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# P-AMENABLE LOCALLY COMPACT HYPERGROUPS

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Abstract. Let $K$ be a locally compact hypergroup with left Haar measure and let $P^1(K) = \{f \in L^1(K) : \ f \geq 0, \ \|f\|_1 = 1 \ \}$. Then $P^1(K)$ is a topological semigroup under the convolution product of $L^1(K)$ induced in $P^1(K)$. We say that $K$ is P-amenable if there exists a left invariant mean on $C(P^1(K))$, the space of all bounded continuous functions on $P^1(K)$. In this note, we consider the P-amenability of hypergroups. The P-amenability of hypergroup joins $K = H \vee J$ where $H$ is a compact hypergroup and $J$ is a discrete hypergroup with $H \cap J = \{e\}$ is characterized. It is also shown that Z-hypergroups are P-amenable if $Z(K) \cap G(K)$ is compact.

## 1. **Introduction**

Throughout this paper, we will consider hypergroups in the sense of [5], which will be referred to for basic definitions and results concerning hypergroups (see also [15] and [12]). We will follow the notations of [5] with the following exceptions:

(i) By $x \longmapsto \check{x}$ we denote the involution on the hypergroup $K$.

(ii) For $(x \in K)$, $\delta_x$ is the Dirac measure concentrated at $x$.

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(iii) $1_X$ will denote the characteristic function of the non-empty set $X$.

(iv) For $A \subset K$ and $x \in K$, let $A * x$ denote the subset $A * \{x\}$ in $K$.

(v) For a locally compact Hausdorff space $X$, $C_{00}(X)$ denotes the set of all continuous functions with compact support on $X$.

Furthermore all hypergroups occurring in this paper are supposed to admit a (left) Haar measure $\lambda$ (it is still unknown if an arbitrary hypergroup has a Haar measure but discrete, compact and commutative hypergroups possess a Haar measure [8]). In this case, one can define the convolution algebra $L^1(K)$ with multiplication

$$f * g(x) = \int_K f(x * y)\, g(\check{y})\, d\lambda(y) \qquad \text{(see [5, §5.5])}$$

for $f, g \in L^1(K)$. Let

$$P^1(K) = \{f \in L^1(K) : f \geq 0, \|f\|_1 = 1\}.$$

Then $P^1(K)$ is a topological semigroup (a semigroup with jointly continuous multiplication and Huasdorff topology) under the convolution product in $L^1(K)$ equipped with the norm topology.

For a topological semigroup $S$, let $l^\infty(S)$ denote the space of all bounded real-valued functions on $S$ with the sup norm. For $f \in S$ and $\theta \in l^\infty(S)$, let ${}_f\theta$ and $\theta_f$ denote, respectively, the *left* and the *right translation* of $\theta$ by $f$, i.e., ${}_f\theta(g) = \theta(fg)$ and $\theta_f(g) = \theta(gf)$, $g \in S$. Let $X$ be a closed subspace of $l^\infty(S)$ containing constants and invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1_S) = 1$; $m$ is called a *left invariant mean [right invariant mean]*, denoted by LIM [RIM] if $m({}_f\theta) = m(\theta) \quad [m(\theta_f) = m(\theta)]$ for all $f \in S, \quad \theta \in X$.

Let $C(S)$ denote the space of all bounded continuous functions on $S$ and let $UC_r(S)$ denote the space of *left uniformly continuous* functions on $S$, i.e., all $F \in C(S)$ so that the mapping $f \longmapsto {}_fF$ from $S$ into $C(S)$ is continuous when $C(S)$ has the sup norm topology. Then both Banach spaces $UC_r(S)$ and $C(S)$ are invariant under translations and contain the constant (see also [7] or [6]). We call $S$ *(left) amenable* if there exists a LIM on $C(S)$.

