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SOME PROPERTIES OF CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS WITH RESPECT TO $2k$-SYMMETRIC CONJUGATE POINTS

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Abstract. We introduce and investigate some new subclasses of close-to-convex and quasi-convex functions with respect to $2k$-symmetric conjugate points. Such results as inclusion relationships, integral representations, convolution properties, integral-preserving properties, growth and covering theorems for these function classes are proved. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are *analytic* in the *open* unit disk,

$$\mathbb{U} := \{z : \; z \in \mathbb{C} \quad \text{and} \quad | z |< 1\}.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$. Also, let $\mathcal{P}$ denote the class of functions of the form

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (z \in \mathbb{U}),$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$\Re(\mathfrak{p}(z)) > 0, \qquad (z \in \mathbb{U})$$

We denote by $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in $\mathbb{U}$. Thus, by definition, we have (see, for details, [2, 4, 7, 9, 11, 14, 15, 16, 17, 18, 19])

$$\mathcal{S}^* := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\},$$

$$\mathcal{K} := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\},$$

$$\mathcal{C} := \left\{ f : f \in \mathcal{A}, \;\; \exists g \in \mathcal{S}^* \;\; \text{such that} \quad \Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\},$$

and

$$\mathcal{C}^* := \left\{ f : f \in \mathcal{A}, \;\; \exists g \in \mathcal{K} \;\; \text{such that} \quad \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\}.$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad | \omega(z) |< 1, \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big), \qquad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z), \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z), \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*(\phi)$ if it satisfies the subordination condition,

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (z \in \mathbb{U};\ \phi \in \mathcal{P}).$$

The class $\mathcal{S}^*(\phi)$ and a corresponding convex class $\mathcal{K}(\phi)$ were defined by Ma and Minda [6]. Furthermore, the results about the convex class $\mathcal{K}(\phi)$ can be easily obtained from the corresponding results of functions in $\mathcal{S}^*(\phi)$.

A function $f \in \mathcal{A}$ is in the class $\mathcal{C}(\alpha, \phi)$ if it satisfies the subordination condition,

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec \phi(z), \qquad (z \in \mathbb{U};\ \alpha \geq 0;\ \phi \in \mathcal{P}).$$

The class $\mathcal{C}(\alpha, \phi)$ is a generalization of the well known α-convex functions which was studied by various authors (see [8, 10, 16, 18, 21]).

Al-Amiri et al. [1] once introduced and investigated a class of functions starlike with respect to $2k$-symmetric conjugate points, which satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f_{2k}(z)}\right) > 0, \qquad (z \in \mathbb{U}),$$

where $k \geq 1$ is a fixed positive integer and f_{2k} is defined by
(1.2)

$$f_{2k}(z) := \frac{1}{2k}\sum_{\nu=0}^{k-1}\left(\varepsilon^{-\nu} f(\varepsilon^{\nu} z) + \varepsilon^{\nu}\overline{f(\varepsilon^{\nu}\overline{z})}\right) \qquad \left(\varepsilon = \exp\left(\frac{2\pi i}{k}\right)\right).$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{sc}^{(k)}(\phi)$ if it satisfies the subordination condition,

$$\frac{zf'(z)}{f_{2k}(z)} \prec \phi(z), \qquad (z \in \mathbb{U}),$$

where, $\phi \in \mathcal{P}$ and f_{2k} is defined by (1.2). Also, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_{sc}^{(k)}(\phi)$ if and only if

$$zf' \in \mathcal{S}_{sc}^{(k)}(\phi), \qquad (z \in \mathbb{U}).$$

The classes $\mathcal{S}_{sc}^{(k)}(\phi)$ of functions starlike with respect to $2k$-symmetric conjugate points and $\mathcal{C}_{sc}^{(k)}(\phi)$ of functions convex with respect to $2k$-symmetric conjugate points were studied recently by Wang and Gao [20].

Motivated by the above-mentioned function classes, we now introduce the following subclasses of analytic functions with respect to $2k$-symmetric conjugate points.

