*Bulletin of the Iranian Mathematical Society Vol.* 34 *No.* 2 *(2008), pp 71-96.*

# DIFFERENTIAL POLYNOMIAL RINGS OF TRIANGULAR MATRIX RINGS

H. GHAHRAMANI AND A. MOUSSAVI*

ABSTRACT. Let $R, S$ be rings with identity and $M$ be a unitary $(R, S)$-bimodule. We characterize homomorphisms and derivations of the generalized matrix ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, and provide a triangular representation of the differential polynomial ring $T[\theta; d]$.

## 1. Introduction

Throughout the paper all rings are assumed to have identity and all modules are unitary. The additive map $\delta : R \to R$ is called a *derivation*, if for each $a, b \in R$, $\delta(ab) = a\delta(b) + \delta(a)b$. For an element $x \in R$, the mapping $I_x : R \to R$, given by $I_x(a) = ax - xa$, for each $a \in R$, is called an *inner derivation* of $R$.

We denote $R[\theta; \delta]$ to be the differential polynomial ring whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication is subject to the relation $\theta a = a\theta + \delta(a)$ for any $a \in R$.

Derivations of the algebra of triangular matrices and some class of their subalgebras have been the object of active research for a long time [1, 4, 5, 7-9]. Coelho and Milies provided in [4] a description of the derivations in $T_n(R)$, the upper triangular matrices over $R$. They proved

MSC(2000): Primary: 16S36 ; Secondary: 16E50
Keywords: Derivation, homomorphism, differential polynomial ring, triangular matrix ring
Received:15 August 2007, Accepted:16 May 2008
∗Corresponding author

that every derivation is the sum of an inner derivation and another one induced from $R$. Jondrup in [7] gave a new proof of this result. A similar result for full matrix rings appears in [5], and the special case where $R$ is an algebra over a field, with $\mathrm{char}(R) \neq 2, 3$ and $n > 2$, is given in [1]. The case of upper triangular matrix rings over a simple algebra finite dimensional over its center appears in [5].

Here we give a description of homomorphisms and derivations of generalized matrix rings $T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ assuming no restrictions on $R, S$ and $M$, other than the existence of the identity element. We shall show that they are obtained in a very natural way. Analysts have studied these derivations in the context of algebras on certain normed spaces. Many widely studied algebras, including upper triangular matrix algebras, nest algebras and triangular Banach algebras, may be viewed as triangular algebras.

A large class of ring extensions which have a generalized triangular matrix representations is investigated by Birkenmeier et al. in [3].

Let $\delta_R : R \to R$ and $\delta_S : S \to S$ be derivations. The additive mapping $\tau : M \to M$ is called a *generalized derivation* with respect to $(\delta_R, \delta_S)$, on $M$, if $\tau(rm) = \delta_R(r)m + r\tau(m)$, $\tau(ms) = \tau(m)s + m\delta_S(s)$, for each $r \in R$,$s \in S$ and $m \in M$.

If $d : T \to T$ is the derivation induced by the generalized derivation $\tau$ with respect to $(\delta_R, \delta_S)$.i.e,

$$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix}, \text{ for each } r \in R, s \in S \text{ and } m \in M,$$

then we provide a triangular representation of the differential polynomial ring $T[\theta; d]$ in terms of the triangular matrix ring. Indeed, we prove the isomorphism:

$$T[\theta; d] \cong \begin{pmatrix} R[x; \delta_R] & M[x, y; \tau] \\ 0 & S[y; \delta_S] \end{pmatrix},$$

where $R[x; \delta_R]$ and $S[y; \delta_S]$ are the differential polynomial rings over $R$ and $S$, and $M[x, y; \tau]$ is an $(R[x; \delta_R], S[y; \delta_S])$-bimodule.

We denote $E_{ij}$ for the matrix units, for all $i, j$.

## 2. Generalized module homomorphisms

In order to describe homomorphisms of the generalized matrix rings, first we introduce and study the notion of generalized module homomorphisms.

**Definition 2.1.** Let $R$, $R', S$ and $S'$ be rings, $M$ an $(R, S)$-bimodule, $N$ an $(R', S')$-bimodule, $\varphi_1 : R \to R'$ and $\varphi_2 : S \to S'$ be ring homomorphisms. Then an additive mapping $T : M \to N$ is called a *generalized module homomorphism related to* $(\varphi_1, \varphi_2)$, if $T(rm) = \varphi_1(r)T(m)$, $T(ms) = T(m)\varphi_2(s)$, for each $r \in R, s \in S$ and $m \in M$.

**Lemma 2.2.** *Let $M$ be an $(R, S)$-bimodule, $N$ be an $(R', S')$-bimodule and*
*$T : M \to N$ be a generalized module homomorphism related to $(\varphi_1, \varphi_2)$. Then, the mapping $\Psi : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} R' & N \\ 0 & S' \end{pmatrix}$, given by $\Psi \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \varphi_1(r) & T(m) \\ 0 & \varphi_2(s) \end{pmatrix}$, is a ring homomorphism.*

**Proof.** Clearly $\Psi$ is additive. We have,

$$\Psi[\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}]$$

$$= \begin{pmatrix} \varphi_1(r)\varphi_2(r') & \varphi_1(r)T(m') + T(m)\varphi_2(s') \\ 0 & \varphi_2(s)\varphi_2(s') \end{pmatrix}$$

$$= \Psi \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \Psi \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}. \qquad \square$$

**Theorem 2.3.** *Let $R$, $R', S$ and $S'$ be rings with identity, $M$ be a unitary $(R, S)$-bimodule and $N$ be a unitary $(R', S')$-bimodule. If $\Psi$ : $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$*
*$\to \begin{pmatrix} R' & N \\ 0 & S' \end{pmatrix}$ is a mapping, then the followings are equivalent:*

*I. $\Psi \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \varphi_1(r) & T(m) \\ 0 & \varphi_2(r) \end{pmatrix}$, where $\varphi_1 : R \to R'$ and $\varphi_2$ : $S \to S'$ are ring homomorphisms and $T : M \to N$ is a generalized module homomorphism related to $(\varphi_1, \varphi_2)$.*

*II*. $\Psi$ *is a ring homomorphism such that* $\Psi(RE_{11}) \subseteq R'E_{11}$, *and* $\Psi(SE_{22}) \subseteq S'E_{22}$.

**Proof.** $(I \Rightarrow II)$. The proof clearly follows from Lemma 2.2.

$(II \Rightarrow I)$. The mappings $\varphi_1 : R \to R'$ and $\varphi_2 : S \to S'$ are defined by

$\Psi(rE_{11}) = \varphi_1(r)E_{11}$ and $\Psi(sE_{22}) = \varphi_2(s)E_{22}$. By considering the effect of $\Psi$ on $\begin{pmatrix} r+r' & 0 \\ 0 & s+s' \end{pmatrix}$ we see that $\varphi_1, \varphi_2$ are additive, and

$\Psi\begin{pmatrix} rr' & 0 \\ 0 & ss' \end{pmatrix} = \Psi\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\Psi\begin{pmatrix} r' & 0 \\ 0 & s' \end{pmatrix}$. So we have,
$\begin{pmatrix} \varphi_1(rr') & 0 \\ 0 & \varphi_2(ss') \end{pmatrix} = \begin{pmatrix} \varphi_1(r)\varphi_1(r') & 0 \\ 0 & \varphi_2(s)\varphi_2(s') \end{pmatrix}$.
Hence, we have,

$\varphi_1(rr') = \varphi_1(r)\varphi_1(r')$ and $\varphi_2(ss') = \varphi_2(s)\varphi_2(s')$, and $\varphi_1, \varphi_2$ are ring homomorphisms.
Now, assume that $\Psi(mE_{12}) = \begin{pmatrix} \alpha(m) & T(m) \\ 0 & \beta(m) \end{pmatrix}$, for some $\alpha : M \to R'$, $T : M \to N$ and $\beta : M \to S'$. Then, for each $m \in M$, we have,
$\Psi(mE_{12}) = \Psi(E_{11}mE_{12}) = \varphi_1(1)E_{11}\begin{pmatrix} \alpha(m) & T(m) \\ 0 & \beta(m) \end{pmatrix}$. So,
$\begin{pmatrix} \alpha(m) & T(m) \\ 0 & \beta(m) \end{pmatrix} = \begin{pmatrix} \varphi_1(1)\alpha(m) & \varphi_1(1)T(m) \\ 0 & 0 \end{pmatrix}$
and hence
$\beta(m) = 0$. So, $\Psi(mE_{12}) = \Psi(mE_{12}E_{22}) = \begin{pmatrix} \alpha(m) & T(m) \\ 0 & \beta(m) \end{pmatrix}\varphi_2(1)E_{22}$.
Thus, we have,

$$\begin{pmatrix} \alpha(m) & T(m) \\ 0 & \beta(m) \end{pmatrix} = \begin{pmatrix} 0 & T(m)\varphi_2(1) \\ 0 & \beta(m)\varphi_2(1) \end{pmatrix},$$
and so $\alpha(m) = 0$, for each $m \in M$.