We say that a hypergroup $K$ is P-*amenable* if $C(P^1(K))$ has a LIM. In this paper we initiate a study of P-amenable hypergroups and generalize some of the results from [3]. Also among other things, we show that if $H$ is a strongly normal sub-hypergroup (for a definition see section 2) of $K$, then $K$ is P-amenable, which in turn implies that $K//H$ (see section 2) is P-amenable. For the properties of strongly normal and normal sub-hypergroups one can consult with [15]. In particular, we show that the hypergroup joins $K = H \vee J$ (see section 2) is P-amenable if and only if $J$ is P-amenable where $H$ is a compact hypergroup and $J$ is a discrete hypergroup with $H \cap J = \{e\}$ (see Corollary 2.12) and a central hypergroup (see 2.13) is P-amenable if $Z(K) \cap G(K)$ ( see 2.13) is compact (see 2.14).

## 2. **P-amenable hypergroups**

We start this section by recalling the definition of hypergroup joins which we use very often in this paper.

Let $H$ be a compact hypergroup and $J$ a discrete hypergroup with $H \cap J = \{e\}$, where $e$ is the identity of both hypergroups. Let $H \cup J$ have the unique topology for which both $H$ and $J$ are closed subspaces of $K$. Let $\sigma$ be the normalized Haar measure on $H$. Define the operation $\bullet$ on $K$ as follows:

(i) If $s, t \in H$, then $\delta_s \bullet \delta_t = \delta_s * \delta_t$;
(ii) If $a, b \in J$, $\quad a \neq \breve{b}$, then $\delta_a \bullet \delta_b = \delta_a * \delta_b$;
(iii) If $s \in H,\ a \in J\ (a \neq e)$, then $\delta_s \bullet \delta_a = \delta_a \bullet \delta_s = \delta_a$;
(iv) If $a \in J,\ a \neq e$, and $\delta_{\breve{a}} * \delta_a = \Sigma_{b \in J} c_b \delta_b$, where $c_b$'s are non-negative, only finitely many of them are non-zero and $\Sigma_{b \in J} c_b = 1$, then

$$\delta_{\breve{a}} \bullet \delta_a = c_e \sigma + \Sigma_{b \in J \setminus \{e\}} c_b \delta_b.$$

We call the hypergroup $K$ the *hypergroup joins* of $H$ and $J$, and write $K = H \vee J$. Observe that $H$ is a sub-hypergroup of $K$, but $J$ is not a sub-hypergroup unless $J$ or $H$ is equal to $\{e\}$. The hypergroup joins always has a left Haar measure [17, Proposition 1.1] and $K//H \cong J$ as hypergroups [17, Proposition 1.3]. Let $H$ be a compact sub-hypergroup of $K$ with the normalized Haar measure $\sigma$. As shown in [5, §14] the double coset space $K//H = \{H * x * H : x \in K\}$ is a hypergroup with convolution defined by

$$\int_{K//H} f \, d\delta_{H*x*H} * \delta_{H*y*H} = \int_K f \circ \pi \, d\delta_x * \sigma * \delta_y,$$

for all positive Borel measurable functions $f$ on $K//H$ where $\pi$ is the canonical projection of $K$ onto $K//H$. A Haar measure on $K//H$ is given by $\dot{\lambda} = \int_K \delta_{H*x*H}\, dx$. By [5, 14.2H] the Haar measure $\dot{\lambda}$ on $K//H$ can be so chosen that

$$\int_{K//H} \int_K f\, d\sigma * \delta_x * \sigma\, d\dot{\lambda}(\dot{x}) = \int_K f\, dx. \tag{2.1}$$

Let $T_H$ be the mapping defined by $T_H f(H * x * H) = \int_K f\, d\sigma * \delta_x * \sigma$ for $f \in L^1(K)$. Then as shown in [9, Theorem 2.4.(ii)], $T_H$ is a bounded linear map of $L^1(K)$ onto $L^1(K//H)$ with norm 1.