Definition 1.1. Let $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ denote the class of functions f in $\mathcal{A}$ satisfying

$$\frac{f(z)f'(z)}{z} \neq 0, \qquad (z \in \mathbb{U}) \tag{1.3}$$

and the subordination condition

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)} \prec \phi(z), \qquad (z \in \mathbb{U}), \tag{1.4}$$

where, $\phi \in \mathcal{P}$, $0 \leq \lambda \leq 1$, $k \geq 1$ is a fixed positive integer and f_{2k} is defined by (1.2). Moreover, a function $f \in \mathcal{A}$ is in the class $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$ if and only if

$$zf' \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi).$$

Definition 1.2. Let $\mathcal{T}_{sc}^{(k)}(\lambda, \phi; g)$ denote the class of functions f in $\mathcal{A}$ satisfying (1.3) and the subordination condition,

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)g_{2k}(z) + \lambda z g'_{2k}(z)} \prec \phi(z) \qquad (z \in \mathbb{U}),$$

where, $\phi \in \mathcal{P}$, $0 \leq \lambda \leq 1$, $k \geq 1$ is a fixed positive integer and

$$g_{2k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} \left(\varepsilon^{-\nu} g(\varepsilon^{\nu} z) + \varepsilon^{\nu} \overline{g(\varepsilon^{\nu} \overline{z})} \right) \tag{1.5}$$

with $g \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$.

Definition 1.3. Let $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ denote the class of functions f in $\mathcal{A}$ satisfying (1.3) and the subordination condition,

$$(1-\alpha)\frac{zf'(z)}{f_{2k}(z)} + \alpha \frac{(zf'(z))'}{f'_{2k}(z)} \prec \phi(z), \quad (z \in \mathbb{U}), \tag{1.6}$$

where, $\phi \in \mathcal{P}$, $\alpha \geq 0$, $k \geq 1$ is a fixed positive integer and f_{2k} is defined by (1.2).

Definition 1.4. Let $\mathcal{H}_{sc}^{(k)}(\alpha, \phi; g)$ denote the class of functions f in $\mathcal{A}$ satisfying (1.3) and the subordination condition,

$$(1-\alpha)\frac{zf'(z)}{g_{2k}(z)} + \alpha\frac{(zf'(z))'}{g'_{2k}(z)} \prec \phi(z), \qquad (z \in \mathbb{U}),$$

where, $\phi \in \mathcal{P}$, $\alpha \geq 0$, $k \geq 1$ is a fixed positive integer and g_{2k} is defined by (1.5) with $g \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi)$.

Remark 1.5. In view of definitions 1.1 and 1.3, we know that the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ and $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ unify the classes of starlike and convex functions with respect to $2k$-symmetric conjugate points.

In our proposed investigation of the function classes

$$\mathcal{S}_{sc}^{(k)}(\lambda, \phi), \quad \mathcal{C}_{sc}^{(k)}(\lambda, \phi), \quad \mathcal{T}_{sc}^{(k)}(\lambda, \phi; g), \quad \mathcal{K}_{sc}^{(k)}(\alpha, \phi) \quad \text{and} \quad \mathcal{H}_{sc}^{(k)}(\alpha, \phi; g),$$

we need the following lemmas.

Lemma 1.6. (See [3, 6]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose that ϕ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \textit{and} \quad \Re(\beta\phi(z) + \gamma) > 0, \qquad (z \in \mathbb{U}).$$

If $\mathfrak{m}$ is analytic in $\mathbb{U}$ with $\mathfrak{m}(0) = 1$, then the subordination

$$\mathfrak{m}(z) + \frac{z\mathfrak{m}'(z)}{\beta\mathfrak{m}(z) + \gamma} \prec \phi(z)$$

implies that

$$\mathfrak{m}(z) \prec \phi(z).$$

Lemma 1.7. (See [12]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose that ϕ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \textit{and} \quad \Re(\beta\phi(z) + \gamma) > 0, \qquad (z \in \mathbb{U}).$$

Also, let

$$\mathfrak{q}(z) \prec \phi(z).$$

If $\mathfrak{u} \in \mathcal{P}$ and satisfies the subordination

$$\mathfrak{u}(z) + \frac{z\mathfrak{u}'(z)}{\beta\mathfrak{q}(z) + \gamma} \prec \phi(z),$$

then,

$$\mathfrak{u}(z) \prec \phi(z).$$

Lemma 1.8. (See [20]) *Let $\phi \in \mathcal{P}$. Then,*

$$\mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Lemma 1.9. (See [20]) *Let* $\phi \in \mathcal{P}$. *Then,*

$$\mathcal{C}_{sc}^{(k)}(\phi) \subset \mathcal{C}^* \subset \mathcal{C}.$$

Lemma 1.10. (See [10]) *Let*

$$\alpha \geq 0 \quad \text{and} \quad \phi \in \mathcal{P}.$$

Then, $f \in \mathcal{C}(\alpha, \phi)$ *if and only if*

$$F(z) = f(z) \left(\frac{zf'(z)}{f(z)} \right)^{\alpha} \in \mathcal{S}^*(\phi).$$