Therefore, $\Psi(mE_{12}) = T(m)E_{12}$. We have $\Psi(rmE_{12}) = \Psi(rE_{11})\Psi(mE_{12})$, and hence $T(rm)E_{12} = \varphi_1(r)T(m)E_{12}$. Thus, $T(rm) = \varphi_1(r)T(m)$. Similarly,
$T(ms) = T(m)\varphi_2(s)$.
Therefore, we have,

$\Psi\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \varphi_1(r) & T(m) \\ 0 & \varphi_2(s) \end{pmatrix}$,
and that $\varphi_1, \varphi_2$ and $T$ satisfy the required conditions. □

**Proposition 2.4.** *If* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *and* $\begin{pmatrix} R' & N \\ 0 & S' \end{pmatrix}$ *have the identity elements and* $\Psi : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} R' & N \\ 0 & S' \end{pmatrix}$ *is a ring homomorphism such that*
$\Psi(E_{11}) = E_{11}$ and $\Psi(E_{22}) = E_{22}$, *then* $\Psi$ *satisfies the conditions I and II of Theorem* 2.3.

**Proof.** Let $\Psi(rE_{11}) = \begin{pmatrix} \alpha(r) & \beta(r) \\ 0 & \gamma(r) \end{pmatrix}$ for some $\alpha : R \to R'$, $\beta : R \to N$ and $\gamma : R \to S'$. We have, $\Psi(rE_{11}) = \Psi(rE_{11})\Psi(E_{11})$. So
$\begin{pmatrix} \alpha(r) & \beta(r) \\ 0 & \gamma(r) \end{pmatrix} = \begin{pmatrix} \alpha(r) & \beta(r) \\ 0 & \gamma(r) \end{pmatrix} E_{11} = \alpha(r)E_{11}$
and hence $\beta(r) = 0, \gamma(r) = 0$.

So, $\Psi(rE_{11}) = \alpha(r)E_{11}$, and $R\Psi(E_{11}) \subseteq R'E_{11}$. We have,
$\Psi(sE_{22}) = \begin{pmatrix} \alpha'(s) & \beta'(s) \\ 0 & \gamma'(s) \end{pmatrix}$,
where $\alpha' : S \to R', \beta' : S \to N$ and $\gamma' : S \to S'$ are additive mappings. But, $\Psi(sE_{22}) = \Psi(E_{22})\Psi(sE_{22})$, and

$\begin{pmatrix} \alpha'(s) & \beta'(s) \\ 0 & \gamma'(s) \end{pmatrix} = E_{22}\begin{pmatrix} \alpha'(s) & \beta'(s) \\ 0 & \gamma'(s) \end{pmatrix} = \gamma'(s)E_{22}$, so $\alpha'(s) = 0$, $\beta'(s) = 0$.

Thus, $\Psi(sE_{22}) = \gamma'(s)E_{22}$, and hence $\Psi(SE_{22}) \subseteq S'E_{22}$. Therefore, $\Psi$ satisfies the condition $II$ of Theorem 2.3. □

**Example 2.5.** The converse of Proposition 2.4 is not true, in general. Let $M$ be a unitary $(R, S)$-bimodule. Then, we make $M$ a unitary $R \times S$-bimodule by defining $(r, s)m := rm, m(r, s) := ms$, for each $r \in R, m \in M$ and $s \in S$.
Define $\varphi_1 : R \to R \times S$ and $\varphi_2 : S \to R \times S$, given by $\varphi_1(r) = (r, 0)$ and $\varphi_2(s) = (0, s)$, for each $r \in R, s \in S$. Then, $\varphi_1$ and $\varphi_2$ are ring homomorphisms. Let $T \in Hom({}_RM_S, {}_RM_S)$. Now we see that $T$ is a generalized module homomorphism related to $(\varphi_1, \varphi_2)$. Since $T(rm) = rT(m)$, then

we have $T(rm) = (r, 0)T(m) = \varphi_1(r)T(m)$ and $T(ms) = T(m)s$, and so $T(ms) = T(m)(0, s) = T(m)\varphi_2(s)$. Thus, the mapping
$\Psi : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} R \times S & M \\ 0 & R \times S \end{pmatrix}$,
given by
$\Psi \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \varphi_1(r) & T(m) \\ 0 & \varphi_2(s) \end{pmatrix}$,
is a ring homomorphism and we have,

$\Psi(E_{11}) = \varphi_1(1)E_{11} = (1, 0)E_{11}$, $\Psi(E_{22}) = (0, 1)E_{22}$. Note that $(1, 0)$ and $(0, 1)$ are not the identity elements of $R \times S$. □

**Lemma 2.6.** *Let $R, R', S$ and $S'$ be rings, $M$ be an $(R, S)$-bimodule, $N$ be an $(R', S')$-bimodule, and $\varphi_1 : R \to R', \varphi_2 : S \to S'$ be ring isomorphisms. Let $T : M \to N$ be a bijective generalized homomorphism related to $(\varphi_1, \varphi_2)$. Then, the mapping defined in Lemma 2.2 is a ring isomorphism.*

**Proof.** By Lemma 2.2, $\Psi$ is a ring homomorphism. We have,
$\Psi \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = 0$,
and so $\varphi_1(r) = 0, T(m) = 0, \varphi_2(m) = 0$. So $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = 0$, and hence $\Psi$ is injective. If $\begin{pmatrix} r' & n \\ 0 & s' \end{pmatrix} \in \begin{pmatrix} R' & N \\ 0 & S' \end{pmatrix}$ and $\varphi_1, \varphi_2, T$ are surjective, then there exist $r \in R, s \in S$ and $m \in M$, such that $\varphi_1(r) = r', \varphi_2(s) = s'$ and $T(m) = n$. So, we have,
$\Psi \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \varphi_1(r) & T(m) \\ 0 & \varphi_2(s) \end{pmatrix} = \begin{pmatrix} r' & n \\ 0 & s' \end{pmatrix}$.
Therefore, $\Psi$ is surjective and hence a ring isomorphism. □

The mapping $T$ in Lemma 2.6 is called a generalized module isomorphism related to $(\varphi_1, \varphi_2)$.

## 3. Derivations on generalized triangular matrix rings

Let $R, S$ be rings with identity and $M$ be an $(R, S)$-bimodule. In this section we determine the derivations of the generalized triangular matrix ring

$T =: \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$,
in terms of derivations of $R$ and $S$ and some special mapping of $M$.

**Definition 3.1.** Let $R, S$ be rings, $M$ be an $(R, S)$-bimodule, $\delta_R : R \to R$ and $\delta_S : S \to S$ be derivations. The additive mapping $\tau : M \to M$ is called a generalized derivation with respect to $(\delta_R, \delta_S)$, on $M$, if $\tau(rm) = \delta_R(r)m + r\tau(m)$ and $\tau(ms) = \tau(m)s + m\delta_S(s)$, for each $r \in R$,$s \in S$ and $m \in M$.

**Theorem 3.2.** *If $d : T \to T$ is a derivation, then $d = \bar{d} + I_A$,* where $I_A$ *is an inner derivation with $A \in T$* and $\bar{d}$, *is given by*

$$\bar{d}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix},$$

*for derivations $\delta_R : R \to R, \delta_S : S \to S$ and a generalized derivation $\tau : M \to M$.*

**Proof.** It is enough to determine $d$ on $rE_{11}, mE_{12}$ and $sE_{22}$, for each $r \in R$, $s \in S$ and $m \in M$. Then, we have,
$d(E_{11}) = d(E_{11}^2) = E_{11}d(E_{11}) + d(E_{11})E_{11}$. $\quad (*)$
Let $d(E_{11}) = \begin{pmatrix} r & b \\ 0 & s \end{pmatrix}$, for some $r \in R, s \in S$ and $b \in M$. From $(*)$, we have,

$$\begin{pmatrix} r & b \\ 0 & s \end{pmatrix} = E_{11}\begin{pmatrix} r & b \\ 0 & s \end{pmatrix} + \begin{pmatrix} r & b \\ 0 & s \end{pmatrix}E_{11} = \begin{pmatrix} 2r & b \\ 0 & 0 \end{pmatrix}.$$

So we have $r = 0$ and $s = 0$, and hence $d(E_{11}) = bE_{12}$. But,

$d(E_{11}) + d(E_{22}) = d(I) = 0$, so $d(E_{22}) = -bE_{12}$. Now we have,
$d(mE_{12}) = d(E_{11}mE_{12}) = E_{11}d(mE_{12}) + d(E_{11})mE_{12}$. $\quad (**)$
Assume that
$d(mE_{12}) = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}$,
for some $y_1 \in R$, $y_3 \in S$ and $y_2 \in M$. From $(**)$,
we have,

$$\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} = E_{11}\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} + bE_{12}mE_{12} = \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}. \text{ So } y_3 = 0.$$

We also have,

$d(mE_{12}) = d(mE_{12}E_{22}) = mE_{12}d(E_{22}) + d(mE_{12})E_{22}$.
So,

$$\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} = \begin{pmatrix} 0 & y_2 \\ 0 & y_3 \end{pmatrix}.$$

Thus, we have $y_1 = 0$ and that $d(mE_{12}) = y_2E_{12} = \tau(m)E_{12}$. So, $\tau : M \to M$ is a mapping.
To determine $d(rE_{11})$ for each $r \in R$, assume $d(rE_{12}) = \begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix}$, with $z_1 \in R$, $z_3 \in S$ and $z_2 \in M$. We have,

$rE_{11} = E_{11}rE_{11} = rE_{11}E_{11}$, so $d(rE_{11}) = d(rE_{11})E_{11} + rE_{11}d(E_{11})$, for each $r \in R$. So,

$$\begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix} = \begin{pmatrix} z_1 & rb \\ 0 & 0 \end{pmatrix}.$$

Hence $z_2 = rb$ and $z_3 = 0$.