A compact sub-hypergroup $H$ of a locally compact hypergroup $K$ is called *strongly normal* if $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in K$ where $\sigma$ is the normalized Haar measure of $H$. In this case, we have $x * H = H * x = H * x * H$ for each $x \in K$ and (1) takes the form

$$\int_{K/H} \int_H f(x * \xi)\, d\sigma(\xi)\, d\dot{\lambda}(\dot{x}) = \int_K f(x)\, d\lambda(x).$$

Moreover, $T_H$ is an algebra homomorphism [9, Theorem 2.4(iv)]. For example, in the hypergroup joins $K = H \vee J$, where $H$ is a compact hypergroup and $J$ a discrete hypergroup with $H \cap J = \{e\}$, $H$ is strongly normal in $K$ [17, Proposition 1.2].

**Definition 2.1.** A hypergroup $K$ is called P-amenable if $P^1(K)$ as a topological semigroup is left amenable.

**Example 2.2.** (a) Every abelian hypergroup $K$ is P-amenable. Indeed if $K$ is abelian, then so is $P^1(K)$ and we know that every abelian topological semigroup is left amenable (see [16] and [2]).

(b) Every compact hypergroup $K$ is P-amenable. To see this, first note that if $K$ is compact then $1_K \in P^1(K)$. Let $m$ be defined on $l^\infty(P^1(K))$ by $m(F) = F(1_K)$, for all $F \in l^\infty(P^1(K))$. Then $m$ is a LIM on $l^\infty(P^1(K))$. Indeed, it is clear that $m$ is a mean on $l^\infty(P^1(K))$. Now by using the fact that $f * 1_K = 1_K$ for all $f \in P^1(K)$, one can see easily that $m$ is left invariant on $l^\infty(P^1(K))$. This implies that $K$ is P-amenable.

**Theorem 2.3.** *Let $H$ be a strongly normal sub-hypergroup of $K$. Then if $K$ is P-amenable, so is $K//H$.*

**Proof.** Let $m$ be a LIM on $C(P^1(K))$. Define $\bar{m} : C(P^1(K//H)) \to I\!R$ by $\bar{m}(F) = m(\bar{F}), \quad (F \in C(P^1(K//H)))$, where $\bar{F} : P^1(K) \to I\!R$ is defined by $\bar{F}(f) = F(T_H f)$ for $f \in P^1(K)$. Then clearly $\bar{F}$ is a bounded continuous function and $\bar{m}$ is a mean on $C(P^1(K//H))$. Indeed, it is easy to check that $\bar{m}$ is linear and positive. Furthermore, for $F = 1_{P^1(K//H)}$, one can see that $\bar{F} = 1_{P^1(K)}$ (see [4, Lemma 1.1 and Remark 1.5]), hence

$$\bar{m}(F) = m(\bar{F}) = 1.$$

If $g \in P^1(K//H)$, then there exists $f \in P^1(K)$ such that $T_H f = g$ (see [4, Lemma 1.1] and [9, Theorem 2.4]). Now observe that for $h \in P^1(K)$,

$$\begin{aligned}\bar{({}_gF)}(h) = {}_gF(T_H h) = F(g * T_H h) = F(T_H f * T_H h) = \\ F(T_H(f * h)) = \bar{F}(f * h) = {}_f(\bar{F})(h).\end{aligned}$$

So $\bar{({}_gF)} = {}_f(\bar{F})$. Hence

$$\langle \bar{m}, {}_gF\rangle = \langle m, \bar{({}_gF)}\rangle = \langle m, {}_f(\bar{F})\rangle = \langle m, \bar{F}\rangle = \langle \bar{m}, F\rangle.$$

Consequently $\bar{m}$ is a LIM on $C(P^1(K//H))$. □

**Definition 2.4.** Let $H$ be a sub-hypergroup of $K$. We say that $H$ is *supernormal* in $K$ if $\check{x} * H * x \subseteq H$. As an example, in hypergroup joins $K = H \vee G$, where $H$ is a compact hypergroup and $G$ is any discrete group with $H \cap G = \{e\}$, $H$ is supernormal.