Here, we aim at proving such results as inclusion relationships, integral representations, convolution properties, integral-preserving properties, growth and covering theorems for the function classes

$$\mathcal{S}_{sc}^{(k)}(\lambda, \phi), \ \mathcal{C}_{sc}^{(k)}(\lambda, \phi), \ \mathcal{T}_{sc}^{(k)}(\lambda, \phi; g), \ \mathcal{K}_{sc}^{(k)}(\alpha, \phi) \quad \text{and} \quad \mathcal{H}_{sc}^{(k)}(\alpha, \phi; g).$$

The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

2. Properties of the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$, $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$ and $\mathcal{T}_{sc}^{(k)}(\lambda, \phi; g)$

First, we give some inclusion relationships for the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$, which tell us that $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$ are subclasses of the classes $\mathcal{C}$ and $\mathcal{C}^*$, respectively.

Theorem 2.1. *Let*

$$\phi \in \mathcal{P} \quad \textit{and} \quad 0 \leq \lambda \leq 1.$$

Then,

$$\mathcal{S}_{sc}^{(k)}(\lambda, \phi) \subset \mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Proof. Let $f \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$. Replacing z by $\varepsilon^{\mu} z$ ($\mu = 0, 1, 2, \ldots, k-1$) in (1.4), then (1.4) also holds true, that is,

$$\frac{\varepsilon^{\mu} z f'(\varepsilon^{\mu} z) + \lambda (\varepsilon^{\mu} z)^2 f''(\varepsilon^{\mu} z)}{(1-\lambda) f_{2k}(\varepsilon^{\mu} z) + \lambda \varepsilon^{\mu} z f'_{2k}(\varepsilon^{\mu} z)} \prec \phi(z). \tag{2.1}$$

It follows from (2.1) that

$$\frac{\overline{\varepsilon^{\mu} \overline{z}} \ \overline{f'(\varepsilon^{\mu} \overline{z})} + \lambda \overline{(\varepsilon^{\mu} \overline{z})^2} \ \overline{f''(\varepsilon^{\mu} \overline{z})}}{(1-\lambda) \overline{f_{2k}(\varepsilon^{\mu} \overline{z})} + \lambda \overline{\varepsilon^{\mu} \overline{z}} \ \overline{f'_{2k}(\varepsilon^{\mu} \overline{z})}} \prec \phi(z). \tag{2.2}$$

Note that

$$f_{2k}(\varepsilon^{\mu} z) = \varepsilon^{\mu} f_{2k}(z), \quad f'_{2k}(\varepsilon^{\mu} z) = f'_{2k}(z),$$
$$\overline{f_{2k}(\varepsilon^{\mu} \overline{z})} = \varepsilon^{-\mu} f_{2k}(z) \quad \text{and} \quad \overline{f'_{2k}(\varepsilon^{\mu} \overline{z})} = f'_{2k}(z).$$

Thus, we know that (2.1) and (2.2) can be written as follows:

$$\frac{zf'(\varepsilon^\mu z) + \lambda z^2 \varepsilon^\mu f''(\varepsilon^\mu z)}{(1-\lambda) f_{2k}(z) + \lambda z f'_{2k}(z)} \prec \phi(z), \tag{2.3}$$

and

$$\frac{z\overline{f'(\varepsilon^\mu \overline{z})} + \lambda z^2 \varepsilon^{-\mu} \overline{f''(\varepsilon^\mu \overline{z})}}{(1-\lambda) f_{2k}(z) + \lambda z f'_{2k}(z)} \prec \phi(z). \tag{2.4}$$

Upon summing (2.3) and (2.4), we obtain

$$\frac{\frac{z}{2}\left(f'(\varepsilon^\mu z) + \overline{f'(\varepsilon^\mu \overline{z})}\right) + \frac{\lambda z^2}{2}\left(\varepsilon^\mu f''(\varepsilon^\mu z) + \varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}\right)}{(1-\lambda) f_{2k}(z) + \lambda z f'_{2k}(z)} \prec \phi(z). \tag{2.5}$$