Now define $\delta_R : R \to R$ given by $\delta_R(r) = z_1$. Then we have,

$$d(rE_{11}) = \begin{pmatrix} \delta_R(r) & rb \\ 0 & 0 \end{pmatrix}.$$

Now we determine $d(sE_{22})$ for each $s \in S$. Assume that $d(sE_{22}) = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}$, with $w_1 \in R$, $w_3 \in S$ and $w_2 \in M$. We have, $d(sE_{22}) = d(E_{22}sE_{22}) = d(E_{22})sE_{22}+E_{22}d(sE_{22})$. So, $\begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix} = \begin{pmatrix} 0 & -bs \\ 0 & w_3 \end{pmatrix}$, and hence $w_2 = -bs$ and $w_1 = 0$. Now we define $\delta_S : S \to S$ given by $\delta_S(s) = w_3$. So, $d(sE_{22}) = \begin{pmatrix} 0 & -bs \\ 0 & \delta_S(s) \end{pmatrix}$. Now by the above computations we get,

$$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & rb \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tau(m) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -bs \\ 0 & \delta_S(s) \end{pmatrix}$$
$$= \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix} + I_A\begin{pmatrix} r & m \\ 0 & s \end{pmatrix},$$

where $A = bE_{12} = d(E_{11})$.

Now we show that $\delta_R$ and $\delta_S$ are derivations of $R$ and $S$ respectively and $\tau$ is a generalized $(\delta_R, \delta_S)$-derivation.
For each $r, r' \in R$, we have $d((r + r')E_{11}) = d(rE_{11}) + d(r'E_{11})$. So $\delta_R(r + r') = \delta_R(r) + \delta_R(r')$.

Now, $d(rr'E_{11}) = \begin{pmatrix} \delta_R(rr') & rr'b \\ 0 & 0 \end{pmatrix}$, and
$d(rr'E_{11}) = d(rE_{11}r'E_{11}) = d(rE_{11})r'E_{11} + rE_{11}d(r'E_{11})$
$= \begin{pmatrix} \delta_R(r)r' + r\delta_R(r') & rr'b \\ 0 & 0 \end{pmatrix}$.
Thus, we have $\delta_R(rr') = \delta_R(r)r' + r\delta_R(r')$ for each $r, r' \in R$, and hence $\delta_R$ is a derivation of $R$. Similarly, $\delta_S$ is a derivation of $S$.
Next, we have that $\tau$ is an additive mapping of $M$. We have,
$d(rmE_{12}) = d(rE_{11}mE_{12}) = d(rE_{11})mE_{12} + rE_{11}d(mE_{12})$. So,

$$\begin{pmatrix} 0 & \tau(rm) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_R(r)m + r\tau(m) \\ 0 & 0 \end{pmatrix}.$$

Thus, for each $r \in R$ and $m \in M$, $\tau(rm) = r\tau(m) + \delta_R(r)m$.
Similarly, $\tau(ms) = \tau(m)s + m\delta_S(s)$, for each $s \in S$ and $m \in M$. Therefore, $\tau$ is a generalized derivation, and $d = \bar{d} + I_A$, where

$$\bar{d}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix}. \qquad \square$$

Now, we show that for every given derivations $\delta_R : R \to R$ and $\delta_S : S \to S$, every generalized derivation $\tau : M \to M$ with respect to $(\delta_R, \delta_S)$, induces a derivation $\bar{d}$ on the formal triangular matrix ring $T$.

**Proposition 3.3.** *Let $M$ be a unitary $(R, S)$-bimodule. If $d$ is a mapping of $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, then the followings are equivalent:*

I. $d = \begin{pmatrix} \delta_R & \tau \\ 0 & \delta_S \end{pmatrix}$, *where $\delta_R : R \to R$, $\delta_S : S \to S$ are derivations and $\tau : M \to M$ is a generalized derivation related to $(\delta_R, \delta_S)$.*

II. *$d$ is a derivation of $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ such that $d(RE_{11}) \subseteq RE_{11}$.*

III. *$d$ is a derivation of $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ such that $d(E_{11}) = 0$.*

IV. *d is a derivation of* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *such that* $d(SE_{22}) \subseteq SE_{22}$.

V. *d is a derivation of* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *such that* $d(E_{22}) = 0$.

**Proof.** (I$\Rightarrow$ II). Since $\delta_R, \delta_S$, and $\tau$ are additive, then $d$ is also additive. So, we have,

$$d[\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}] =$$

$$\begin{pmatrix} r\delta_R(r') + \delta_R(r)r' & r\tau(m') + \delta_R(r)m' + \tau(m)s' + m\delta_S(s') \\ 0 & s\delta_S(s') + \delta_S(s)s' \end{pmatrix} =$$

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} d\begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} + d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}.$$

So, $d$ is a derivation. We have $d(rE_{11}) = \delta_R(r)E_{11}$ for each $r \in R$, and so $d(RE_{11}) \subseteq RE_{11}$.

(II$\Rightarrow$ III). By Theorem 3.2, we have,
$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix} + I_{bE_{12}}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ with $b \in M$. We have, $d(E_{11}) = bE_{12}$. Since $d(RE_{11}) \subseteq RE_{11}$, then $b = 0$. So, $d(E_{11}) = 0$.

(III $\Rightarrow$ IV). We have,
$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix} + (rb - bs)E_{12}, d(E_{11}) = 0$. So, $b = 0$.
We have $d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix}$,
$d(SE_{22}) = \delta_S(s)E_{22}$, and so $d(SE_{22}) \subseteq SE_{22}$.

(IV$\Rightarrow$ V). It is similar to (II$\Rightarrow$ III).

(IV$\Rightarrow$ I). We have,
$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix} + (rb - bs)E_{12}$, and so $d(E_{22}) = -bE_{12}$.
Thus, $b = 0$, and hence $d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix}$.

$\square$

By the above result we see that any generalized derivation $\tau$ induces a derivation $\bar{d}$ on $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, which satisfies one of the above equivalent conditions; and every derivation on $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, is a sum of an inner derivation and a derivation $\bar{d}$ induced by $\tau$.

**Proposition 3.4.** *Let $M$ be a unitary $(R,S)$-bimodule. If $d$ is a mapping of $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, then the followings are equivalent:*

I. $d = I_{bE_{12}}$, *where* $0 \neq b \in M$.

II. *$d$ is a nonzero derivation of* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ , $d\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \subseteq ME_{12}$ *and* $d(mE_{12}) = 0$, *for each* $m \in M$.

**Proof.** (I$\Rightarrow$ II). It is clear. (II$\Rightarrow$ I). We have,

$$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix} + (rb - bs)E_{12}, \text{ with } b \in M.$$

We see that $\delta_R = 0$, since otherwise, if for some $r \in R$, $\delta_R(r) \neq 0$, then $d(rE_{11}) = \delta_R(r)E_{12} + rbE_{12} \notin ME_{12}$, which contradicts the assumption.

Similarly, $\delta_S = 0$. We also have $\tau = 0$, since otherwise, if for some $m \neq 0$, $\tau(m) \neq 0$, then $d(mE_{12}) = \tau(m)E_{12} \neq 0$. So,

$$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = (rb - bs)E_{12} = I_{bE_{12}}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}.$$

Since $d \neq 0$, we have $b \neq 0$. $\square$

By the following example we can not weaken the condition II in Proposition 3.4.

