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STABLE EXPONENTIALLY HARMONIC MAPS BETWEEN FINSLER MANIFOLDS

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Abstract. We derive the first and second variation formulas for exponentially harmonic maps between Finsler manifolds, and prove that there is no non-degenerate stable exponentially harmonic map between a compact convex hypersurface of the Euclidean space and any compact Finsler manifold.

1. Introduction

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M, Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM\backslash 0$.
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$, for $\lambda > 0$.
(iii) Strong convexity: The fundamental quadratic form $g = g_{ij}dx^i \otimes dx^j$ is positive definite, where $g_{ij} = \frac{\partial^2(F^2)}{2\partial Y^i \partial Y^j}$.

Let $\phi : M \to \overline{M}$ be a non-degenerate smooth map between Finsler manifolds; i.e., $ker(d\phi) = 0$. Harmonic maps between Finsler manifolds

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are defined as the critical points of energy functionals. They are important in both classical and modern differential geometry. The first and second variation formulas of non-degenerate harmonic maps between Finsler manifolds was given in [4] and [8]. As for stability of harmonic maps between Finsler manifolds, He and Shen [4] proved that there is no non-degenerate stable harmonic map between a Riemannian unit sphere $S^n (n > 2)$ and any compact Finsler manifold.

For the exponentially harmonic maps, Eells and Lemaire [2] introduced the definition of exponentially harmonic maps between Riemannian manifolds; i.e., exponentially harmonic maps between Riemannian manifolds are defined as the critical points of exponential energy functional. Stability of exponentially harmonic maps between Riemannian manifolds was discussed in [6] and [7]. On the other hand, Riemannian manifold is a special case of Finsler manifold. A natural question is how to generalize exponentially harmonic maps from the Riemannian case to the Finsler case.

Here, we are concerned with an exponentially harmonic map between Finsler manifolds. These are critical points of the exponential energy functional; cf. (3.2). We derive the first and second variation formulas for exponentially harmonic maps between Finsler manifolds (see Lemma 3.4 and Theorem 4.1) which generalize results of [7] from the Riemannian case to the Finsler case, and we prove the following result.

Theorem 1.1. *Let M^n be a compact convex hypersurface of the Euclidean space E^{n+1}, with its principal curvatures sorted as $0 < \lambda_1 \leq \cdots \leq \lambda_n$, satisfying $\lambda_n < \sum_{j=1}^{n-1} \lambda_j$. There is no non-degenerate stable exponentially harmonic map ϕ between M^n and any compact Finsler manifold with $|d\phi|^2 < \frac{1}{\lambda_n^2} min_{1\leq i \leq n}\{\lambda_i(\sum_{j=1}^{n} \lambda_j - 2\lambda_i)\}$.*

Corollary 1.2. *There is no non-degenerate stable exponentially harmonic map ϕ between a Riemannian unit sphere S^n and any compact Finsler manifold with $|d\phi|^2 < n - 2$.*

Remark 1.3. This result generalizes Theorem 1 of [7] from the Riemannian case to the Finsler case.

2. Preliminaries

We shall use the following convention for index ranges unless otherwise stated:

$$1 \leq i, j, \cdots \leq n;\ 1 \leq \alpha, \beta, \cdots \leq m;\ 1 \leq a, b, \cdots \leq n-1.$$

Let (M, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert* form and *Cartan* tensor as follows:

$$\omega^n = \frac{\partial F}{\partial Y^i} dx^i, A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, A_{ijk} = F \frac{\partial g_{ij}}{\partial Y^k}, g_{ij} = [\frac{1}{2} F^2]_{Y^i Y^j}.$$

It is well known that there exists uniquely the Chern connection ∇ on $\pi^* TM$ with $\nabla \frac{\partial}{\partial x^i} = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying

$$dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta Y^k}{F}, \tag{2.1}$$

where $\delta Y^i = dY^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F} A_{jk}^i \gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

On the other hand, by [1], $\nabla e_n = \frac{\delta Y^k}{F} \frac{\partial}{\partial x^k}$, where $e_n = \frac{Y^i}{F} \frac{\partial}{\partial x^i}$, so (2.1) is equivalent to:

$$X\langle U, V\rangle = \langle \nabla_X U, V\rangle + \langle U, \nabla_X V\rangle + 2C(U, V, \nabla_X (Fe_n)), \tag{2.2}$$

where $A_{ijk} = FC_{ijk}$ and $X, U, V \in \Gamma(\pi^* TM)$.

The curvature 2-forms of the Chern connection ∇ are:

$$\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \frac{1}{2} R_{jkl}^i dx^k \wedge dx^l + \frac{1}{F} P_{jkl}^i dx^k \wedge \delta Y^l. \tag{2.3}$$

Take a g-orthonormal frame $\{e_i = u_i^j \frac{\partial}{\partial x^j}\}$ with $e_n = \frac{Y^i}{F}\frac{\partial}{\partial x^i}$, for each fibre of $\pi^* TM$ and $\{\omega^i\}$ its dual coframe. The collection $\{\omega^i, \omega_n^i\}$ forms an orthonormal basis for $T^*(TM\backslash\{0\})$ with respect to the Sasakitype metric $g_{ij} dx^i \otimes dx^j + g_{ij}\delta Y^i \otimes \delta Y^j$. The pull-back of the Sasaki metric from $TM\backslash\{0\}$ to SM is a Riemannian metric,

$$\widehat{g} = g_{ij} dx^i \otimes dx^j + \delta_{ab}\omega_n^a \otimes \omega_n^b. \tag{2.4}$$

Then, we have the following result.

Lemma 2.1. [4] *For* $\psi = \psi_i \omega^i \in \Gamma(\pi^* T^* M)$, *we have*

$$div_{\widehat{g}}\psi = \sum_i \psi_{i|i} + \sum_{a,b} \psi_a P_{bba} = (\nabla_{e_i^H}\psi) e_i + \sum_{a,b} \psi_a P_{bba}.$$

where "|" denotes the horizontal covariant differentials with respect to the Chern connection, $e_i^H = u_i^j \frac{\delta}{\delta x^j} = u_i^j(\frac{\partial}{\partial x^j} - N_j^k \frac{\partial}{\partial Y^k})$ denotes the horizontal part of e_i and $P_{bba} = P^n_{bba}$.

3. The first variation formula

Let $\phi : M^n \to \overline{M}^m$ be a non-degenerate smooth map. The exponential energy density of ϕ is the function $e(\phi) : SM \to R$ defined by

$$e(\phi)(x, Y) = \exp(\frac{1}{2}|d\phi|^2), \tag{3.1}$$

where $|d\phi|^2 = g^{ij}(x, Y)\phi_i^\alpha \phi_j^\beta \overline{g}_{\alpha\beta}(\overline{x}, \overline{Y})$, $d\phi(\frac{\partial}{\partial x^i}) = \phi_i^\alpha \frac{\partial}{\partial \overline{x}^\alpha}$ and $\overline{Y} = Y^i \phi_i^\alpha \frac{\partial}{\partial \overline{x}^\alpha}$.

We define the exponential energy functional $E(\phi)$ by

$$E(\phi) = \frac{1}{C_{n-1}} \int_{SM} e(\phi) dV_{SM}, \tag{3.2}$$

where $dV_{SM} = \Omega d\tau \wedge dx, dx = dx^1 \wedge \cdots \wedge dx^n$ and C_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1}.