Letting $\mu = 0, 1, 2, \ldots, k-1$ in (2.5), repeatedly, and summing the resulting equations, we easily get:

$$\frac{\frac{z}{2k}\sum_{\mu=0}^{k-1}\left(f'(\varepsilon^\mu z) + \overline{f'(\varepsilon^\mu \overline{z})}\right) + \frac{\lambda z^2}{2k}\sum_{\mu=0}^{k-1}\left(\varepsilon^\mu f''(\varepsilon^\mu z) + \varepsilon^{-\mu}\overline{f''(\varepsilon^\mu \overline{z})}\right)}{(1-\lambda) f_{2k}(z) + \lambda z f'_{2k}(z)} \prec \phi(z),$$

or equivalently,

$$\frac{zf'_{2k}(z) + \lambda z^2 f''_{2k}(z)}{(1-\lambda) f_{2k}(z) + \lambda z f'_{2k}(z)} \prec \phi(z). \tag{2.6}$$

If we set

$$q(z) = \frac{zf'_{2k}(z)}{f_{2k}(z)}, \qquad (z \in \mathbb{U}),$$

then (2.6) can be written as follows:

$$\frac{zf'_{2k}(z) + \lambda z^2 f''_{2k}(z)}{(1-\lambda) f_{2k}(z) + \lambda z f'_{2k}(z)} = q(z) + \frac{\lambda z q'(z)}{(1-\lambda) + \lambda q(z)} \prec \phi(z). \tag{2.7}$$

We now prove

$$q(z) \prec \phi(z), \tag{2.8}$$

considering the following two cases.

(1) If $\lambda = 0$, from (2.7), we know that (2.8) obviously holds true.

(2) If $0 < \lambda \leq 1$, by noting that

$$\Re\left(\phi(z) + \frac{1}{\lambda} - 1\right) > 0,$$

in view of Lemma 1.6, we readily get (2.8).

By setting

$$p(z) = \frac{zf'(z)}{f_{2k}(z)} \qquad (z \in \mathbb{U}),$$

we get

$$\begin{aligned}
&\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)} \\
&\quad = \frac{(1-\lambda)zf'(z) + \lambda z (zf'(z))'}{(1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)} \\
&\quad = \frac{(1-\lambda)p(z)f_{2k}(z) + \lambda z \left[p'(z)f_{2k}(z) + p(z)f'_{2k}(z)\right]}{(1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)} \\
&\quad = \frac{(1-\lambda)p(z) + \lambda z p'(z) + \frac{\lambda z f'_{2k}(z)}{f_{2k}(z)} \cdot p(z)}{(1-\lambda) + \frac{\lambda z f'_{2k}(z)}{f_{2k}(z)}} \qquad (2.9) \\
&\quad = \frac{\lambda z p'(z) + p(z)\left[(1-\lambda) + \frac{\lambda z f'_{2k}(z)}{f_{2k}(z)}\right]}{(1-\lambda) + \frac{\lambda z f'_{2k}(z)}{f_{2k}(z)}} \\
&\quad = p(z) + \frac{\lambda z p'(z)}{(1-\lambda) + \lambda q(z)}.
\end{aligned}$$

Now, by similarly applying the above proof for

$$q(z) \prec \phi(z),$$

and using Lemma 1.7 in (2.9), we know that

$$p(z) = \frac{zf'(z)}{f_{2k}(z)} \prec \phi(z),$$

which implies that

$$\mathcal{S}_{sc}^{(k)}(\lambda, \phi) \subset \mathcal{S}_{sc}^{(k)}(\phi).$$

Furthermore, by Lemma 1.8, we find that

$$\mathcal{S}_{sc}^{(k)}(\lambda, \phi) \subset \mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

□

Remark 2.2. Indeed, if we set

$$F_{2k}(z) = (1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z)$$

with $f \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$, then it follows from (2.6) that

$$\frac{zF'_{2k}(z)}{F_{2k}(z)} \prec \phi(z),$$

which implies that

$$F_{2k}(z) = z + \sum_{l=1}^{\infty}(1+\lambda lk)\Re(a_{lk+1})z^{lk+1} \in \mathcal{S}^*(\phi).$$

Thus, the functions with missing and real coefficients belonging to the class $\mathcal{S}^*(\phi)$ are in the class $\mathcal{S}_{sc}^{(k)}(\lambda,\phi)$.