**Example 3.5.** If $0 \neq T \in Hom({}_RM_S, {}_RM_S)$, then $T$ is a generalized $(I_0, I_0)$-derivation. Since $T(rm) = rT(m) = rT(m) + I_0(r)m$, $T(ms) = T(m)s = T(m)s + mI_0(s)$, and the mapping $\Delta$ on $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, given by

$$\Delta\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & T(m) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & rb - bs \\ 0 & 0 \end{pmatrix}$$

is a derivation. So, we have, $\Delta(E_{11}) = bE_{12} \neq 0$ and $\Delta m(E_{12}) = \tau(m)E_{12}$.
But, $I_{dE_{12}}mE_{12} = 0$, with $m \in M$. So, this contradicts the fact that $\Delta$ is of the form of $I_{dE_{12}}$.
Let $\delta_S : S \to S$ be a nonzero derivation with $\delta_S(S) \subseteq ann_S M$. Then, the mapping $\Delta : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, given by
$\Delta \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \delta_S(s)E_{22} + I_{bE_{12}} \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$, $b \neq 0$, is a derivation.
We have $\Delta(E_{11}) = bE_{12} \neq 0$, and $\Delta(RE_{11}) \subseteq ME_{12}, \Delta(mE_{12}) = 0$. But, $\Delta \neq I_{dE_{12}}$, since $\Delta(sE_{22}) = \begin{pmatrix} 0 & -bs \\ 0 & \delta_S(s) \end{pmatrix}$. We also have $I_{dE_{12}}(sE_{22}) = -dsE_{12}$, as $\Delta SE_{22} \not\subseteq ME_{12}$ and $I_{dE_{12}}(SE_{22}) \subseteq ME_{12}$. $\square$

Let $a \in R$ and $b \in S$ be fixed elements. Define the mapping $\tau_{(a,b)} : M \to M$ given by $\tau_{(a,b)}(m) = am - mb$ for each $m \in M$. Then, $\tau_{(a,b)}$ is a generalized derivation with respect to $(I_{-a}, I_{-b})$, on $M$, where $I_{-a}, I_{-b}$ are the inner derivations.

For each $r \in R$ and $m \in M$, we have,
$\tau_{(a,b)}(rm) = arm - rmb = arm + ram - ram - rmb =$
$(ar - ra)m + r(am - mb) = I_{-a}(r)m + r\tau_{(a,b)}(m)$.

Similarly, we have $\tau_{(a,b)}(ms) = \tau_{(a,b)}(m)s + mI_{-b}(s)$, for each $s \in S$ and $m \in M$.

We call $\tau_{(a,b)}$ *the generalized inner derivation* on $M$.

**Lemma 3.6.** *The induced derivation of $\tau_{(a,b)}$ on $T$ with respect to $(I_{-a}, I_{-b})$ is an inner derivation.*

**Proof.** It is clear.

We notice that the notion of the generalized derivation defined on a module is a generalization of the notion of the derivation defined on a ring. If $R$ is a ring, $d : R \to R$ a derivation and $R$ is considered as an $(R, R)$-bimodule, then $d$ is a $(d, d)$-generalized derivation on $R$, and every inner derivation $I_a$ of $R$ is the inner generalized derivation $\tau_{(a,a)}$ on $R$.

If $T : M \to M$ is an $(R, S)$-bimodule homomorphism, that is $T$ is an additive mapping and $T(rm) = rT(m), T(ms) = T(m)s$, for each $r \in R, m \in M$, and $s \in S$, then $T$ is a generalized derivation with respect to $(I_0, I_0)$ on $M$. So, the generalized derivation is a generalization of bimodule homomorphisms.

**Lemma 3.7.** *If $\tau$ is a generalized derivation with respect to $(\delta_R, \delta_S)$ on $M$, then we have the Libnietz formula as follows:*
$\tau^n(rm) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \delta_R^k(r)\tau^{n-k}(m),$
$\tau^n(ms) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \tau^{n-k}(m)\delta_S^k(s),$
*for each $r \in R, s \in S$ and $m \in M$.*

**Proof.** It is clear. $\square$

Now, we provide another proof of the main result due to Coelho and Milies [4], which is different from the one due to Jondrup [7]. This is a corollary and an application of Theorem 2.2.

**Theorem 3.8.** *Let $R$ be a ring with identity. Every derivation $\triangle$ on $T_n(R)$, with $n \geq 2$, is of the from: $\triangle(r_{ij})_{i,j} = (\delta(r_{ij}))_{i,j} + I_A(r_{ij})_{i,j}$, where $\delta : R \to R$ is a derivation and $I_A$ is the inner derivation induced by $A$.*

**Proof.** We first consider the case $n = 2$. Let $T_2(R) = \left(\begin{array}{cc} R & R \\ 0 & R \end{array}\right)$,
and $\triangle : \left(\begin{array}{cc} R & R \\ 0 & R \end{array}\right) \to \left(\begin{array}{cc} R & R \\ 0 & R \end{array}\right)$ be a derivation. We have,
$\triangle\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right) = \left(\begin{array}{cc} \delta(r_1) & \tau(r_2) \\ 0 & \delta'(r_3) \end{array}\right) + I_{bE_{12}}\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right),$
and $\tau : R \to R$ is a generalized derivation related to $(\delta, \delta')$. We have, $\tau(r) = \tau(r1) = r\tau(1) + \delta(r) = ra + \delta(r)$, with $a = \tau(1)$, and $r \in R$. So,

$\tau(r) = \tau(1r) = \tau(1)r + \delta'(r) = ar + \delta'(r)$, and $ra + \delta(r) = ar + \delta'(r)$. Hence $\delta(r) = ar - ra + \delta'(r)$. So,

$$\triangle\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right) = \left(\begin{array}{cc} \delta(r_1) & \tau(r_2) - r_2a + r_2a \\ 0 & \delta'(r_3) - I_a(r_3) + I_a(r_3) \end{array}\right) + I_{bE_{12}}\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right)$$
$$= \left(\begin{array}{cc} \delta(r_1) & \delta(r_2) \\ 0 & \delta(r_3) \end{array}\right) + I_{aE_{22}}\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right) + I_{bE_{12}}\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right)$$
$$= \left(\begin{array}{cc} \delta(r_1) & \delta(r_2) \\ 0 & \delta(r_3) \end{array}\right) + I_{bE_{12}+aE_{22}}\left(\begin{array}{cc} r_1 & r_2 \\ 0 & r_3 \end{array}\right).$$

So, $\triangle$ satisfies the required conditions.

Now, we prove the theorem by induction on $n$, for $n \geq 2$. Assume inductively that the result holds for $n$. Now, we have the ring isomorphism

$$T_{n+1}(R) \cong \begin{pmatrix} R & R^n \\ 0 & T_n(R) \end{pmatrix},$$

with $R^n = R \times \cdots \times R$, where multiplication on $R^n$ is given by $r(r_1, \cdots, r_n) = (rr_1, \cdots, rr_n)$, with $r, r_1, \cdots, r_n \in R$, and multiplication by $T_n(R)$ to $R^n$ is the matrix multiplication. The derivation $\triangle$ on $\begin{pmatrix} R & R^n \\ 0 & T_n(R) \end{pmatrix}$
is $\triangle = \begin{pmatrix} \delta_1 & \tau' \\ 0 & \bar{\delta_2} \end{pmatrix} + I_B$, where $\delta_1 : R \to R$, $\bar{\delta_2} : T_n(R) \to T_n(R)$ are derivations and $\tau' : R^n \to R^n$ is the generalized derivation. By the hypothesis,

$\bar{\delta_2}(r_{ij})_{i,j} = (\delta_2(r_{ij}))_{i,j} + I_A(r_{ij})$, $A \in T_n(R)$, we have,

$\triangle = \begin{pmatrix} \delta_1 & \tau' \\ 0 & (\delta_2)_{ij} + I_A \end{pmatrix} + I_B.$

Since $I_A$ is an inner derivation on $T_n(R)$ , then $\tau_{(0,A)}$ is a generalized derivation with respect to $(I_0, I_A)$ and $\begin{pmatrix} 0 & \tau_{(0,A)} \\ 0 & I_A \end{pmatrix}$ is an inner derivation on $\begin{pmatrix} R & R^n \\ 0 & T_n(R) \end{pmatrix}$. So, we have,
$\triangle = \begin{pmatrix} \delta_1 & \tau' - \tau_{(0,A)} \\ 0 & (\delta_2)_{ij} \end{pmatrix} + \begin{pmatrix} 0 & \tau_{(0,A)} \\ 0 & I_A \end{pmatrix} + I_B,$
where $\tau = \tau' - \tau_{(0,A)}$ is the generalized derivation related to $(\delta_1, (\delta_2)_{(i,j)})$. We determine the structure of the derivation $\begin{pmatrix} \delta_1 & \tau \\ 0 & (\delta_2) \end{pmatrix}$ in terms of $\delta_1$. We have,
$\tau(0, \cdots, r_i, \cdots, 0) = \tau[(0, \cdots, r_i, \cdots, 0)e_{ii}] =$
$\tau(0, \cdots, r_i, \cdots, 0)e_{ii} + (0, \cdots, r_i, \cdots, 0)(\delta_2(1)e_{ii})$.
If $\tau(0, \cdots, r_i, \cdots, 0) = (u_1, \cdots, u_n)$, then we have,
$(u_1, \cdots, u_n) = (u_1, \cdots, u_n)e_{ii} = (0, \cdots, u_i, \cdots, 0)$.
So for each $j \neq i$, $u_j = 0$, and hence $\tau(0, \cdots, r_i, \cdots, 0) = (0, \cdots, \tau_i(r_i), \cdots, 0)$. By the definition, $\tau_i : R \to R$ is additive, for $i = 1, \cdots, n$. We have,

