = \frac{R}{S} \subseteq V^*(\frac{F}{S}) = V^*(G^*) \text{ and } A = \frac{R}{S} \subseteq \frac{V(F)S}{S} = V(\frac{F}{S}) = V(G^*).$$

Hence $G^* = F/S$ is a $\mathcal{V}$-covering group of G.

(ii) Let F be the free group freely generated by the set X and let $\pi : F \longrightarrow G$ be an epimorphism with $R = ker\pi$. Let G^* be a $\mathcal{V}$-covering group of G, with the $\mathcal{V}$-stem cover $1 \longrightarrow A \longrightarrow G^* \xrightarrow{\phi} G \longrightarrow 1$. Clearly for any $x \in X$, there exists $g_x \in G^*$ such that $\phi(g_x) = \pi(x)$. Now, we put $H =< g_x \in G^* \mid x \in X >$, hence $G^* = HA$. But using Lemma 1.2, $A \subseteq V^*(G^*) = V(H)$, so $G^* = H$. We consider the homomorphism $\psi : F \longrightarrow G^*$ given by $\psi(x) = g_x, x \in X$. Then ψ is onto and $\pi = \phi \circ \psi$. It is easily seen that $\psi(R) \subseteq A$, so

$$\psi([RV^*F]) \subseteq [\psi(R)V^*G^*] = 1.$$

Thus ψ induces a homomorphism $\bar{\psi}$ from $\frac{F}{[RV^*F]}$ onto G^*, which is the required assertion.

(iii) Let $a \in A$ and $a = \psi(x)$, for some $x \in F$. Then $1 = \phi(a) = \pi(x)$. So $x \in R$ and hence $A \subseteq \psi(R)$. One can easily see that $A = \psi(R)$. Now observe that

$$\psi(R \cap V(F)) \subseteq \psi(R) \cap \psi(V(F)) = A \cap V(G^*) = A.$$

To prove the converse, suppose that $z = \psi(x) = \psi(y)$, for some $x \in V(F)$ and $y \in R$, whence $x^{-1}y \in ker\psi$. Thus $\pi(x^{-1}y) = 1$ and $x^{-1}y \in R$. It follows that $x \in R$ and $z \in \psi(R \cap V(F))$, which shows that $A \subseteq \psi(R \cap V(F))$. Hence $A = \psi(R \cap V(F))$. Therefore $\bar{\psi}$ restricts to an isomorphism from $\frac{R\cap V(F)}{[RV^*F]}$ onto A. Let $S = ker\psi$. Then $\frac{S}{[RV^*F]}$ is the kernel of the restriction of $\bar{\psi}$ to $\frac{R}{[RV^*F]}$ and the image of this restriction is A. Thus one may conclude that

$$\frac{R}{[RV^*F]} = \frac{R\cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]}.$$

Now, part (i) implies that F/S is a $\mathcal{V}$-covering group of G. Let $\theta : F/S \longrightarrow G^*$ be the homomorphism induced by ψ. Using the fact that ψ is onto and $\psi(R) = A$, it follows that $\theta(R/S) = A$, which completes the proof. ■

In general, it is not true that any two covering groups of a given group G are isomorphic (see [6]). However using the above theorem we deduce that any tow covering groups are $\mathcal{V}$-isologic, which generalizes a theorem of Bioch and van der Waall [2].

Corollary 3.2 (see also [15]) *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and G be a group. Then all $\mathcal{V}$-covering groups of G are $\mathcal{V}$-isologic.*

In [16], by imposing some condition on homomorphisms we give a sufficient condition for tow $\mathcal{V}$-covering groups of a given group to be isomorphic. Also we deduce that all $\mathcal{V}$-covering groups of a group in $\mathcal{V}$ are Hopfian.

Covering groups have been studied for the abelian case, by several authors. See for instance [6], [10] or [26].

In the following we deal with the property of covering groups in an arbitrary variety of groups $\mathcal{V}$, which generalizes the work of E.W. Read [24], extensively.