Let $\widetilde{\nabla}$ be the pullback Chern connection on $\pi^*(\phi^{-1}T\overline{M})$ and $\widetilde{\Omega}$ be the curvature form of the pullback connection $\widetilde{\nabla}$. We have the following result, from (2.2) and $d\phi(Fe_n) = \overline{F}\overline{e_m}$

Lemma 3.1.

$$\begin{aligned} X\langle d\phi U, d\phi V\rangle = & \langle \widetilde{\nabla}_X(d\phi U), d\phi V\rangle + \langle d\phi U, \widetilde{\nabla}_X(d\phi V)\rangle \\ & + 2\overline{C}(d\phi U, d\phi V, (\widetilde{\nabla}_X(d\phi Fe_n)). \end{aligned}$$

Let $\widetilde{\Omega}$ be the curvature form of the pullback connection $\widetilde{\nabla}$. We have the following result.

Lemma 3.2.

$$\widetilde{\Omega}^\alpha_\beta(U, V) = \overline{R}^\alpha_\beta(d\phi U, d\phi V) + \frac{F}{\overline{F}}\overline{P}^\alpha_\beta(d\phi U, \widetilde{\nabla}_V d\phi e_n) - \frac{F}{\overline{F}}\overline{P}^\alpha_\beta(d\phi V, \widetilde{\nabla}_U d\phi e_n),$$

where $\overline{R}^\alpha_\beta = \overline{R}^\alpha_{\beta\gamma\sigma} d\overline{x}^\gamma \otimes d\overline{x}^\sigma$ and $\overline{P}^\alpha_\beta = \overline{P}^\alpha_{\beta\gamma\sigma} d\overline{x}^\gamma \otimes d\overline{x}^\sigma$.

We call ϕ a exponentially harmonic map if it is a critical point of exponential energy functional. Let ϕ_t be a smooth variation of ϕ with

$\phi_0 = \phi$ and $\phi_t|_{\partial M} = \phi|_{\partial M}$. $\{\phi_t\}$ induces a vector field V along ϕ by

$$V = \frac{\partial \phi_t}{\partial t}|_{t=0} = v^\alpha \frac{\partial}{\partial \overline{x}^\alpha},\ V|_{\partial M} = 0.$$

Lemma 3.3.

$$\begin{aligned} &\sum_i \int_{SM} \langle \widetilde{\nabla}_{e_i^H} d\phi \frac{\partial}{\partial t}, \exp(\frac{1}{2}|d\phi|^2) d\phi e_i \rangle dV_{SM} \\ = \quad & -\int_{SM} \Big\{ \sum_i [\langle d\phi \frac{\partial}{\partial t}, (\widetilde{\nabla}_{e_i^H} \exp(\frac{1}{2}|d\phi|^2) d\phi) e_i \rangle \\ & +2\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi \frac{\partial}{\partial t}, d\phi e_i, \widetilde{\nabla}_{e_i^H} d\phi F e_n)] \\ & + \textstyle\sum_{a,b} \exp(\frac{1}{2}|d\phi|^2) \langle d\phi \frac{\partial}{\partial t}, d\phi e_b \rangle P_{aab} \Big\} dV_{SM}. \end{aligned}$$

Proof. Let $\psi = \exp(\frac{1}{2}|d\phi|^2)\langle d\phi \frac{\partial}{\partial t}, d\phi e_i\rangle \omega^i$. From Lemma 2.1, we obtain:

$$\begin{aligned} div_{\hat{g}}\psi = & \langle \widetilde{\nabla}_{e_i^H} d\phi \tfrac{\partial}{\partial t}, \exp(\tfrac{1}{2}|d\phi|^2) d\phi e_i \rangle + \langle d\phi \tfrac{\partial}{\partial t}, (\widetilde{\nabla}_{e_i^H} \exp(\tfrac{1}{2}|d\phi|^2) d\phi) e_i \rangle \\ & +2\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi \frac{\partial}{\partial t}, d\phi e_i, (\widetilde{\nabla}_{e_i^H} d\phi) F e_n) \\ & + \textstyle\sum_{a,b} \exp(\frac{1}{2}|d\phi|^2) \langle d\phi \frac{\partial}{\partial t}, d\phi e_b \rangle P_{aab}. \end{aligned} \tag{3.5}$$

By integrating (3.5), we get the result. □

Similarly, let $\psi = \exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \frac{d\phi}{dt}) F \omega^n$, which is a global section on $T^*(S_x M)$. By $P_{aan} = 0$, we also have the following result.

Lemma 3.4.

$$\begin{aligned} &\sum_i \int_{SM} \exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{F e_n^H} \frac{d\phi}{dt}) dV_{SM} \\ = \quad & -\sum_i \int_{SM} \Big\{ (\widetilde{\nabla}_{F e_n^H} \exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, \frac{d\phi}{dt}) \\ & + \exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{F e_n^H} \overline{C})(d\phi e_i, d\phi e_i, \tfrac{d\phi}{dt}) \\ & +2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{F e_n^H} d\phi e_i, d\phi e_i, \tfrac{d\phi}{dt}) \Big\} dV_{SM}. \end{aligned}$$

It follows from Lemma 3.3 and Lemma 3.4 that

$$
\begin{aligned}
&\frac{d}{dt}E(\phi)\\
(3.6)\qquad =\ & \frac{1}{C_{n-1}}\sum_i\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi e_i, d\phi e_i\rangle\\
&+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi(Fe_n))\Big\}dV_{SM}\\
=\ & -\frac{1}{C_{n-1}}\int_{SM}\langle\frac{d\phi}{dt},\tau\rangle dV_{SM},
\end{aligned}
$$

where,

$$
\begin{aligned}
\tau =\ & \sum_i(\widetilde{\nabla}_{e_i^H}\exp(\frac{1}{2}|d\phi|^2)d\phi)e_i\\
&+\textstyle\sum_{i,\alpha}\Big\{2\exp(\frac{1}{2}|d\phi|^2)\overline{C}(\overline{e}_\alpha, d\phi e_i, \widetilde{\nabla}_{e_i^H}d\phi Fe_n)\overline{e}_\alpha\\
(3.7)\qquad &+(\widetilde{\nabla}_{Fe_n^H}\exp(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha\\
&+\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha\\
&+2\exp(\frac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{Fe_n^H}d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha\Big\}\\
&+\textstyle\sum_{a,b}\exp(\frac{1}{2}|d\phi|^2)\langle\overline{e}_\alpha, d\phi e_b\rangle\overline{e}_\alpha P_{aab}.
\end{aligned}
$$

From (3.6), we get immediately the following.

Theorem 3.5. *ϕ is exponentially harmonic map if and only if*

$$\int_{SM}\langle V,\tau\rangle dV_{SM} = 0,$$

for any vector $V\in\Gamma(\phi^{-1}T\overline{M})$.

Remark 3.6. Theorem 3.5 generalizes the result of [6] from the Riemannian case to the Finsler case.