$\tau(0,\cdots,rr'_i,\cdots,0)$
$=\tau[r(0,\cdots,r'_i,\cdots,0)]$
$=r\tau(0,\cdots,r'_i,\cdots,0)+\delta_1(r)(0,\cdots,r'_i,\cdots,0)$.
So, $(0,\cdots,\tau_i(rr'_i),\cdots,0)=(0,\cdots,r\tau_i(r'_i)+\delta_1(r)r'_i,\cdots,0)$, and hence $\tau_i(rr'_i)=r\tau_i(r'_i)+\delta_1(r)r'_i$. Thus, $\tau_i(r)=\tau_i(r1)=r\tau_i(1)+\delta_1(r)$. We have,
$\tau(0,\cdots,r'_ir,\cdots,0)=\tau[(0,\cdots,r'_i,\cdots,0)re_{ii}]$, and
$(0,\cdots,\tau_i(r'_ir),\cdots,0)=(0,\cdots,\tau_i(r'_i),\cdots,0)re_{ii}+(0,\cdots,r'_i,\cdots,0)\delta_2(r)e_{ii}$
$=(0,\cdots,\tau_i(r'_i)r+r'_i\delta_2(r),\cdots,0)$. So $\tau_i(r'_ir)=\tau_i(r'_i)r+r'_i\delta_2(r)$.
Hence, $\tau_i(r)=\tau_i(1r)=\tau_i(1)r+\delta_2(r)$. Therefore, $\tau_i$ is a $(\delta_1,\delta_2)$ generalized derivation of $R$. For each $r\in R$, $(r,0,\cdots,0)e_{1i}=(0,\cdots,r,\cdots,0)$, and hence $\tau[(r,0,\cdots,0)e_{1i}]=\tau(0,\cdots,r,\cdots,0)$. So, we have,
$(0,\cdots,\tau_i(r),\cdots,0)=(\tau_1(r),0,\cdots,0)e_{1i}+(r,0,\cdots,0)\delta_2(1)e_{1i}$. Thus,
$(0,\cdots,\tau_i(r),\cdots,0)=(0,\cdots,\tau_1(r),\cdots,0)$. So $\tau_i(r)=\tau_1(r)$. Hence, all $\tau_i$ are equal. Assume that $\tau_i(1)=a$. So,
$\tau(r_1,\cdots,r_n)=(\tau_1(r_1),\cdots,\tau_1(r_n))=(r_1a+\delta_1(r_1),\cdots,r_na+\delta_1(r_n))=$
$=(r_1,\cdots,r_n)aI_n+(\delta_1(r_1),\cdots,\delta_1(r_n))$. So, we have,
$\tau_1(r)=ra+\delta_1(r)=ar+\delta_2(r)$. Hence, $\delta_2(r)=ra-ar+\delta_1(r)$. Thus,

$$\begin{pmatrix}\delta_2(r_{11}) & \cdots & \delta_2(r_{1n})\\ & \ddots & \vdots\\ 0 & & \delta_2(r_{nn})\end{pmatrix}=-\begin{pmatrix}ar_{11} & \cdots & ar_{1n}\\ & \ddots & \vdots\\ 0 & & ar_{nn}\end{pmatrix}$$
$$+\begin{pmatrix}r_{11}a & \cdots & r_{1n}a\\ & \ddots & \vdots\\ 0 & & r_{nn}a\end{pmatrix}+\begin{pmatrix}\delta_1(r_{11}) & \cdots & \delta_1(r_{1n})\\ & \ddots & \vdots\\ 0 & & \delta_1(r_{nn})\end{pmatrix}$$
$$=-aI_n\begin{pmatrix}r_{11} & \cdots & r_{1n}\\ & \ddots & \vdots\\ 0 & & r_{nn}\end{pmatrix}+\begin{pmatrix}r_{11} & \cdots & r_{1n}\\ & \ddots & \vdots\\ 0 & & r_{nn}\end{pmatrix}aI_n+(\delta_1(r_{ij})_{i,j}).$$

So if $d=\begin{pmatrix}\delta_1 & \tau\\ 0 & (\delta_2)\end{pmatrix}$, then we have

$$d\begin{pmatrix}r_{11} & (r_{12},\cdots,r_{1,n+1})\\ 0 & \begin{pmatrix}r_{22} & \cdots & r_{2,n+1}\\ & \ddots & \vdots\\ 0 & \cdots & r_{n+1,n+1}\end{pmatrix}\end{pmatrix}=$$

$$\begin{pmatrix} \delta_1(r_{11}) & \tau(r_{12},\cdots,r_{1,n+1}) \\ 0 & \begin{pmatrix} \delta_2(r_{22}) & \cdots & \delta_2(r_{2,n+1}) \\ & \ddots & \vdots \\ 0 & \cdots & \delta_2(r_{n+1,n+1}) \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} \delta_1(r_{11}) & (r_{12},\cdots,r_{1,n+1})\mathbb{A} + (\delta_1(r_{12}),\cdots,\delta_1(r_{1,n+1})) \\ 0 & \mathbb{B}\mathbb{A} - \mathbb{A}\mathbb{B} + (\delta_1(r_{ij})_{i,j}) \end{pmatrix} =$$

$$\begin{pmatrix} \delta_1(r_{11}) & (\delta_1(r_{12}),\cdots,\delta_1(r_{1,n+1})) \\ 0 & \begin{pmatrix} \delta_1(r_{22}) & \cdots & \delta_1(r_{2,n+1}) \\ & \ddots & \vdots \\ 0 & \cdots & \delta_1(r_{n+1,n+1}) \end{pmatrix} \end{pmatrix} +$$

$$I_{\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{A} \end{pmatrix}} \begin{pmatrix} r_{11} & (r_{12},\cdots,r_{1,n+1}) \\ 0 & \mathbb{B} \end{pmatrix},$$

where $\mathbb{A} = aI_n$ and $\mathbb{B} = \begin{pmatrix} r_{22} & \cdots & r_{2,n+1} \\ & \ddots & \vdots \\ 0 & \cdots & r_{n+1,n+1} \end{pmatrix}$. Thus, we have,

$$\Delta(\mathbb{C}) = \begin{pmatrix} \delta_1(r_{11}) & (\delta_1(r_{12}),\cdots,\delta_1(r_{1,n+1})) \\ 0 & \begin{pmatrix} \delta_1(r_{22}) & \cdots & \delta_1(r_{2,n+1}) \\ & \ddots & \vdots \\ 0 & \cdots & \delta_1(r_{n+1,n+1}) \end{pmatrix} \end{pmatrix} +$$

$I_{\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{A} \end{pmatrix}}(\mathbb{C}) + \begin{pmatrix} 0 & \tau_{(0,A)} \\ 0 & I_A \end{pmatrix}(\mathbb{C}) + I_B(\mathbb{C})$, where

$$\mathbb{C} = \begin{pmatrix} r_{11} & (r_{12},\cdots,r_{1,n+1}) \\ 0 & \begin{pmatrix} r_{22} & \cdots & r_{2,n+1} \\ & \ddots & \vdots \\ 0 & \cdots & r_{n+1,n+1} \end{pmatrix} \end{pmatrix}.$$

So, by the mentioned isomorphism, the derivation $\Delta$ on $T_{n+1}(R)$ is given by:

$\Delta(r_{ij})_{i,j} = (\delta_1(r_{ij}))_{i,j} + I_D(r_{ij})_{i,j}$,

with $\delta_1 : R \to R$ a derivation. So the result follows. □

## 4. **Differential polynomial rings of triangular matrix rings**

In this section, we study the differential polynomial extension of generalized matrix rings.

In [3], Birkenmeier and Park studied the condition of having a generalized triangular matrix representation to pass between a ring $R$ and some of its ring extensions.

If $R$ and $S$ are rings and $M$ is an $(R, S)$-bimodule, then we provide a triangular representation of differential polynomial ring $T[\theta; d]$.