Note that if $H$ is a supernormal sub-hypergroup in $K$, then it is also strongly normal but the converse is not true in general. In fact, $\{e\}$ ($e$ the identity element of $K$) is supernormal in $K$ if and only if $K$ is a group. Also when $H$ is a supernormal sub-hypergroup, we have $K//H = K/H$ [1, p. 549]. In this case $K/H$ is a group under the convolution

$$\int f \, d\delta_{x*H} * \delta_{y*H} = \int f \circ \pi \, d\delta_x * \delta_y = \int_H f \circ \pi(x * t * y) \, d\lambda(t),$$

for all $f \in C_{00}(K/H),\ x, y \in K$, where $\pi$ is the canonical map of $K$ onto $K/H$ (see [18, Theorem 2.1]).

**Corollary 2.5.** *Let $K$ be a P-amenable hypergroup and $H$ a compact supernormal sub-hypergroup in $K$. Then $K/H$ is P-amenable.*

**Proof.** This can be concluded from the fact that any super-normal sub-hypergroup is strongly normal, and an application of Theorem 2.3. □

**Definition 2.6.** A subgroup $N$ of a locally compact hypergroup $K$ is called *normal* if $xN = Nx$ for all $x \in K$. In this case $K/N$ is a hypergroup with convolution defined by $\int_{K/N} f\, d\delta_{xN} * \delta_{yN} = \int_N f \circ \pi d\delta_x * \delta_y$ for all $x, y \in K$ and $f \in C_{00}(K/N)$ (see [4, p 84]).

Note that any compact normal subgroup $N$ of a hypergroup $K$ is strongly normal because, $xN = Nx = NxN$, for all $x \in K$.

**Corollary 2.7.** *Let $N$ be a compact normal subgroup of a locally compact hypergroup $K$. If $K$ is P-amenable then so is $K/N$.*

In order to prove the next theorem, we first need to prove the following Lemma.

**Lemma 2.8.** Let $S$ be a topological semigroup and $I$ a non-void left ideal of $S$. If there exists a left invariant mean on $l^\infty(I)$ then there exists a left invariant mean on $l^\infty(S)$.

**Proof.** We choose $b \in I$ to be a fixed element. For $f \in l^\infty(S)$, let $f' = ({}_b f)|_I$; then $f' \in l^\infty(I)$. Since $I$ is a left ideal of $S$, for any $a \in S$, we have $ab \in I$. Also for $x \in I$ we establish the following:

$$({}_a f)'(x) = ({}_b({}_a f))(x) = f(abx) = {}_{(ab)}f(x).$$

Now for a left invariant mean $m$ on $l^\infty(I)$ we have

$$m(({}_a f)') = m({}_{(ab)}(f|_I)) = m(({}_b f)|_I) = m(f').$$

At this point we define $n(f) = m(f')$ for $f \in l^\infty(S)$. From this definition we conclude that $n$ is a mean on $l^\infty(S)$ and $n({}_a f) = n(f)$, for all $f \in l^\infty(S)$ and $a \in S$. $\square$

**Theorem 2.9.** *Let $H$ be a strongly normal compact sub-hypergroup in hypergroup $K$. If there exists a left invariant mean on $l^\infty(P^1(K/H))$, then there exists a left invariant mean on $l^\infty(P^1(K))$.*

**Proof.** Let $I$ denote the set of all elements $f$ in $P^1(K)$ which are constant on the cosets. Then $I$ is a left ideal in $P^1(K)$, i.e., $P^1(K) * I \subseteq I$. Indeed, for $f \in I$ and $g \in P^1(K)$ we have

$$g * f(x) = \int_K g(x * y) f(\check{y}) dy = \int_K g(x * h * y) f(\check{y} * \check{h}) dy =$$

$$= \int_K g(x * h * y) f(\check{y}) dy = g * f(x * h).$$

Then by Theorem 2.4 in [9], $I \neq \emptyset$. According to Lemma 2.8 it is enough to prove that there exists a left invariant mean on $l^\infty(I)$. Let $\theta$ be a real valued function on $I$. Define