4. The second variation formula

First, from Lemma 3.2 , we obtain immediately,

$$
\begin{aligned}
&\widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{e_i^H}d\phi\frac{\partial}{\partial t}-\widetilde{\nabla}_{e_i^H}\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}\\
(4.1)\qquad =\ & -\overline{R}(d\phi e_i, d\phi\frac{\partial}{\partial t})d\phi\frac{\partial}{\partial t}\\
&+\tfrac{F}{\overline{F}}\overline{P}(d\phi\tfrac{\partial}{\partial t},(\nabla_{e_i}d\phi)e_n)d\phi\tfrac{\partial}{\partial t}-\tfrac{F}{\overline{F}}\overline{P}(d\phi e_i,\widetilde{\nabla}_{e_n^H}\tfrac{d\phi}{dt})d\phi\tfrac{\partial}{\partial t},
\end{aligned}
$$

where $\overline{R} = \overline{R}^{\alpha}_{\beta\gamma\sigma}\frac{\partial}{\partial\overline{x}^{\alpha}} \otimes d\overline{x}^{\beta} \otimes d\overline{x}^{\gamma} \otimes d\overline{x}^{\sigma}$ and $\overline{P} = \overline{P}^{\alpha}_{\beta\gamma\sigma}\frac{\partial}{\partial\overline{x}^{\alpha}} \otimes d\overline{x}^{\beta} \otimes d\overline{x}^{\gamma}\otimes d\overline{x}^{\sigma}$.

On putting $\psi = \exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, d\phi e_i\rangle\omega^i$, we obtain from Lemma 2.1,

$$
\begin{aligned}
div_{\widehat{g}}\psi = \ & \sum_i \exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, d\phi e_i\rangle \\
& +\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, (\widetilde{\nabla}_{e_i^H}\exp(\tfrac{1}{2}|d\phi|^2)d\phi)e_i\rangle \\
& +2\sum_i \exp(\frac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, d\phi e_i, \widetilde{\nabla}_{e_i^H}d\phi F e_n) \\
& +\textstyle\sum_{a,b}\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, d\phi e_b\rangle P_{aab}.
\end{aligned}
\tag{4.2}
$$

It can be seen from (4.2) that

$$
\begin{aligned}
& \sum_i\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}), d\phi e_i\rangle dV_{SM} \\
= \ & -\int_{SM}\Big\{\sum_i\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, (\widetilde{\nabla}_{e_i^H}\exp(\frac{1}{2}|d\phi|^2)d\phi)e_i\rangle \\
& +2\sum_i\exp(\frac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, d\phi e_i, \widetilde{\nabla}_{e_i^H}d\phi F e_n) \\
& +\sum_{a,b}\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, d\phi e_b\rangle P_{aab}\Big\}dV_{SM}.
\end{aligned}
\tag{4.3}
$$

And similarly, we have

$$
\begin{aligned}
& \int_{SM}\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H}\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t})dV_{SM} \\
= \ & -\int_{SM}\Big\{(\widetilde{\nabla}_{Fe_n^H}\exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}) \\
& +\exp(\frac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}) \\
& +2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{Fe_n^H}d\phi e_i, d\phi e_i, \widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\tfrac{\partial}{\partial t})\Big\}dV_{SM}.
\end{aligned}
\tag{4.4}
$$

It follows from (4.3) and (4.4) that

$$
\begin{aligned}
& \int_{SM}\Big\{\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}), d\phi e_i\rangle \\
& +\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H}\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t})\Big\}dV_{SM} \\
= \ & -\int_{SM}\langle\widetilde{\nabla}_{d\phi\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}, \tau\rangle dV_{SM}.
\end{aligned}
\tag{4.5}
$$

On the other hand, we need the following lemma.

Lemma 4.1.

$$\begin{aligned}&\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\\ =\ &\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{\frac{d\phi}{dt}^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\\ &+\exp(\tfrac{1}{2}|d\phi|^2)\overline{\mathcal{C}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}),\end{aligned}$$

where $\overline{\mathcal{C}} = \overline{\mathcal{C}}^{\alpha}_{\beta\gamma\sigma} d\overline{x}^{\alpha}\otimes d\overline{x}^{\beta}\otimes d\overline{x}^{\gamma}\otimes d\overline{x}^{\sigma},\ \overline{\mathcal{C}}^{\alpha}_{\beta\gamma\sigma} = \frac{\partial \overline{C}^{\alpha}_{\beta\gamma}}{\partial \overline{Y}^{\sigma}}$.

Proof. By $\frac{\partial \overline{Y}^{\sigma}}{\partial t} = Y^i \frac{\partial V^{\sigma}}{\partial x^i}$, we obtain:

(4.6)
$$\begin{aligned}&(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\\ =\ &V^{\sigma}\tfrac{\partial \overline{C}}{\partial \overline{x}^{\sigma}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}) + \overline{\mathcal{C}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}, Y^i\tfrac{\partial V^{\sigma}}{\partial x^i}\tfrac{\partial}{\partial \overline{x}^{\sigma}}),\end{aligned}$$

and

(4.7)
$$\begin{aligned}&(\widetilde{\nabla}_{\frac{d\phi}{dt}^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\\ =\ &V^{\sigma}\tfrac{\partial \overline{C}}{\partial \overline{x}^{\sigma}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}) - \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}, V^{\sigma}\overline{N}^{\sigma}_{\tau}\tfrac{\partial}{\partial \overline{x}^{\tau}}).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}&\overline{\mathcal{C}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\\ =\ &\overline{\mathcal{C}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}, Y^i\tfrac{\partial V^{\sigma}}{\partial x^i}\tfrac{\partial}{\partial \overline{x}^{\sigma}} + V^{\sigma}\overline{N}^{\tau}_{\sigma}\tfrac{\partial}{\partial \tau}).\end{aligned} \tag{4.8}$$

By (4.6) − (4.8), the result follows. □

Lemma 4.2.

$$\begin{aligned}&\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\frac{d}{dt}\langle\widetilde{\nabla}_{e_i}d\phi\frac{\partial}{\partial t}, d\phi e_i\rangle dV_{SM}\\ =\ &\frac{1}{C_{n-1}}\int_{SM}\Big\{\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t}), d\phi e_i\rangle\\ &+2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{e_i^H}d\phi\tfrac{\partial}{\partial t}, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}dV_{SM} + \Xi_2 + \Xi_4,\end{aligned}$$

where,

$$\Xi_2 = \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}\frac{d\phi}{dt}, \widetilde{\nabla}_{e_i^H}\frac{d\phi}{dt}\rangle dV_{SM}, \tag{4.9}$$

and

$$\begin{aligned}\Xi_4 = \ & \frac{1}{C_{n-1}}\int_{SM}\Big\{-\exp(\frac{1}{2}|d\phi|^2)\langle\overline{R}(d\phi e_i,\frac{d\phi}{dt})\frac{d\phi}{dt},d\phi e_i\rangle \\ & +\exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\langle\overline{P}(\tfrac{d\phi}{dt},(\widetilde{\nabla}_{e_i}d\phi)e_n)\tfrac{d\phi}{dt},d\phi e_i\rangle \\ & -\exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\langle\overline{P}(d\phi e_i,\widetilde{\nabla}_{e_n^H}\tfrac{d\phi}{dt})\tfrac{d\phi}{dt},d\phi e_i\rangle\Big\}dV_{SM}.\end{aligned}\tag{4.10}$$