**Lemma 4.1.** *Let $\delta$ be a derivation of $R$ and $S = R[x; \delta]$.*

(I). *We consider the ring $T$ and ring homomorphism $\Phi : R \to T$ , so that for each $r \in R$ and an element $y \in T$, $y\Phi(r) = \Phi(r)y + \Phi(\delta(r))$. In this case, there exists a unique ring homomorphism $\Psi : S \to T$, such that $\Psi\mid_R = \Phi$, and $\Psi(x) = y$. Indeed, we have $\Psi(\Sigma_i r_i x^i) = \sum_i \Phi(r_i)y^i$.*

(II). *If $S' = R[x'; \delta]$, then there exists a unique ring isomorphism $\Psi : S \to S'$ with $\Psi(x) = x'$ and $\Psi\mid_R$ is the identity map on $R$.*

**Proof.** [6, page 10, Exercise 1H].

**Proposition 4.2.** *Let $R$ be a ring, $\delta_1, I_a$ and $\delta$ be derivations on $R$ such that $\delta = \delta_1 + I_a$. In this case, we have $R[x; \delta_1] \cong R[\theta; \delta]$. Indeed, we have $R[\theta; \delta] = R[\theta + a; \delta_1]$.*

**Proof.** We consider the mapping $\phi : R \to R[\theta; \delta]$, where $\phi(r) = r$ and $y = \theta + a \in R[\theta; \delta]$. In this case, we have, $y\phi(r) = (\theta + a)r = \theta r + ar = r\theta + \delta(r) + ar$
$= r\theta + \delta_1(r) + I_a(r) + ar = = r\theta + \delta_1(r) + ra - ar + ar = r(\theta + a) + \delta_1(r) = \phi(r)y + \phi(\delta_1(r))$.
So, by Lemma 4.1, there exists a unique ring homomorphism $\psi : R[x; \delta] \to R[\theta; \delta]$, with $\psi\mid_R = \phi$. So, for each $r \in R$, $\psi(r) = \phi(r) = r$, and $\psi(x) = y = \theta + a$, $\psi(\sum_i r_i x^i) = \sum_i r_i(\theta + a)^i$.
Applying Lemma 3.1 on $\phi : R \to R[x; \delta_1]$, with $\phi(r) = r$ and $y = x - a \in R[x; \delta_1]$, we then have,
$y\phi(r) = (x - a)r = xr - ar = rx + \delta_1(r) - ar = rx + \delta_1(r) + ra - ra - ar$
$= r(x - a) + \delta_1(r) + I_a(r) = \phi(r)y + \phi(\delta(r))$.
So, there exists a unique ring homomorphism $\psi' : R[\theta; \delta] \to R[x; \delta_1]$, where $\psi'(r) = \phi(r) = r$, $r \in R$, and $\psi'(\theta) = y = x - a$, $\psi'(\sum_i r_i \theta^i) = \sum_i r_i(x - a)^i$.

So, we have
$\psi o\psi'(\sum_i r_i\theta^i) = \psi(\sum_i r_i(x-a)^i) = \sum_i \psi(r_i)\psi(x-a)^i = \sum r_i(\psi(x-a))^i = \sum_i r_i\theta^i$.
We have $\psi(x-a) = \theta$ and $\psi o\psi' = id_{R[\theta;\delta]}$. Similarly, we get $\psi' o\psi = id_{R[x;\delta_1]}$. Therefore, we have $R[x;\delta_1] \cong R[\theta;\delta]$. Thus, $\psi$ is an isomorphism, and hence
$R[\theta;\delta] = R[\theta + a;\delta_1]$. □

If $\delta = I_a$ is an inner derivation on $R$, then $\delta = I_0 + I_a$, where $I_0$ is the zero derivation on $R$. By Proposition 4.2, we have $R[\theta;\delta] = R[\theta+a;I_0] = R[\theta+a]$.

Now, let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ and $d$ be a derivation of $T$. Then, we have, $d = \bar{d} + I_A$, where $\bar{d} = \begin{pmatrix} \delta_R & \tau \\ 0 & \delta_S \end{pmatrix}$. By Proposition 4.2, we have $T[\theta;d] \cong T[x;\bar{d}]$.
Thus, to determine the structure of $T[\theta;d]$, it is enough to take $d$ the derivation induced by a generalized derivation such as $d = \begin{pmatrix} \delta_R & \tau \\ 0 & \delta_S \end{pmatrix}$.

If $T$ is a ring with identity and $e \in T$ is an idempotent such that $e'Te = 0$, where $e' = 1 - e$, then $R = Te$ and $S = e'T$ are subrings of $T$, and $M = eTe'$ is an additive subgroup of $T$ which is also an $(R, S)$-bimodule. We have $eTe = Te$, $e'Te' = e'T$ , $e$ and $e'$ are the identity elements of $R$ and $S$, respectively.

**Proposition 4.3.** *Let $R, S, M, e$ and $e'$ be as mentioned above. Then, the mapping $g : T \to \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, given by $g(t) = \begin{pmatrix} te & ete' \\ 0 & e't \end{pmatrix}$, for each $t \in R$, is a ring isomorphism.*

**Proof.** See [2, Proposition 1.3].

Following Birkenmeier and Park [3], we provide conditions of having a generalized triangular matrix representation of the differential polynomial rings.

**Theorem 4.4.** *Let $M$ be a unitary $(R,S)$-bimodule and* $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be the generalized triangular matrix ring and $d : T \to T$ be the derivation induced by the generalized derivation $\tau$ with respect to $\delta_R$ and $\delta_S$; i.e.,* $d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix}$*, for each $r \in R, s \in S,$ ,and $m \in M$.*
*In this case, we have the isomorphism,*

$$T[\theta;\delta] \cong \begin{pmatrix} R[x;\delta_R] & M[x,y;\tau] \\ 0 & S[y;\delta_S] \end{pmatrix},$$

*where $R[x;\delta_R]$ and $S[y;\delta_S]$ are differential polynomial rings over $R$ and $S$, and $M[x,y;\tau]$ is an $(R[x;\delta_R], S[y;\delta_S])-$bimodule which satisfying,*
I. *$M[x,y;\tau]$ contains $M$ as an $(R,S)-$subbimodule.*
II. *For each $m \in M$, we have $xm = my + \tau(m)$.*
III. *Each element $p \in M[x,y;\tau]$ is uniquely written as:*
*$p = m_0 + m_1 y + m_2 y^2 + \cdots + m_k y^k$, with $m_j \in M$, $1 \le j \le k$ and $y^j \in S[y;\delta_S]$.*

**Proof.** We have $T \subseteq T[\theta;\delta]$ and that $e = E_{11}, e' = E_{22}$, are idempotents. For each $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in T$, and each positive integer $n$,
$e'\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\theta^n e = sE_{22}\theta^n E_{11} = sE_{22}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} d^k(E_{11})\theta^{n-k}$
$= sE_{22}E_{11}\theta^n = 0$.
So, for each $p \in T[\theta;\delta]$, we have,
$e'pe = e'(\sum_i \begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix}\theta^i)e = \sum_i e'\begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix}\theta^i e = 0.$
So, we have,
$e'T[\theta;\delta]e = 0$, and by Proposition 4.3,

$$T[\theta;d] \cong \begin{pmatrix} T[\theta;d]e & eT[\theta;d]e' \\ 0 & e'T[\theta;d] \end{pmatrix}.$$

So $T[\theta;d]$ is isomorphic to a generalized triangular matrix ring. Next, we show that

$$T[\theta;d]e \cong R[x;\delta_R], e'T[\theta;d] \cong S[y;\delta_S].$$

We have the following computations:
$\theta e = E_{11} \in T[\theta;d]e, n \ge 0$;
$\theta^n e = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} d^k E_{11}\theta^{n-k} = E_{11}\theta^n = e\theta^n$, and similarly,
$\theta^n e' = e'\theta^n$;

$rE_{11} = E_{11}e \in T[\theta; d]e$.
So, $RE_{11} \subseteq T[\theta; d]e$.
Now, the map $\Phi : R \to T[\theta; d]e$, given by $\Phi(r) = rE_{11}$, is a ring homomorphism. If $y = \theta e$, then we have,
$y\Phi(r) = \theta E_{11} r E_{11} = \theta r E_{11} = rE_{11}\theta + \delta_R(r)E_{11} = rE_{11}E_{11}\theta + \delta_R(r)E_{11} = rE_{11}\theta E_{11} + \delta_R(r)E_{11} = \Phi(r)y + \Phi(\delta_R(r))$.
So, by Lemma 4.1, there exists a unique ring homomorphism,
$\Psi : R[x; \delta_R] \to T[\theta; d]e$
such that,
$\Psi(r) = rE_{11}$, for each r,
$\Psi(x) = \theta e$,
$\Psi(\sum_i r_i x^i) = \sum_i r_i E_{11}(\theta e)^i = \sum_i r_i E_{11}\theta^i e$.
Now, we show that $\Psi$ is a bijection.
Assume that $\Psi(\sum_i r_i x^i) = 0$. So, $\sum_i r_i E_{11}\theta^i e = 0$, and thus we have, $\sum_i r_i E_{11}E_{11}\theta^i = 0$, and hence $\sum_i r_i E_{11}\theta^i = 0$. Thus, for each $i$, $r_i E_{11} = 0$ and hence $r_i = 0$, for each $i$. Therefore $\sum_i r_i x^i = 0$, and $\Psi$ is injective.
Next, let $pe = (\sum_i \begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix} \theta^i)E_{11} \in T[\theta; d]e$. Now, consider the element $q = \sum_i r_i x^i \in R[x; \delta_R]$. We have,
$\Psi(q) = \Psi(\sum_i r_i x^i) = \sum_i r_i E_{11}\theta^i E_{11} = pe$.
Therefore, $\Psi$ is onto and hence $R[x; \delta_R] \cong T[\theta; d]e$. By a similar method we can show that there exists an isomorphism $\Psi' : S[y; \delta_S] \to e'T[\theta; d]$, given by $\Psi'(\sum_j s_j y^j) = \sum_j s_j E_{22}\theta^j$, and that $S[y; \delta_S] \cong e'T[\theta; d]$.
Next, we take $eT[\theta; d]e'$ as $M[x, y; \tau]$, and that $eT[\theta; d]e'$ is an $(T[\theta; d]e, e'T[\theta; d])-$bimodule. So, by the above isomorphisms, $M[x, y; \tau]$ is an $(R[x; \delta_R], S[y; \delta_S])-$bimodule, by the following operations:
$(\sum_j r_j x^j)(epe') = \Psi(\sum_j r_j x^j)(epe')$.
So, we have,
$(\sum_j r_j x^j)[e(\sum_i \begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix} \theta^i)e'] = (\sum_j r_j E_{11}\theta^j e)(epe') = eqepe'$, and
$(epe')(\sum_j s_j y^j) = (epe')\Psi'(\sum_j s_j y^j)$.
So, we have,
$(epe')(\sum_j s_j y^j) = (epe')(\sum_j s_j E_{22}e'\theta^j) = epe'q'e'$,
where $p = \sum_i \begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix} \theta^i$, $q = \sum_j r_j E_{11}\theta^j e$ and $q' = \sum_j s_j E_{22}e'\theta^j$.
So, $M[x, y; \tau]$ is an $(R, S)$-bimodule, and for each $m \in M$ we have,
$emE_{12}e' = mE_{12} \in M[x, y; \tau]$.
Since $ME_{12}$ is an $(R, S)$-bimodule, and consider $M$ as $mE_{12}$, then it is