By means of Lemma 1.9, and making use of similar arguments given in the proof for Theorem 2.1, we easily get the following inclusion relationship for the class $\mathcal{C}_{sc}^{(k)}(\lambda,\phi)$.

Corollary 2.3. *Let*

$$\phi \in \mathcal{P} \quad \textit{and} \quad 0 \leq \lambda \leq 1.$$

Then,

$$\mathcal{C}_{sc}^{(k)}(\lambda,\phi) \subset \mathcal{C}_{sc}^{(k)}(\phi) \subset \mathcal{C}^* \subset \mathcal{C}.$$

We now give some integral representations for the function classes $\mathcal{S}_{sc}^{(k)}(\lambda,\phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda,\phi)$.

Theorem 2.4. *Let* $f \in \mathcal{S}_{sc}^{(k)}(\lambda,\phi)$ *with* $0 < \lambda \leq 1$. *Then,*

(2.10)

$$f_{2k}(z) =$$
$$\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z \exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^u \frac{\left(\phi(\omega(\varepsilon^\mu\zeta))+\overline{\phi(\omega(\varepsilon^\mu\overline{\zeta}))}\right)-2}{\zeta}d\zeta\right)u^{\frac{1}{\lambda}-1}du,$$

where, f_{2k} *is defined by* (1.2), ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\,\omega(z)\,| < 1, \quad (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{S}_{sc}^{(k)}(\lambda,\phi)$. We know that the condition (1.4) can be written as follows:

$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)} = \phi(\omega(z)), \qquad (z \in \mathbb{U}), \tag{2.11}$$

where, ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\,\omega(z)\,| < 1, \quad (z \in \mathbb{U}).$$

By similarly applying the arguments given in the proof for Theorem 2.1 to (2.11), we obtain:

(2.12)

$$\frac{(1-\lambda)zf'_{2k}(z)+\lambda z(zf'_{2k}(z))'}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\left(\phi(\omega(\varepsilon^\mu z))+\overline{\phi(\omega(\varepsilon^\mu\overline{z}))}\right).$$

It now follows from (2.12) that
(2.13)

$$\frac{(1-\lambda)f'_{2k}(z)+\lambda(zf'_{2k}(z))'}{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}-\frac{1}{z}=\frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{\left(\phi(\omega(\varepsilon^{\mu}z))+\overline{\phi(\omega(\varepsilon^{\mu}\overline{z}))}\right)-2}{z}.$$

Upon integrating (2.13), we readily get

$$\log\left(\frac{(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)}{z}\right)=$$
$$\tfrac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{\left(\phi(\omega(\varepsilon^{\mu}\zeta))+\overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\right)-2}{\zeta}d\zeta,$$

that is,

$$(1-\lambda)f_{2k}(z)+\lambda zf'_{2k}(z)=$$
$$z\cdot\exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{\left(\phi(\omega(\varepsilon^{\mu}\zeta))+\overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\right)-2}{\zeta}d\zeta\right). \tag{2.14}$$

The assertion (2.10) in Theorem 2.4 can now easily be derived from (2.14). □

Theorem 2.5. *Let $f\in\mathcal{S}_{sc}^{(k)}(\lambda,\phi)$ with $0<\lambda\leq 1$. Then,*

(2.15)

$$f(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z\int_0^u\exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^{\xi}\frac{\left(\phi(\omega(\varepsilon^{\mu}\zeta))+\overline{\phi(\omega(\varepsilon^{\mu}\overline{\zeta}))}\right)-2}{\zeta}d\zeta\right)$$
$$\cdot\,\phi(\omega(\xi))d\xi u^{\frac{1}{\lambda}-2}du.$$

where, ω is analytic in $\mathbb{U}$ with

$$\omega(0)=0\quad\textit{and}\quad|\omega(z)|<1,\quad(z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$. Then, in light of (2.11) and (2.14), we have

(2.16)
$$\begin{aligned}(1-\lambda)f'(z) &+ \lambda(zf'(z))' \\ &= \frac{(1-\lambda)f_{2k}(z) + \lambda z f_{2k}'(z)}{z} \cdot \phi(\omega(z)) \\ &= \exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{\left(\phi(\omega(\varepsilon^\mu\zeta)) + \overline{\phi(\omega(\varepsilon^\mu\overline{\zeta}))}\right) - 2}{\zeta} d\zeta\right) \cdot \phi(\omega(z)).\end{aligned}$$

Upon integrating (2.16) two times, we easily get (2.15). This completes the proof of Theorem 2.5. □

By similarly applying the arguments given in the proof for theorems 2.1 and 2.4 for the class $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$, we get the following results.