Proof. By (4.1), we have

$$\begin{aligned}&\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\frac{d}{dt}\langle\widetilde{\nabla}_{e_i}d\phi\frac{\partial}{\partial t},d\phi e_i\rangle dV_{SM}\\ = \ & \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\langle\widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{e_i^H}d\phi\frac{\partial}{\partial t},d\phi e_i\rangle\\ & +\langle\widetilde{\nabla}_{e_i^H}d\phi\tfrac{\partial}{\partial t},\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi e_i\rangle+2\overline{C}(\widetilde{\nabla}_{e_i^H}d\phi\tfrac{\partial}{\partial t},d\phi e_i,\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi F e_n)\Big\}dV_{SM}\\ = \ & +\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\langle\widetilde{\nabla}_{e_i^H}\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\phi\frac{\partial}{\partial t},d\phi e_i\rangle\\ & -\langle\overline{R}(d\phi e_i,d\phi\tfrac{\partial}{\partial t})d\phi\tfrac{\partial}{\partial t},d\phi e_i\rangle+\tfrac{F}{\overline{F}}\langle\overline{P}(d\phi\tfrac{\partial}{\partial t},(\widetilde{\nabla}_{e_i}d\phi)e_n)d\phi\tfrac{\partial}{\partial t},d\phi e_i\rangle\\ & -\tfrac{F}{\overline{F}}\langle\overline{P}(d\phi e_i,\widetilde{\nabla}_{e_n^H}\tfrac{d\phi}{dt})d\phi\tfrac{\partial}{\partial t},d\phi e_i\rangle+\langle\widetilde{\nabla}_{e_i^H}d\phi\tfrac{\partial}{\partial t},\widetilde{\nabla}_{e_i}d\phi\tfrac{\partial}{\partial t}\rangle\\ & +2\overline{C}(\widetilde{\nabla}_{e_i^H}d\phi\tfrac{\partial}{\partial t},d\phi e_i,\widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}dV_{SM},\end{aligned}\tag{4.11}$$

which completes the proof. □

Lemma 4.3.

$$\begin{aligned}&\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\frac{d}{dt}[\overline{C}(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n}\frac{d\phi}{dt})]dV_{SM}\\ = \ & \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\overline{C}(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n}\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{d\phi}{dt})\\ & -2\overline{C}(\widetilde{\nabla}_{e_i^H}\tfrac{d\phi}{dt},d\phi e_i,\widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}dV_SM+\Xi_3+\Xi_5,\end{aligned}$$

where,

$$\begin{aligned}\Xi_3 = \ & \frac{1}{C_{n-1}}\int_{SM}\Big\{4\exp(\frac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{e_i^H}\frac{d\phi}{dt},d\phi e_i,\widetilde{\nabla}_{Fe_n}\frac{d\phi}{dt})\\ & +\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{(\frac{d\phi}{dt})^H}\overline{C})(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\\ & +\exp(\tfrac{1}{2}|d\phi|^2)\overline{\mathcal{C}}(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt},\widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}dV_{SM},\end{aligned}\tag{4.12}$$

and

(4.13)

$$\Xi_5 = \frac{1}{C_{n-1}}\int_{SM}\Big\{-\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \overline{R}(d\phi F e_n, \frac{d\phi}{dt})\frac{d\phi}{dt}) + \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(\tfrac{d\phi}{dt}, (\widetilde{\nabla}_{Fe_n} d\phi)e_n)\tfrac{d\phi}{dt}) - \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(d\phi F e_n, \widetilde{\nabla}_{e_n^H}\tfrac{d\phi}{dt})\tfrac{d\phi}{dt})\Big\}dV_{SM}.$$

Proof.

(4.14)

$$\begin{aligned} &\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\frac{d}{dt}[\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\frac{d\phi}{dt})]dV_{SM} \\ =\ & \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\frac{d\phi}{dt}) \\ &+2\overline{C}(\widetilde{\nabla}_{\frac{\partial}{\partial t}} d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}) + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}dV_{SM} \\ =\ & \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\frac{d\phi}{dt}) \\ &+2\overline{C}(\widetilde{\nabla}_{e_i^H}\tfrac{d\phi}{dt}, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt}) + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\widetilde{\nabla}_{\frac{\partial}{\partial t}}\tfrac{d\phi}{dt}) \\ &-\overline{C}(d\phi e_i, d\phi e_i, \overline{R}(d\phi F e_n, d\phi\tfrac{\partial}{\partial t})d\phi\tfrac{\partial}{\partial t}) \\ &+\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(d\phi\tfrac{\partial}{\partial t}, (\widetilde{\nabla}_{Fe_n}d\phi)e_n)d\phi\tfrac{\partial}{\partial t}) \\ &-\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(d\phi F e_n, \widetilde{\nabla}_{e_n^H}\tfrac{d\phi}{dt})d\phi\tfrac{\partial}{\partial t})\Big\}dV_{SM}. \end{aligned}$$

The proof follows from (4.14) and Lemma 4.1. □

So, we get from (4.5), Lemma 4.2 and Lemma 4.3,

(4.15)

$$\begin{aligned} \tfrac{d^2}{dt^2}E(\phi) =\ & \frac{1}{C_{n-1}}\frac{d}{dt}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\langle\widetilde{\nabla}_{e_i^H} d\phi\frac{\partial}{\partial t}, d\phi e_i\rangle \\ &+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}dV_{SM} \\ =\ & \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\langle\widetilde{\nabla}_{e_i^H} d\phi\frac{\partial}{\partial t}, d\phi e_i\rangle \\ &+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})\Big\}^2 dV_{SM} \\ &+\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\frac{d}{dt}\langle\widetilde{\nabla}_{e_i} d\phi\frac{\partial}{\partial t}, d\phi e_i\rangle \\ &+\tfrac{d}{dt}[\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}\tfrac{d\phi}{dt})]\Big\}dV_{SM} \\ =\ & \Xi_1+\Xi_2+\Xi_3+\Xi_4+\Xi_5+\Xi_6, \end{aligned}$$

where,

$$\Xi_1 = \tfrac{1}{C_{n-1}} \int_{SM} \exp(\tfrac{1}{2}|d\phi|^2) \Big\{\langle \widetilde{\nabla}_{e_i^H} V, d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} V)\Big\}^2 dV_{SM}, \tag{4.16}$$

and

$$\Xi_6 = -\frac{1}{C_{n-1}} \int_{SM} \langle \widetilde{\nabla}_{d\phi \frac{\partial}{\partial t}} d\phi \frac{\partial}{\partial t}, \tau\rangle dV_{SM}. \tag{4.17}$$

From (4.15), we get the second variation formula as follows.

Theorem 4.4. *Let $\phi : M^n \to \overline{M}^m$ be a non-degenerate smooth map. Let ϕ_t be a smooth variation of ϕ with $\phi_0 = \phi$ and $V = \frac{\partial \phi_t}{\partial t}|_{t=0}$. Then, the second variation of the exponentially energy functional for ϕ is:*

$$I(V,V) = \frac{d^2}{dt^2} E(\phi_t)|_{t=0} = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 + \Xi_6,$$

where,

$$\Xi_1 = \tfrac{1}{C_{n-1}} \int_{SM} \exp(\tfrac{1}{2}|d\phi|^2) \Big\{\langle \widetilde{\nabla}_{e_i^H} V, d\phi e_i\rangle + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} V)\Big\}^2 dV_{SM}, \tag{4.18}$$

$$\Xi_2 = \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2) \langle \widetilde{\nabla}_{e_i^H} V, \widetilde{\nabla}_{e_i^H} V\rangle dV_{SM}, \tag{4.19}$$

$$\begin{aligned} \Xi_3 = \;& \frac{1}{C_{n-1}} \int_{SM} \Big\{ 4 \exp(\frac{1}{2}|d\phi|^2) \overline{C}(\widetilde{\nabla}_{e_i^H} V, d\phi e_i, \widetilde{\nabla}_{Fe_n} V) \\ & + \exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V^H} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} V) \\ & + \exp(\tfrac{1}{2}|d\phi|^2) \overline{\mathcal{C}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} V, \widetilde{\nabla}_{Fe_n} V)\Big\} dV_{SM}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \Xi_4 = \;& \frac{1}{C_{n-1}} \int_{SM} \Big\{ -\exp(\frac{1}{2}|d\phi|^2) \langle \overline{R}(d\phi e_i, V)V, d\phi e_i\rangle \\ & + \exp(\tfrac{1}{2}|d\phi|^2) \tfrac{F}{\overline{F}} \langle \overline{P}(V, (\widetilde{\nabla}_{e_i} d\phi) e_n)V, d\phi e_i\rangle \\ & - \exp(\tfrac{1}{2}|d\phi|^2) \tfrac{F}{\overline{F}} \langle \overline{P}(d\phi e_i, \widetilde{\nabla}_{e_n^H} V)V, d\phi e_i\rangle \Big\} dV_{SM}, \end{aligned} \tag{4.21}$$