an $(R,S)$-subbimodule of $M[x,y;\tau]$. Now, we have,
$xm = \theta em = \theta mE_{12} = mE_{12}\theta + \tau(m)E_{12} = mE_{12}E_{22}\theta + \tau(m)E_{12} = me'\theta + \tau(m) = my + \tau(m)$.
Next, we show that each element $p \in M[x,y;\tau]$ can be written as $\sum_{i=0}^{k} m_i y^i$ with $m_i \in M$. We have,

$$p = E_{11}(\sum_{i=0}^{k} \begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix} \theta^i)E_{22}$$
$$= \sum_{i=0}^{k} E_{11} \begin{pmatrix} r_i & m_i \\ 0 & s_i \end{pmatrix} E_{22}\theta^i$$
$$= \sum_{i=0}^{k} m_i E_{12}\theta^i = \sum_{i=0}^{k} m_i E_{12}E_{22}\theta^i$$
$$= \sum_{i=0}^{k} m_i(e'\theta)^i = \sum_{i=0}^{k} m_i y^i.$$

Now, to show the uniqueness, it is enough to see that, if $\sum_{i=0}^{k} m_i y^i = 0$, then $m_i = 0$ for each $i$. So, we have,
$0 = \sum_{i=0}^{k} m_i E_{12}(e'\theta)^i = \sum_{i=0}^{k} m_i E_{12}\theta^i$. So, we have, $m_i E_{12} = 0$. Thus, for each $i$, $m_i = 0$. Therefore, each element $p \in M[x,y;\tau]$ can be written as $\sum_{i=0}^{k} m_i y^i$ with $m_i \in M$. Now, if we consider the identity mapping $id : M[x,y;\tau] \to M[x,y;\tau]$, then we have,
$id(qp) = qp = \Psi(q)p = \Psi(q)id(p)$, with $q \in R[x;\delta_R], p \in M[x,y;\theta]$.
We have $id(pq') = pq' = p\Psi'(q') = id(p)\Psi'(q')$, with $q' \in S[y;\delta_S]$. But, $id$ is bijective, and so $id$ is a generalized module isomorphism related to $(\Psi,\Psi')$ and satisfies Lemma 2.6. So, we have,

$$T \cong \begin{pmatrix} Te & eTe' \\ 0 & e'T \end{pmatrix} \cong \begin{pmatrix} R[x;\delta_R] & M[x,y;\tau] \\ 0 & S[y;\delta_S] \end{pmatrix},$$

and the result follows. □

Notice that, by the isomorphism mentioned in Theorem 4.4, the element $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ is mapped to $\begin{pmatrix} \theta e & 0 \\ 0 & e'\theta \end{pmatrix}$ and $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ to $\begin{pmatrix} rE_{11} & mE_{12} \\ 0 & sE_{22} \end{pmatrix}$. Therefore in the isomorphism mentioned in the Theorem 4.4, $\theta$ is corresponds to
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and the isomorphism restricted to $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is the identity.

**Theorem 4.5.** *Let $M$ be a unitary $(R,S)$-bimodule, $\delta_R : R \to R$, $\delta_S : S \to S$ be derivations, $\tau : M \to M$ be a generalized derivation and $M[x,y;\tau]$ be as in Theorem 4.4. Let $N$ be a unitary $(R[x;\delta_R], S[y;\delta_S])$-bimodule and $\phi : M \to N$ be an $(R,S)$-homomorphism such that for* *each $m \in M$, $x\phi(m) = \phi(m)y + \phi o \tau(m)$. Then, there exists a unique $(R[x;\delta_R], S[y;\delta_S])$-bimodule homomorphism $\Phi : M[x,y;\tau] \to N$ such that $\Phi\mid_M = \phi$.*

**Proof.** Define $\varphi : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} R[x;\delta_R] & N \\ 0 & S[y;\delta_S] \end{pmatrix}$, given by $\varphi\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r & \phi(m) \\ 0 & s \end{pmatrix}$. We have that $\phi : M \to N$ is a generalized module homomorphism related to $i_R : R \to R[x;\delta_R]$, and $i_S : S \to S[x;\delta_S]$ with $i_R(r) = r$,$i_S(s) = s$, for each $r \in R$ and $s \in S$. So, $\varphi$ is a ring homomorphism and $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \begin{pmatrix} R[x;\delta_R] & N \\ 0 & S[y;\delta_S] \end{pmatrix}$.
We have,
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\varphi\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} r & \phi(m) \\ 0 & s \end{pmatrix} =$
$\begin{pmatrix} xr & x\phi(m) \\ 0 & ys \end{pmatrix} = \begin{pmatrix} rx + \delta_R(r) & \phi(m)y + \phi(\tau(m)) \\ 0 & sy + \delta_S(s) \end{pmatrix} =$
$\begin{pmatrix} r & \phi(m) \\ 0 & s \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \varphi\begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix} =$
$\varphi\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \varphi o d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$,
where $d$ is the derivation induced by $\tau$ on $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ and
$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau(m) \\ 0 & \delta_S(s) \end{pmatrix}$. So, by Lemma 4.1, and the isomorphism in Theorem 4.4, we have the unique ring homomorphism defined as:

$$\psi : \begin{pmatrix} R[x;\delta_R] & M[x,y;\tau] \\ 0 & S[y;\delta_S] \end{pmatrix} \to \begin{pmatrix} R[x;\delta_R] & N \\ 0 & S[y;\delta_S] \end{pmatrix},$$

such that $\psi \mid_{\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}} = \varphi$, and $\psi\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. So, we have,
$\psi\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \varphi\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r & \phi(m) \\ 0 & s \end{pmatrix}$,
and hence
$\psi(rE_{11}) = E_{11}, \psi(E_{22}) = E_{22}$. So, by Proposition 2.4, $\psi$ can be given by:

$\psi = \begin{pmatrix} \varphi_1 & \Phi \\ 0 & \varphi_2 \end{pmatrix}$,
where $\varphi_1 : R[x;\delta_R] \to R[x;\delta_R]$ and $\varphi_2 : S[y;\delta_S] \to S[x;\delta_S]$ are ring homomorphisms and $\Phi : M[x,y;\tau] \to N$ is the generalized module homomorphism related to $(\varphi_1, \varphi_2)$. We have,
$\psi(rE_{11}) = \varphi_1(r)E_{11} = rE_{11}$ and $\psi(sE_{22}) = \varphi_2(s)E_{22} = sE_{22}$.
We have that $\varphi_1 : R[x;\delta_R] \to R[x;\delta_R]$ is a ring homomorphism such that $\varphi_1(R) \subseteq R$ and $\varphi_1(x) = x$. So, by Lemma 4.1 and the uniqueness, $\varphi$ must be the identity, and by a similar argument $\varphi_2 : S[y;\delta_S] \to S[y;\delta_S]$ is also the identity. Hence, we have,
$\Phi(q_1p) = \varphi_1(q_1)\Phi(p) = q_1\Phi(p)$, and $\Phi(pq_2) = \Phi(p)\varphi_2(q_2) = \Phi(p)q_2$, for $q_1 \in R[x;\delta_R]$, $q_2 \in S[y;\delta_S]$ and $p \in M[x,y;\tau]$. So, $\Phi$ is a $(R[x;\delta_R], S[y;\delta_S])$-bimodule homomorphism and we have,
$\psi(mE_{12}) = \Phi(m)E_{12}$. Therefore, $\Phi\mid_M = \phi$. Now if $\Phi' : M[x,y;\tau] \to N$ is an $(R[x;\delta_R], S[y;\delta_S])$-bimodule homomorphism such that $\Phi'\mid_M = \phi$. Then, we consider the following mapping,
$\psi' : \begin{pmatrix} R[x;\delta_R] & M[x,y;\tau] \\ 0 & S[y;\delta_S] \end{pmatrix} \to \begin{pmatrix} R[x;\delta_R] & N \\ 0 & S[y;\delta_S] \end{pmatrix}$,
given by $\psi' \begin{pmatrix} q_1 & p \\ 0 & q_2 \end{pmatrix} = \begin{pmatrix} q_1 & \Phi'(p) \\ 0 & q_2 \end{pmatrix}$. In this case, $\psi'$ is a ring homomorphism and
$\psi' \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r & \phi(m) \\ 0 & s \end{pmatrix}$, $\psi' \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.
So, by the uniqueness of $\psi$ we must have $\psi' = \psi$, and hence $\Phi' = \Phi$. Therefore, $\Phi$ is unique. $\square$