Theorem 3.3 *Let $\mathcal{V}$ be a variety of groups, and let G_1 and G_2 be two $\mathcal{V}$-covering groups of a given group G. Let*

$$1 \longrightarrow A_i \longrightarrow G_i \longrightarrow G \longrightarrow 1 \quad , \; i = 1,2$$

be a $\mathcal{V}$-stem cover of G. Then

$$\frac{V^*(G_1)}{A_1} \cong \frac{V^*(G_2)}{A_2}.$$

Proof. Let $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ be a free presentation for the group G, and G^* be a fixed covering group of G with respect to a given variety $\mathcal{V}$. By the definition, there is an exact sequence $1 \longrightarrow A \longrightarrow G^* \longrightarrow G \longrightarrow 1$ such that $A \subseteq V^*(G^*) \cap V(G^*)$ and $A \cong \mathcal{V}M(G)$. To prove the result it suffices to show that isomorphism class of groups $\frac{V^*(G^*)}{A}$ are determined uniquely by the presentation $F/R \cong G$. Using Theorem 3.1, we may assume that $G^* \cong F/S$, $A \cong R/S$ for some normal subgroup S of F such that

$$\frac{R}{[RV^*F]} \cong \mathcal{V}M(G) \times \frac{S}{[RV^*F]}.$$

Put $V^*(\frac{F}{[RV^*F]}) = \frac{L}{[RV^*F]}$, then clearly $[LV^*F] \subseteq [RV^*F] \subseteq S$ and hence $L/S \subseteq V^*(F/S)$. Now, if $xS \in V^*(F/S)$ then for every $v \in V$ and $f_1, \ldots, f_r \in F$, we have

$$v(f_1, \ldots, f_i x, \ldots, f_r)v(f_1, \ldots, f_r)^{-1} \in S \cap V(F) = [RV^*F].$$

So $x[RV^*F] \in V^*(\frac{F}{[RV^*F]})$. This implies that $V^*(F/S) \subseteq L/S$. Thus $V^*(F/S) = L/S$. Hence $\frac{V^*(G^*)}{A} \cong L/R$. But the factor group L/R is only determined by the free presentation $F/R \cong G$, and hence the result follows. ■

Remark. In [7], Leedham-Green and McKay introduced the generalized version of the Baer-invariant of a group G with respect to two varieties of groups. We have proved, a result similsr to Theorem 3.3 in this generalized version (see [21] and [22] for more details).

4. Subvarieties of abelian groups

In this final section we consider $\mathcal{V}$-covering groups and $\mathcal{V}$-marginal extensions of a $\mathcal{V}$-perfect group with respect to a subvariety of abelian groups $\mathcal{V}$, say.

In [15], we have shown the following theorem, yielding the existence of $\mathcal{V}$-covering groups for $\mathcal{V}$-perfect groups with respect to an arbitrary variety of groups $\mathcal{V}$.

Theorem 4.1 *Let $\mathcal{V}$ be a variety of groups and let G be a $\mathcal{V}$-perfect group with a free presentation $G \cong F/R$. Then $\dfrac{V(F)}{[RV^*F]}$ is a $\mathcal{V}$-covering group of G.*

Theorem 4.1 is also generalized Theorem 2.1 of [11], in which the variety $\mathcal{V}$ was defined by the set of outer commutator words.

Now in the following main result (Theorem 4.3), it is shown that if $\mathcal{V}$ is a variety of groups contained in the abelian variety $\mathcal{A}$, say, then the $\mathcal{V}$-marginal extensions of a $\mathcal{V}$-perfect group G are homomorphic images of a $\mathcal{V}$-stem cover. Of course, we have also proved such a result in [14] for any group with respect to the variety of abelian groups of exponent m, where m is a square free positive integer.

The following lemma shortens the proof of the main theorem considerably, and its proof is straightforward.

Lemma 4.2 *Let $\mathcal{V}$ be a variety of groups and G be any group with a free presentation $1 \longrightarrow R \longrightarrow F \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$, and let $1 \longrightarrow A \longrightarrow H \longrightarrow \bar{G} \longrightarrow 1$ be a $\mathcal{V}$-marginal extension of another group $\bar{G}$. If $\alpha : G \longrightarrow \bar{G}$ is an isomorphism, then there exists an epimorphism β :*

$\frac{F}{[RV^*F]} \longrightarrow H$ *such that the following diagram commutes*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{R}{[RV^*F]} & \longrightarrow & \frac{F}{[RV^*F]} & \xrightarrow{\bar{\pi}} & G & \longrightarrow & 1 \\ & & \downarrow \beta_1 & & \downarrow \beta & & \downarrow \alpha & & \\ 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & \bar{G} & \longrightarrow & 1 \end{array}$$

where $\bar{\pi}$ is the natural homomorphism induced by π and β_1 is the restriction of β.