(4.22)

$$\Xi_5 = \frac{1}{C_{n-1}} \int_{SM} \Big\{ -\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \overline{R}(d\phi F e_n, V)V) \\ + \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(V, (\widetilde{\nabla}_{Fe_n} d\phi)e_n)V) \\ - \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(d\phi F e_n, \widetilde{\nabla}_{e_n^H} V)V) \Big\} dV_{SM},$$

$$\Xi_6 = -\frac{1}{C_{n-1}} \int_{SM} \langle \widetilde{\nabla}_V V, \tau \rangle dV_{SM}. \tag{4.23}$$

Remark 4.5. Theorem 4.4 generalizes the results of [6] from the Riemannian case to Finsler case.

5. Stability in the case source manifold is a convex hypersurface

Let M^n be a compact Riemannian hypersurface in the Euclidean space E^{n+1}. We choose a local field of orthonormal frames $\{e_1, \cdots, e_{n+1}\}$ in the Euclidean space E^{n+1} such that $\{e_1, \cdots, e_n\}$ are tangent to M and $h_{ij}^{n+1} = \lambda_i \delta_{ij}$, where $B = h_{ij}^{n+1} \omega^i \otimes \omega^j \otimes e_{n+1}$ is the second fundamental form of M in E^{n+1}. Let $\{\Lambda_1, \cdots, \Lambda_{n+1}\}$ be the constant orthonormal basis in E^{n+1} and $V_\lambda = \langle \Lambda_\lambda, e_i \rangle e_i$, $\lambda = 1, \cdots, n+1$. Then, a straightforward computation shows:

$$\nabla_{e_i} V_\lambda = v_\lambda^{n+1} \lambda_i e_i, \tag{5.1}$$

where, $v_\lambda^\mu = \langle \Lambda_\lambda, e_\mu \rangle$, $\mu = 1, \cdots, n+1$.

The second variation formula of the exponentially harmonic map $\phi : M^n \to \overline{M}^m$ can be written as:

$$\sum_\lambda I(d\phi V_\lambda, d\phi V_\lambda) = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 + \Xi_6,$$

where,

$$\Xi_1 = \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2) \Big\{ \langle \widetilde{\nabla}_{e_i^H} d\phi V_\lambda, d\phi e_i \rangle \\ + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda) \Big\}^2 dV_{SM}, \tag{5.2}$$

$$\Xi_2 = \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2) \langle \widetilde{\nabla}_{e_i^H} d\phi V_\lambda, \widetilde{\nabla}_{e_i^H} d\phi V_\lambda \rangle dV_{SM}, \tag{5.3}$$

(5.4)

$$\Xi_3 = \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \Big\{ 4\exp(\frac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{e_i^H} d\phi V_\lambda, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda) + \exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{(d\phi V_\lambda)^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda) + \exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda) \Big\} dV_{SM},$$

(5.5)

$$\Xi_4 = \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \Big\{ -\exp(\frac{1}{2}|d\phi|^2)\langle \overline{R}(d\phi e_i, d\phi V_\lambda) d\phi V_\lambda, d\phi e_i\rangle + \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\langle \overline{P}(d\phi V_\lambda, (\widetilde{\nabla}_{e_i} d\phi)e_n) d\phi V_\lambda, d\phi e_i\rangle - \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\langle \overline{P}(d\phi e_i, \widetilde{\nabla}_{e_n^H} d\phi V_\lambda) d\phi V_\lambda, d\phi e_i\rangle \Big\} dV_{SM},$$

(5.6)

$$\Xi_5 = \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \Big\{ -\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \overline{R}(d\phi F e_n, d\phi V_\lambda) d\phi V_\lambda) + \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(d\phi V_\lambda, (\widetilde{\nabla}_{Fe_n} d\phi)e_n) d\phi V_\lambda) - \exp(\tfrac{1}{2}|d\phi|^2)\tfrac{F}{\overline{F}}\overline{C}(d\phi e_i, d\phi e_i, \overline{P}(d\phi F e_n, \widetilde{\nabla}_{e_n^H} d\phi V_\lambda) d\phi V_\lambda) \Big\} dV_{SM},$$

$$\Xi_6 = -\sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \langle \widetilde{\nabla}_{d\phi V_\lambda} d\phi V_\lambda, \tau\rangle dV_{SM}. \tag{5.7}$$

We have [3]

(5.8)

$$\begin{aligned}(\widetilde{\nabla}_{X^H}\widetilde{\nabla}_Z d\phi)Y = & -d\phi R(X,Y)Z + (\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_Z d\phi)X \\ & +(\widetilde{\nabla}_Y d\phi)(\nabla_{X^H} Z) - (\widetilde{\nabla}_X d\phi)(\nabla_{Y^H} Z) \\ & +\overline{R}(d\phi X, d\phi Y)d\phi Z + \frac{F}{\overline{F}}\overline{P}(d\phi X, (\widetilde{\nabla}_{e_n} d\phi)Y)d\phi Z \\ & -\frac{F}{\overline{F}}\overline{P}(d\phi Y, (\widetilde{\nabla}_{e_n} d\phi)X)d\phi Z.\end{aligned}$$

Set $X = Z = V, Y = e_i$ in (5.8). We obtain:

(5.9)

$$\begin{aligned} & -\langle \overline{R}(d\phi e_i, d\phi V)d\phi V, d\phi e_i\rangle + \frac{F}{\overline{F}}\langle \overline{P}(d\phi V, (\widetilde{\nabla}_{e_i} d\phi)e_n)d\phi V, d\phi e_i\rangle \\ & -\tfrac{F}{\overline{F}}\langle \overline{P}(d\phi e_i, (\widetilde{\nabla}_{e_i} d\phi)V)d\phi V, d\phi e_i\rangle \\ = & -\langle d\phi R(e_i, V)V, d\phi e_i\rangle + \langle (\widetilde{\nabla}_{V^H}\widetilde{\nabla}_V d\phi)e_i, d\phi e_i\rangle \\ & -\langle (\widetilde{\nabla}_{e_i^H}\widetilde{\nabla}_V d\phi)V, d\phi e_i\rangle - \langle (\widetilde{\nabla}_{e_i} d\phi)(\nabla_{V^H} V), d\phi e_i\rangle \\ & +\langle (\widetilde{\nabla}_V d\phi)(\nabla_{e_i^H} V), d\phi e_i\rangle.\end{aligned}$$

We need following lemma.