**Corollary 4.6.** *Let $M$ be a unitary $(R,S)$-bimodule, $\delta_R : R \to R$, $\delta_S : S \to S$ be derivations and $\tau : M \to M$ be a $(\delta_R, \delta_S)$-generalized derivation and $M[x,y;\tau]$, $M[x,y;\tau]'$ be $(R[x;\delta_R], S[y;\delta_S])$-bimodule satisfying conditions in Theorem 4.4. Then, there exists a unique $(R[x;\delta_R], S[y;\delta_S])$-bimodule isomorphism,*
*$\Lambda : M[x,y;\tau] \to M[x,y;\tau]'$ such that $\Lambda\mid_M = I_M$, where $I_M$ is the identity mapping of $M$.*

**Proof.** Consider $\phi : M \to M[x,y;\tau]'$, with $\phi(m) = m$ for each $m \in M$. Then, $\phi$ is an $(R,S)$-bimodule homomorphism such that.
$x\phi(m) = xm = my + \tau(m) = \phi(m)y + \phi o\tau(m)$.
So, $\phi$ satisfies the conditions of Theorem 4.5, and hence there exists a unique

$(R[x;\delta_R], S[y;\delta_S])$-bimodule homomorphism $\Lambda : M[x,y;\tau] \to M[x,y;\tau]'$ such that $\Lambda\mid_M = \phi$ so $\Lambda(m) = m$, for each $m \in M$. Similarly, by Theorem 4.5, there exists a unique $(R[x;\delta_R], S[y;\delta_S])$-bimodule homomorphism $\Lambda' : M[x,y;\tau]' \to M[x,y;\tau]$ such that $\Lambda'\mid_M = \phi$ so $\Lambda'(m) = m$, for each $m \in M$. We have,
$\Lambda o\Lambda'(p) = \Lambda o\Lambda'(m_0 + m_1 y + \cdots + m_k y^k) = \Lambda(\sum_{i=0}^{k} \Lambda'(m_i) y^i) =$
$\Lambda(\sum_{i=0}^{k} m_i y^i) = \sum_{i=0}^{k} \Lambda(m_i) y^i = \sum_{i=0}^{k} m_i y^i.$
So, $\Lambda o\Lambda' = I$ and similarly $\Lambda' o\Lambda = I$. Therefore, $\Lambda$ is a bimodule isomorphism such that $\Lambda(m) = m$, for each $m \in M$. The uniqueness follows from Theorem 4.5. □

By the proof of Corollary 4.6, we observe that the bimodule isomorphism $\Lambda$ is defined by:
$\Lambda(m_0 + m_1 y + \cdots + m_k y^k) = m_0 + m_1 y + \cdots + m_k y^k.$
We also observe that the bimodule $M[x,y;\tau]$ in Corollary 4.6, is unique up to isomorphism. Therefore, we can define the following definition.

**Definition 4.7.** Let $M$ be a unitary $(R,S)$-bimodule, $\delta_R : R \to R$, $\delta_S : S \to S$ be derivations and $\tau : M \to M$ be a $(\delta_R, \delta_S)$-generalized derivation. We define $M[x,y;\tau]$ as:

I. $M[x,y;\tau]$ is a unitary $(R[x;\delta_R], S[y;\delta_S])$-bimodule, which contains $M$ as an $(R,S)$-subbimodule.

II. For each $m \in M$, we have $xm = my + \tau(m)$.

III. Each element of $p \in M[x,y;\tau]$ is uniquely written as:
$p = m_0 + m_1 y + \cdots + m_k y^k$, with $m_j \in M$, $y^j \in S[y;\delta_S]$, $1 \le j \le k$.

If the module $M[x,y;\tau]$ exists, then by Corollary 4.6, it is unique up to isomorphism and is called *the module of differential polynomials* over ${}_RM_S$.

By Theorem 4.4, for each $(R,S)$-bimodule $M$ and generalized derivation $\tau$ on $M$, the module $M[x,y;\tau]$ exists.

Let $R$ be a ring and $\delta : R \to R$ a derivation. Consider $R$ as an $(R,R)$-bimodule. Then, the differential polynomial module $R[x,x;\delta]$, is an $(R[x;\delta], R[x;\delta])$-bimodule, and satisfies the conditions in Definition 4.7. On the other hand, $R[x;\delta]$ as $(R[x;\delta], R[x;\delta])$-bimodule, satisfies the conditions in Definition 4.7, and so $R[x,x;\delta]$ is isomorphic to $R[x;\delta]$, as $(R[x;\delta], R[x;\delta])$-bimodule, by Corollary 4.6.

Some properties of the module of differential polynomials are similar to those of the ring of differential polynomials, such as what follows next.

**Lemma 4.8.** *Let $M[x, y; \tau]$ be the module of differential polynomials. Then, for each $m \in M$ we have,*
$x^k m = \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \tau^i(m) y^{k-i}$,

*for each $x^k \in R[x; \delta_R]$, $m \in M$, $\tau^0(m) = m$, and $k \geq 0$.*

**Proof.** We proceed by induction on $k$. If $k = 1$, then $xm = my + \tau(m) = \sum_{i=0}^{1} \begin{pmatrix} 1 \\ i \end{pmatrix} \tau^i(m) y^{1-i}$,

Assume that the result is true for $k \leq n$. Now, we have,
$x^{n+1}m = x^n(xm) = x^n(my + \tau(m) = x^n my + x^n \tau(m) =$
$(\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \tau^i(m) y^{n-i}) y + \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \tau^i(\tau(m)) y^{n-i} =$
$my^{n+1} + \begin{pmatrix} n \\ 1 \end{pmatrix} \tau(m) y^n + \cdots + \tau^n(m) y + \tau(m) y^n + \begin{pmatrix} n \\ 1 \end{pmatrix} \tau^2(m) y^{n-1} +$
$\cdots + \tau^{n+1} = my^{n+1} + [\begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix}] \tau(m) y^n + [\begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix}] \tau^2(m) y^{n-1}$
$+ \cdots + \tau^{n+1}(m) = \sum_{i=0}^{n+1} \begin{pmatrix} n+1 \\ i \end{pmatrix} \tau^i(m) y^{(n+1)-i}$. □

## Acknowledgment

The authors are thankful to the referee for his careful reading of the paper and for some helpful comments and suggestions.

## References

[1] G. M. Benkart and J. M. Osborn, Derivations and automorphisms of non associative matrix algebras, *Trans. Amer. Math. Soc.* **263** (1981) 411-430.

[2] G. F. Birkenmeier, H. E. Heatherly, J. Y. Kim and J. K. Park, Triangular matrix representations, *J. Algebra*, **230** (2000) 558-595.

[3] G. F. Birkenmeier and J. K. Park, Triangular matrix representations of ring extensions, *J. Algebra*, **265** (2003) 457-477.

[4] S. P. Coelho and C. P. Milies, Derivations of upper triangular matrix rings, *Linear Alg. Appl.* **187** (1993) 263-267.

[5] Giambruno and I. N. Herstein, Derivations with nilpotent values, *Rendicoti, Del Circolo Mathematico Di palermo Serie II, Tommo XXX,* (1981) 199-206.
[6] K. R. Gooderl and R. B. Warfield, An introduction to noncommutative Noetherian rings, Cambridge University Press, Cambridge, (1989).
[7] S. Jöndrup, Automorphism and derivations of triangular matrices, *Linear Alg. Appl.* **22** (1995) 205-215.
[8] S. Jöndrup, Automorphism of upper triangular matrix rings, *Arch. Math.* **49** (1987) 497-502.
[9] D. Mathis, Differential polynomial rings and Morita equivalence, *Comm. Algebra.* **10** (1982) 2001-2017.

**Hoger Ghahramani**

**A. Moussavi**

Department of Mathematics, Tarbiat Modares University, Tehran, Iran, P.O. Box:14115-170.

Email: `hoger_gh@yahoo.com`

Email: `moussavi_a5@yahoo.com`