Theorem 4.3 *Let $\mathcal{V}$ be a variety contained in the variety of abelian groups and let*

$$1 \longrightarrow A \longrightarrow H \longrightarrow G \longrightarrow 1$$

be a $\mathcal{V}$-marginal extension of a $\mathcal{V}$-perfect group G. Then there exists a $\mathcal{V}$-covering group G^* of G such that H is a homomorphic image of G^*.

Proof. Let $1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$ be a free presentation of the group G. By Lemma 4.2, there exists an epimorphism $\beta : \frac{F}{[RV^*F]} \longrightarrow H$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{R}{[RV^*F]} & \longrightarrow & \frac{F}{[RV^*F]} & \xrightarrow{\bar{\pi}} & G & \longrightarrow & 1 \\ & & \downarrow \beta_1 & & \downarrow \beta & & || & & \\ 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where β_1 is the restriction of β. Put

$$ker\beta_1 = ker\beta = \frac{T}{[RV^*F]},$$

where T is a normal subgroup of R and $T(R \cap V(F)) = R$. Hence, since G is $\mathcal{V}$-perfect we have

$$\frac{T}{T \cap V(F)} \cong \frac{T(R \cap V(F))}{R \cap V(F)} = \frac{R}{R \cap V(F)} \cong \frac{RV(F)}{V(F)} \cong \frac{F}{V(F)}.$$

Thus the following exact sequence splits

$$1 \longrightarrow \frac{T \cap V(F)}{[RV^*F]} \longrightarrow \frac{T}{[RV^*F]} \longrightarrow \frac{T}{T \cap V(F)} \longrightarrow 1,$$

where $\frac{T}{[RV^*F]}$ is an abelian group, and hence

$$\frac{T}{[RV^*F]} \cong \frac{T \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]},$$

where $\frac{S}{[RV^*F]} = \frac{T}{T \cap V(F)}$. Now we have

$$S \cap (R \cap V(F)) = S \cap (T \cap V(F)) = [RV^*F],$$

and

$$\begin{aligned} S(R \cap V(F)) = S(T(R \cap V(F)) \cap V(F)) &= (R \cap V(F))(S(T \cap V(F))) \\ &= (R \cap V(F))T = R, \end{aligned}$$

which implies that

$$\frac{R}{[RV^*F]} = \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]}.$$

Hence by Theorem 2.1, F/S is a $\mathcal{V}$-covering group of G. Moreover

$$\frac{F/S}{T/S} \cong \frac{\frac{F}{[RV^*F]}}{\frac{T}{[RV^*F]}} \cong H,$$

which completes the proof. ∎

Now, we obtain the following corollary which is of interest in its own account.

Corollary 4.4 *Let $\mathcal{V}$ be a variety contained in the variety of abelian groups and let*
$(e) : 1 \longrightarrow A \longrightarrow H \longrightarrow G \longrightarrow 1$ *be a $\mathcal{V}$-marginal extension of a $\mathcal{V}$-perfect group G such that every other $\mathcal{V}$-marginal extension of G is a homomorphic image of the extension (e). Then (e) is a $\mathcal{V}$-stem cover of G.*

Proof. By Theorem 4.3, there exists a $\mathcal{V}$-stem cover $(e') : 1 \longrightarrow A_1 \longrightarrow G^* \longrightarrow G \longrightarrow 1$ and an epimorphism $\psi : G^* \longrightarrow H$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 & \longrightarrow & G^* & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \psi_1 & & \downarrow \psi & & \| & & \\ 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where ψ_1 is the restriction of ψ to A_1. Now by using [15, Theorem 3.4], we obtain that ψ is an isomorphism which gives the result. ∎

In the context of $\mathcal{V}$-perfect groups, we have proved several other results, for instance we have shown that any automorphism of a finite $\mathcal{V}$-perfect group can be lifted to an automorphism of its $\mathcal{V}$-covering group (see [15]). This result is a vast generalization of Alperin and Gorenstein [1].

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