Lemma 5.1.

$$\begin{aligned}\Xi_4 = & \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2)\Big\{ - \langle d\phi R(e_i, V_\lambda)V_\lambda, d\phi e_i\rangle \\ & -\langle \widetilde{\nabla}_{e_i^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)V_\lambda\}, d\phi e_i\rangle \\ & +\langle \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda} d\phi)e_i], d\phi e_i\rangle - \langle (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{V_\lambda^H} e_i), d\phi e_i\rangle \Big\} dV_{SM}.\end{aligned}$$

Proof. Using the fact $\widetilde{\nabla}_{\frac{\partial}{\partial Y^i}} V_\lambda = 0$ and (5.1), we obtain:

$$\sum_\lambda \widetilde{\nabla}_{V_\lambda^H} V_\lambda = \sum_\lambda \widetilde{\nabla}_{V_\lambda} V_\lambda = \sum_\lambda v_\lambda^i v_\lambda^{n+1} \lambda_i e_i = 0, \tag{5.10}$$

$$\sum_\lambda (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{e_i^H} V_\lambda) = \sum_\lambda (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{e_i} V_\lambda) = \sum_\lambda v_\lambda^j v_\lambda^{n+1} \lambda_i (\widetilde{\nabla}_{e_j} d\phi) e_i = 0, \tag{5.11}$$

$$\begin{aligned}\sum_\lambda (\widetilde{\nabla}_{e_i^H} \widetilde{\nabla}_{V_\lambda} d\phi)V_\lambda = & \textstyle\sum_\lambda \widetilde{\nabla}_{e_i^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)V_\lambda\} - (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{e_i^H} V_\lambda) \\ = & \textstyle\sum_\lambda \widetilde{\nabla}_{e_i^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)V_\lambda\},\end{aligned} \tag{5.12}$$

$$\sum_\lambda (\widetilde{\nabla}_{V_\lambda^H} \widetilde{\nabla}_{V_\lambda} d\phi)e_i = \sum_\lambda \widetilde{\nabla}_{V_\lambda^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)e_i\} - (\widetilde{\nabla}_{V_\lambda} d\phi)(\nabla_{V_\lambda^H} e_i). \tag{5.13}$$

By (5.10) $-$ (5.13), the result follows directly. □

Similarly, we also have the following result, by $\nabla_{V^H} e_n = 0$.

Lemma 5.2.

$$\begin{aligned}\Xi_5 = & \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2)\Big\{ - \overline{C}(d\phi e_i, d\phi e_i, d\phi R(Fe_n, V_\lambda)V_\lambda) \\ & -\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)V_\lambda\}) \\ & +\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda} d\phi)Fe_n)]\Big\} dV_{SM}.\end{aligned}$$

Lemma 5.3.

$$\begin{aligned}
&-\sum_{\lambda}\int_{SM}\Big\{\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda],d\phi e_i\rangle\\
&+\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda])\Big\}dV_{SM}+\Xi_6\\
=\ &0.
\end{aligned}$$

Proof. Putting $\psi=\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)F\omega^n$, using Lemma 2.1 and $\nabla_{V^H}F\omega^n=0$, we get:

$$\begin{aligned}
div_{\widehat{g}}\psi=\ &(\widetilde{\nabla}_{Fe_n^H}\exp(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)\\
&+\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)\\
&+2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{Fe_n^H}d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)\\
&+\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda]).
\end{aligned}\tag{5.14}$$

By integrating (5.14), we obtain:

$$\begin{aligned}
&\sum_{\lambda}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i,d\phi e_i,\widetilde{\nabla}_{Fe_n^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda])dV_{SM}\\
=\ &-\sum_{\lambda}\int_{SM}\Big\{(\widetilde{\nabla}_{Fe_n^H}\exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)\\
&+\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)\\
&+2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{Fe_n^H}d\phi e_i,d\phi e_i,(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)\Big\}dV_{SM}.
\end{aligned}\tag{5.15}$$

On the other hand, let $\psi=\exp(\frac{1}{2}|d\phi|^2)\langle(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda,d\phi e_i\rangle\omega^i$. We get:

$$\begin{aligned}
div_{\widehat{g}}\psi=\ &\exp(\tfrac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda],d\phi e_i\rangle\\
&+\langle(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda,(\widetilde{\nabla}_{e_i^H}\exp(\tfrac{1}{2}|d\phi|^2)d\phi)e_i\rangle\\
&+2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda,d\phi e_i,\widetilde{\nabla}_{e_i^H}d\phi Fe_n).
\end{aligned}\tag{5.16}$$

It follows from (5.16) that

$$\begin{aligned}
&\sum_{\lambda}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{e_i^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda],d\phi e_i\rangle dV_{SM}\\
=\ &-\sum_{\lambda}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\langle(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda,(\widetilde{\nabla}_{e_i^H}\exp(\frac{1}{2}|d\phi|^2)d\phi)e_i\rangle\\
&+2\overline{C}((\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda,d\phi e_i,\widetilde{\nabla}_{e_i^H}d\phi Fe_n)\Big\}dV_{SM}.
\end{aligned}\tag{5.17}$$

Because ϕ is an exponentially harmonic map, by (5.15) and (5.17), we obtain immediately:

$$\begin{aligned}
& -\sum_{\lambda}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{\overline{C}(d\phi e_i, d\phi e_i, \nabla_{Fe_n^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda)] \\
(5.18)\quad & +\langle\widetilde{\nabla}_{e_i^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)V_\lambda], d\phi e_i\rangle\Big\}dV_{SM}+\Xi_6 \\
= & \sum_{\lambda}\int_{SM}\langle\widetilde{\nabla}_{V_\lambda}d\phi V_\lambda-\widetilde{\nabla}_{d\phi V_\lambda}d\phi V_\lambda,\tau\rangle dV_{SM} \\
= & \ 0.
\end{aligned}$$

□

By the Gauss equation, we have

$$\sum_{\lambda}\overline{C}(d\phi e_i, d\phi e_i, d\phi R(Fe_n, V_\lambda)V_\lambda)=0. \tag{5.19}$$

It follows from (5.19), Lemmas 5.1 – 5.3 that

$$\begin{aligned}
& \Xi_4+\Xi_5+\Xi_6 \\
= & \sum_{\lambda}\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{-\langle d\phi R(e_i,V_\lambda)V_\lambda, d\phi e_i\rangle \\
(5.20)\quad & +\langle\widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)e_i], d\phi e_i\rangle-\langle(\widetilde{\nabla}_{V_\lambda}d\phi)(\nabla_{V_\lambda^H}e_i), d\phi e_i\rangle \\
& +\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)]\Big\}dV_{SM}.
\end{aligned}$$

Lemma 5.4.

$$\begin{aligned}
& \sum_{\lambda}\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\langle\widetilde{\nabla}_{V_\lambda^H}\{(\widetilde{\nabla}_{V_\lambda}d\phi)e_i\}, d\phi e_i\rangle dV_{SM}+\Xi_2 \\
= & \frac{1}{C_{n-1}}\int_{SM}\Big\{-\sum_{\lambda}(\widetilde{\nabla}_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2))\langle(\widetilde{\nabla}_{V_\lambda}d\phi)e_i, d\phi e_i\rangle \\
& +\exp(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle\lambda_i^2 \\
& -\sum_{\lambda}\exp(\frac{1}{2}|d\phi|^2)\langle d\phi(\nabla_{V_\lambda^H}e_i), (\widetilde{\nabla}_{V_\lambda}d\phi)e_i)\rangle \\
& -\textstyle\sum_{\lambda}2\exp(\frac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda}d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}d\phi Fe_n)\Big\}dV_{SM}.
\end{aligned}$$