$$\bar{\theta} : P^1(K/H) \to \mathbb{R}$$

by

$$\bar{\theta}(\bar{f}) = \theta(\bar{f} \circ \pi_H),$$

where $\pi_H : K \to K/H$ is the canonical map (see [5, §14.1]). Then by Theorem 2.4(i) in [9], $\bar{f} \circ \pi_H \in P^1(K)$ is constant on cosets and therefore it belongs to $I$. Hence, $\bar{\theta}$ is well defined. Let $\bar{m}$ be a LIM on $l^\infty(P^1(K/H))$ and define a linear functional $m$ on $l^\infty(I)$ by $\langle m, \theta \rangle = \langle \bar{m}, \bar{\theta} \rangle$. Then $m$ is a mean on $l^\infty(I)$. Note that if $f \in I$, then $f = \bar{f} \circ \pi_H$ for some $\bar{f} \in P^1(K/H)$ (by definition of $I$). Now, by Theorem 2.4(i(c)) in [9], we have $\overline{({}_f\theta)} = {}_{\bar{f}}\bar{\theta}$. In fact for any $\bar{g} \in P^1(K/H)$,

$$\overline{({}_f\theta)}(\bar{g}) = {}_f\theta(\bar{g} \circ \pi_H) = \theta(f * \bar{g} \circ \pi_H) = \theta(\bar{f} \circ \pi_H * \bar{g} \circ \pi_H) =$$
$$= \theta((\bar{f} * \bar{g}) \circ \pi_H) = \bar{\theta}(\bar{f} * \bar{g}) = {}_{\bar{f}}\bar{\theta}(\bar{g}).$$

Hence

$$\langle m, {}_f\theta \rangle = \langle \bar{m}, \overline{({}_f\theta)} \rangle = \langle \bar{m}, {}_{\bar{f}}\bar{\theta} \rangle = \langle \bar{m}, \bar{\theta} \rangle = \langle m, \theta \rangle,$$

for any $f \in I$, i.e., $m$ is a LIM on $l^\infty(I)$. □

**Remark 2.10.** Since a compact normal subgroup of a hypergroup is strongly normal (see Definition 2.6) and more generally each compact normal sub-hypergroup is strongly normal, the statement in Theorem 2.9 holds for normal subgroups and compact normal sub-hypergroups (see also Lemma 1.5 in [15]).

The following example shows that we can not remove the condition 'strongly normal' in Theorem 2.9.

**Example 2.11.** Let $SL(2, \mathbb{R})$ be the locally compact group (with the usual topology) of $2 \times 2$ matrices with determinant 1 and let $SO(2)$ be the compact subgroup of unitary matrices in $SL(2, \mathbb{R})$. Then $SL(2, \mathbb{R})$ contains the discrete (closed) free subgroup $F_2$ on two generators [10, Corollary 14.6], so by corollary 2.3 in [3], it is not P-amenable. But the hypergroup $SL(2,\mathbb{R})/SO(2)$ is commutative [5, 15.5] and hence P-amenable.

The following Corollary shows that there are non-compact, non-abelian P-amenable hypergroups.

**Corollary 2.12.** Let $K = H \vee J$ be a hypergroup joins, where $H$ is a compact hypergroup and $J$ a discrete hypergroup with $H \cap J = \{e\}$. Then $K$ is P-amenable if and only if $J$ is P-amenable.

**Proof.** We know that $K//H \cong J$ (see [17, Proposition 1.3]) and $H$ is strongly normal compact sub-hypergroup of $K$ [17, Proposition 1.2]. Now by Theorems 2.3 and 2.9, we are done. $\square$

**Definition 2.13.** Let $K$ be a hypergroup, and let $Z(K) = \{x \in K : \delta_y * \delta_x = \delta_x * \delta_y$ for each $y \in K\}$. Then $K$ is called a *central hypergroup* or *Z-hypergroup* if $K/(Z(K) \cap G(K))$ is compact where $G(K) = \{x \in K : \delta_x * \delta_{\check{x}} = \delta_e\}$ is the maximal subgroup of $K$ [4]. Central hypergroups admit left Haar measures and are unimodular (see [4, p. 93] and [11, §4]).

**Corollary 2.14.** Any central hypergroup $K$ (Z-hypergroup) is P-amenable if $Z(K) \cap G(K)$ is compact.

**Proof.** It follows from Example 2.2 (b) and Theorem 2.9. $\square$

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