Corollary 2.6. *Let $f \in \mathcal{C}_{sc}^{(k)}(\lambda, \phi)$ with $0 < \lambda \le 1$. Then*
$$f_{2k}(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi \frac{\left(\phi(\omega(\varepsilon^\mu\zeta)) + \overline{\phi(\omega(\varepsilon^\mu\overline{\zeta}))}\right) - 2}{\zeta} d\zeta\right) d\xi u^{\frac{1}{\lambda}-2} du,$$
where, f_{2k} is defined by (1.2), ω is analytic in $\mathbb{U}$ with
$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1, \quad (z \in \mathbb{U}).$$

Corollary 2.7. *Let $f \in \mathcal{C}_{sc}^{(k)}(\lambda, \phi)$ with $0 < \lambda \le 1$. Then*
$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \frac{1}{t} \int_0^t \exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi \frac{\left(\phi(\omega(\varepsilon^\mu\zeta)) + \overline{\phi(\omega(\varepsilon^\mu\overline{\zeta}))}\right) - 2}{\zeta} d\zeta\right) \cdot \phi(\omega(\xi)) d\xi dt u^{\frac{1}{\lambda}-2} du, \tag{2.17}$$
where, ω is analytic in $\mathbb{U}$ with
$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1, \quad (z \in \mathbb{U}).$$

In view of theorem 2.1 and Corollary 2.3, we get other integral representations for the classes $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$.

Theorem 2.8. *Let $f \in \mathcal{C}_{sc}^{(k)}(\lambda, \phi)$ with*

$$0 \leq \lambda \leq 1 \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then,

$$f(z) = \int_0^z \frac{1}{\xi} \int_0^\xi \exp\left(\int_0^\zeta \frac{\phi(\omega_1(t)) - 1}{t} dt\right) \cdot \phi(\omega_2(\zeta)) d\zeta d\xi, \tag{2.18}$$

where, ω_j $(j = 1, 2)$ are analytic in $\mathbb{U}$ with

$$\omega_j(0) = 0 \quad \textit{and} \quad |\omega_j(z)| < 1, \quad (z \in \mathbb{U};\ j = 1, 2).$$

Proof. Suppose that $f \in \mathcal{C}_{sc}^{(k)}(\lambda, \phi)$. By Corollary 2.3, we know that

$$f_{2k}(z) \in \mathcal{K}(\phi),$$

that is,

$$\frac{(z f_{2k}'(z))'}{f_{2k}'(z)} = \phi(\omega_1(z)), \tag{2.19}$$

where, ω_1 is analytic in $\mathbb{U}$ with

$$\omega_1(0) = 0 \quad \text{and} \quad |\omega_1(z)| < 1, \quad (z \in \mathbb{U}).$$

By similarly applying the arguments given in the proof for Theorem 2.4 to (2.19), we get

$$\log\left(f_{2k}'(z)\right) = \int_0^z \frac{\phi(\omega_1(t)) - 1}{t} dt,$$

or equivalently,

$$f_{2k}'(z) = \exp\left(\int_0^z \frac{\phi(\omega_1(t)) - 1}{t} dt\right). \tag{2.20}$$

On the other hand, if $f \in \mathcal{C}_{sc}^{(k)}(\lambda, \phi)$, by Corollary 2.3 we also have $f \in \mathcal{C}_{sc}^{(k)}(\phi)$, which implies that

$$\frac{(z f'(z))'}{f_{2k}'(z)} = \phi(\omega_2(z)), \qquad (z \in \mathbb{U}), \tag{2.21}$$

where, ω_2 is analytic in $\mathbb{U}$ with

$$\omega_2(0) = 0 \quad \text{and} \quad |\omega_2(z)| < 1, \quad (z \in \mathbb{U}).$$

Therefore, in light of (2.20) and (2.21), we readily arrive at the assertion (2.18) in Theorem 2.8. $\square$

By similarly applying the arguments given in the proof for Theorem 2.8 for the class $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$, we can prove the following result.