Proof. Let $\psi = \sum_{\lambda,i,j} \exp(\frac{1}{2}|d\phi|^2)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle v_\lambda^j \omega^j$, where $v_\lambda^i = \langle \Lambda_\lambda, e_i\rangle$. We have

$$\begin{aligned} div_{\widehat{g}}\psi = & \sum_\lambda \Big\{\nabla_{e_j^H}\{\exp(\frac{1}{2}|d\phi|^2)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle\} v_\lambda^j \\ & + \exp(\tfrac{1}{2}|d\phi|^2)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle \nabla_{e_j^H}(v_\lambda^k \omega^k)e_j\Big\} \\ = & \sum_\lambda \Big\{(\nabla_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2))\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle \\ & + \exp(\tfrac{1}{2}|d\phi|^2)\langle \nabla_{V_\lambda^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)e_i\}, d\phi e_i\rangle \\ & + \exp(\frac{1}{2}|d\phi|^2)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, \nabla_{V_\lambda^H} d\phi e_i\rangle \\ & +2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H} d\phi F e_n)\Big\}. \end{aligned} \tag{5.21}$$

Integrating (5.21), yields:

$$\begin{aligned} & \sum_\lambda \int_{SM} \langle \widetilde{\nabla}_{V_\lambda^H}\{(\widetilde{\nabla}_{V_\lambda} d\phi)e_i\}, d\phi e_i\rangle dV_{SM} \\ = & -\sum_\lambda \int_{SM} \Big\{(\nabla_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2))\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle \\ & + \exp(\frac{1}{2}|d\phi|^2)\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, \nabla_{V_\lambda^H} d\phi e_i\rangle \\ & +2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H} d\phi F e_n)\Big\} dV_{SM}. \end{aligned} \tag{5.22}$$

On the other hand, we also have

$$\begin{aligned} & \Xi_2 \\ = & \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2)\Big\{\langle(\widetilde{\nabla}_{e_i^H} d\phi)V_\lambda, (\widetilde{\nabla}_{e_i} d\phi)V_\lambda\rangle + \langle d\phi e_i, d\phi e_i\rangle \lambda_i^2\Big\} \\ = & \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2)\Big\{\langle \widetilde{\nabla}_{V_\lambda^H}(d\phi e_i), (\widetilde{\nabla}_{V_\lambda} d\phi)e_i)\rangle \\ & -\langle d\phi(\nabla_{V_\lambda^H} e_i), (\widetilde{\nabla}_{V_\lambda} d\phi)e_i)\rangle + \langle d\phi e_i, d\phi e_i\rangle \lambda_i^2\Big\} dV_{SM}. \end{aligned} \tag{5.23}$$

The result follows by (5.22) and (5.23). $\square$

Applying Lemma 5.4 to (5.20), we get:

(5.24)
$$\begin{aligned}
&\Xi_2+\Xi_4+\Xi_5+\Xi_6\\
&=\frac{1}{C_{n-1}}\int_{SM}\Big\{-\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)\langle d\phi R(e_i,V_\lambda)V_\lambda, d\phi e_i\rangle\\
&+\exp(\frac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle\lambda_i^2-\sum_\lambda(\widetilde{\nabla}_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2))\\
&\langle(\widetilde{\nabla}_{V_\lambda}d\phi)e_i, d\phi e_i\rangle\\
&+\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n])\\
&-\sum_\lambda 2\exp(\frac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda}d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}d\phi Fe_n)\Big\}dV_{SM}.
\end{aligned}$$

Let $\psi=\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)v_\lambda^k\omega^k$. We obtain:

(5.25)
$$\begin{aligned}
&div_{\widehat{g}}\psi\\
=\;&(\widetilde{\nabla}_{V_\lambda^H}\exp(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\\
&+\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\\
&+2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{V_\lambda^H}d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\\
&+\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n]).
\end{aligned}$$

Integrating (5.25), implies:

(5.26)
$$\begin{aligned}
&\sum_\lambda\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n])dV_{SM}\\
=\;&-\sum_\lambda\frac{1}{C_{n-1}}\int_{SM}\Big\{(\widetilde{\nabla}_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\\
&+\exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\\
&+2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{V_\lambda^H}d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\Big\}dV_{SM}.
\end{aligned}$$

On the other hand, by (5.1), we get:

(5.27)
$$\begin{aligned}
&\sum_\lambda\exp(\frac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda}d\phi)Fe_n)\\
=\;&\sum_\lambda\exp(\frac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda)-d\phi(\nabla_{Fe_n}V_\lambda))\\
=\;&\sum_\lambda\exp(\frac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda)),
\end{aligned}$$

and

$$(5.28)\quad \begin{aligned} & 2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{V_\lambda^H} d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda} d\phi)Fe_n) \\ = \; & 2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{V_\lambda^H} d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{Fe_n} d\phi)V_\lambda). \end{aligned}$$

Substituting (5.27) and (5.28) into (5.26), we get:

$$(5.29)\quad \begin{aligned} & \sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \exp(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}[(\widetilde{\nabla}_{V_\lambda} d\phi)Fe_n])dV_{SM} \\ = \; & -\sum_\lambda \frac{1}{C_{n-1}} \int_{SM} \Big\{ (\widetilde{\nabla}_{V_\lambda^H} \exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda} d\phi)Fe_n) \\ & + \exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda)) \\ & + 2\exp(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{V_\lambda^H} d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{Fe_n} d\phi)V_\lambda) \Big\} dV_{SM}. \end{aligned}$$

It is easy to see that

$$(5.30)\quad \begin{aligned} & \sum_\lambda 2\exp(\frac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H} d\phi Fe_n) \\ = \; & \sum_\lambda 2\exp(\frac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda^H} d\phi)Fe_n) \\ = \; & \sum_\lambda 2\exp(\frac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{e_i}(d\phi V_\lambda), d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda)). \end{aligned}$$

Substituting (5.29) and (5.30) into (5.24), we get:

$$(5.31)\quad \begin{aligned} & \Xi_2 + \Xi_4 + \Xi_5 + \Xi_6 \\ = \; & \frac{1}{C_{n-1}} \int_{SM} \Big\{ -\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)\langle d\phi R(e_i, V_\lambda)V_\lambda, d\phi e_i\rangle \\ & + \exp(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle \lambda_i^2 \\ & - \sum_\lambda (\widetilde{\nabla}_{V_\lambda^H} \exp(\frac{1}{2}|d\phi|^2))\langle(\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle \\ & - \sum_\lambda (\widetilde{\nabla}_{V_\lambda^H} \exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda} d\phi)Fe_n) \\ & - \textstyle\sum_\lambda \exp(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{Fe_n} d\phi)V_\lambda) \\ & - \sum_\lambda 4\exp(\frac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H} d\phi Fe_n) \Big\} dV_{SM}. \end{aligned}$$

It can be seen from Lemma 4.1 that

$$(5.32)\quad \begin{aligned} &\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{V_\lambda^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda)) \\ = &\sum_\lambda \Big\{ \exp(\frac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{(d\phi V_\lambda)^H}\overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda) \\ &+ \exp(\tfrac{1}{2}|d\phi|^2)\overline{\mathcal{C}}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda, \widetilde{\nabla}_{Fe_n} d\phi V_\lambda)\Big\}, \end{aligned}$$

which together with (5.31), (5.4) yields:
(5.33)

$$\begin{aligned} &\Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 + \Xi_6 \\ = &\frac{1}{C_{n-1}}\int_{SM}\Big\{ -\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)\langle d\phi R(e_i, V_\lambda)V_\lambda, d\phi e_i\rangle \\ &+ \exp(\frac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle \lambda_i^2 - \sum_\lambda (\widetilde{\nabla}_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2)) \\ &\langle (\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle \\ &- \sum_\lambda (\widetilde{\nabla}_{V_\lambda^H}\exp(\frac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_\lambda} d\phi)Fe_n)\Big\} dV_{SM} \\ = &\frac{1}{C_{n-1}}\int_{SM}\Big\{ -\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)\langle d\phi R(e_i, V_\lambda)V_\lambda, d\phi e_i\rangle \\ &+ \exp(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle\lambda_i^2 \\ &- \sum_\lambda \exp(\frac{1}{2}|d\phi|^2)[\langle (\widetilde{\nabla}_{V_\lambda^H} d\phi)e_i + d\phi(\nabla_{V_\lambda^H} e_i), d\phi e_i\rangle \\ &+ \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda))][\langle (\widetilde{\nabla}_{V_\lambda} d\phi)e_i, d\phi e_i\rangle \\ &+ \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda))]\Big\} dV_{SM}. \end{aligned}$$

Putting $e_i = u_i^j \frac{\partial}{\partial x^j}$, we have
(5.34)

$$\begin{aligned} &\sum_i \langle d\phi(\nabla_{V^H} e_i), d\phi e_i\rangle \\ = &-\tfrac{1}{2} g^{kl}\langle \Lambda, \tfrac{\partial}{\partial X^k}\rangle g^{si} g^{tj} \tfrac{\partial g^{ij}}{\partial x^l}\langle d\phi \tfrac{\partial}{\partial x^s}, d\phi \tfrac{\partial}{\partial x^t}\rangle \\ &+ \frac{1}{2} g^{kl}\langle \Lambda, \frac{\partial}{\partial X^k}\rangle g^{ht} g^{is}\{\frac{\partial g_{il}}{\partial x^h} + \frac{\partial g_{ih}}{\partial x^l} - \frac{\partial g_{hl}}{\partial x^i}\}\langle d\phi \frac{\partial}{\partial x^s}, d\phi \frac{\partial}{\partial x^t}\rangle \\ = &\ 0. \end{aligned}$$

Substituting (5.34) into (5.33), we obtain:

$$
\begin{aligned}
&\Xi_2+\Xi_3+\Xi_4+\Xi_5+\Xi_6\\
= &\frac{1}{C_{n-1}}\int_{SM}\Big\{-\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)\langle d\phi R(e_i,V_\lambda)V_\lambda, d\phi e_i\rangle\\
&+\exp(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle\lambda_i^2\\
&-\sum_\lambda \exp(\frac{1}{2}|d\phi|^2)[\langle(\widetilde{\nabla}_{V_\lambda^H}d\phi)e_i, d\phi e_i\rangle\\
&+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_\lambda))]^2\Big\}dV_{SM}.
\end{aligned}
\tag{5.35}
$$

On the other hand, we also have

$$
\begin{aligned}
&\Xi_1\\
= &\sum_\lambda \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{(\langle\widetilde{\nabla}_{e_i}d\phi)V_\lambda + d\phi(\nabla_{e_i}V_\lambda), d\phi e_i\rangle\\
&+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi Fe_n))\Big\}^2 dV_{SM}\\
= &\sum_\lambda \frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\Big\{(\langle\widetilde{\nabla}_{e_i}d\phi)V_\lambda, d\phi e_i\rangle\\
&+\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_\lambda^H}(d\phi Fe_n))\Big\}^2 dV_{SM}\\
&+\frac{1}{C_{n-1}}\int_{SM}\exp(\frac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle^2\lambda_i^2 dV_{SM}.
\end{aligned}
\tag{5.36}
$$

It is easy to see from (5.35) and (5.36) that

$$
\begin{aligned}
&\textstyle\sum_\lambda I(d\phi V_\lambda, d\phi V_\lambda)\\
= &\frac{1}{C_{n-1}}\int_{SM}\Big\{-\exp(\frac{1}{2}|d\phi|^2)(\lambda_i\sum_k\lambda_k-\lambda_i^2)\langle d\phi e_i, d\phi e_i\rangle\\
&+\exp(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle\lambda_i^2+\exp(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_i\rangle^2\lambda_i^2\Big\}dV_{SM}\\
\leq &\frac{1}{C_{n-1}}\int_{SM}|d\phi|^2\exp(\frac{1}{2}|d\phi|^2)\Big\{(2\lambda_i^2-\lambda_i\sum_k\lambda_k)+|d\phi|^2\lambda_n^2\Big\}dV_{SM}.
\end{aligned}
\tag{5.37}
$$

From (5.37), we get immediately the following result.

Theorem 5.5. *Let M^n be a compact convex hypersurface of the Euclidean space E^{n+1}, with its principal curvatures sorted as $0<\lambda_1\leq\cdots\leq\lambda_n$, satisfying $\lambda_n<\sum_{j=1}^{n-1}\lambda_j$. There is no non-degenerate stable exponentially harmonic map ϕ from M^n to any Finsler manifold with $|d\phi|^2<\frac{1}{\lambda_n^2}min_{1\leq i\leq n}\{\lambda_i(\sum_{j=1}^n\lambda_j-2\lambda_i)\}$.*

6. Stability in the case target manifold is a convex hypersurface

When the target manifold is M^n, let $\{\overline{e_\alpha}\}$ and $\{e_i\}$ be the orthonormal frame of $\overline{M}$ and M^n. The second variation formula of the exponentially harmonic map $\phi : \overline{M}^m \to M^n$ can be written as:

(6.1)

$$\begin{aligned} &\sum_\lambda I(V_\lambda, V_\lambda) \\ = \; & \frac{1}{C_{n-1}} \int_{S\overline{M}} \Big\{ \exp(\frac{1}{2}|d\phi|^2)[\langle \widetilde{\nabla}_{\overline{e}_\alpha} V_\lambda, \widetilde{\nabla}_{\overline{e}_\alpha} V_\lambda \rangle - \langle R(d\phi \overline{e}_\alpha, V_\lambda)V_\lambda, d\phi \overline{e}_\alpha \rangle] \\ & + \exp(\tfrac{1}{2}|d\phi|^2)' \langle \widetilde{\nabla}_{\overline{e}_\alpha} V_\lambda, d\phi \overline{e}_\alpha \rangle^2 \Big\} dV_{S\overline{M}} \\ \leq \; & \frac{1}{C_{n-1}} \int_{S\overline{M}} |d\phi|^2 \exp(\frac{1}{2}|d\phi|^2) \Big\{ (2\lambda_i^2 - \lambda_i \sum_k \lambda_k) + |d\phi|^2 \lambda_n^2 \Big\} dV_{S\overline{M}}. \end{aligned}$$

So, we obtain immediately the following result.

Theorem 6.1. *Let M^n be a compact convex hypersurface of the Euclidean space E^{n+1}, with its principal curvatures sorted as $0 < \lambda_1 \leq \cdots \leq \lambda_n$, satisfying $\lambda_n < \sum_{j=1}^{n-1} \lambda_j$. There is no non-degenerate stable exponentially harmonic map from a compact Finsler manifold $\overline{M}$ to M^n with $|d\phi|^2 < \frac{1}{\lambda_n^2} min_{1\leq i \leq n}\{\lambda_i(\sum_{j=1}^n \lambda_j - 2\lambda_i)\}$.*

Combining Theorem 5.5 and Theorem 6.1 completes the proof of Theorem 1.1.

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