Corollary 2.9. *Let* $f \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ *with*

$$0 \leq \lambda \leq 1 \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then,

$$f(z) = \int_0^z \exp\left(\int_0^\zeta \frac{\phi(\omega_1(t)) - 1}{t} dt\right) \cdot \phi(\omega_2(\zeta)) d\zeta,$$

where, the ω_j *are analytic in* $\mathbb{U}$ *with*

$$\omega_j(0) = 0 \quad \textit{and} \quad |\omega_j(z)| < 1, \quad (j = 1, 2).$$

Let $f,\ \mathfrak{h} \in \mathcal{A}$, where f is given by (1.1) and $\mathfrak{h}$ is defined by

$$\mathfrak{h}(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then, the Hadamard product (or convolution) $f * \mathfrak{h}$ is defined (as usual) by

$$(f * \mathfrak{h})(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (\mathfrak{h} * f)(z).$$

We now derive some convolution properties for the function classes $\mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda, \phi)$.

Theorem 2.10. *Let*

$$f \in \mathcal{A} \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then, $f \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$ *if and only if*

(2.22)

$$\frac{1}{z}\Bigg\{ f * \left[(1-\lambda)\left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2} h\right) + \lambda z \left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2} h\right)'\right] - \phi(e^{i\theta}) \cdot \overline{f * \left(\frac{1-\lambda}{2} h + \frac{\lambda}{2} z h'\right)(\overline{z})} \Bigg\} \neq 0,$$

for all $z \in \mathbb{U}$ *and* $0 \leq \theta < 2\pi$*, where* h *is given by* (2.27)*.*

Proof. Suppose that $f \in \mathcal{S}_{sc}^{(k)}(\lambda, \phi)$. Since the statement,

$$\frac{z f'(z) + \lambda z^2 f''(z)}{(1-\lambda) f_{2k}(z) + \lambda z f_{2k}'(z)} \prec \phi(z)$$

is equivalent to

$$\frac{z f'(z) + \lambda z^2 f''(z)}{(1-\lambda) f_{2k}(z) + \lambda z f_{2k}'(z)} \neq \phi(e^{i\theta}), \tag{2.23}$$

for all $z \in \mathbb{U}$ and $0 \leq \theta < 2\pi$, it is easy to know that the condition (2.23) can be written as follows:

(2.24)
$$\frac{1}{z}\left\{(1-\lambda)zf'(z) + \lambda z(zf'(z))' - [(1-\lambda)f_{2k}(z) + \lambda z f_{2k}'(z)]\phi(e^{i\theta})\right\} \neq 0.$$

On the other hand, it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{2.25}$$

Moreover, from the definition of f_{2k}, we know that

$$f_{2k}(z) = \frac{1}{2}\left((f*h)(z) + \overline{(f*h)(\overline{z})}\right), \tag{2.26}$$

where,

$$h(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{z}{1-\varepsilon^{\upsilon} z}. \tag{2.27}$$

Upon substituting (2.25) and (2.26)) into (2.24), we easily get (2.22). The proof of Theorem 2.10 is evidently complete. □

By similarly applying the arguments given in the proof for Theorem 2.10 for the class $\mathcal{C}_{sc}^{(k)}(\lambda,\phi)$, we prove the following convolution property.

Corollary 2.11. *Let*

$$f \in \mathcal{A} \quad \text{and} \quad \phi \in \mathcal{P}.$$

Then, $f \in \mathcal{C}_{sc}^{(k)}(\lambda,\phi)$ *if and only if*

$$\frac{1}{z}\left\{ f*\left\{ z\left[(1-\lambda)\left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2}h\right) + \lambda z\left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2}h\right)'\right]'\right\}\right.$$
$$\left. -\phi(e^{i\theta}) \cdot \overline{f*\left[z\left(\frac{1-\lambda}{2}h + \frac{\lambda}{2}zh'\right)'\right](\overline{z})}\right\} \neq 0,$$

for all $z \in \mathbb{U}$ *and* $0 \leq \theta < 2\pi$*, where* h *is given by* (2.27).

Next, we provide the growth and covering theorems for the classes $\mathcal{S}_{sc}^{(k)}(\lambda,\phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda,\phi)$. For this purpose, we assume that the function ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, $\phi(\mathbb{U})$ is convex and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. It is easy to check that the functions $k_{\phi n}$ $(n = 2, 3, \ldots)$ defined by

$$k_{\phi n}(0) = k_{\phi n}'(0) - 1 = 0,$$

and

$$1+\frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)}=\phi(z^{n-1}), \qquad (z\in\mathbb{U}),$$

are important examples of functions in $\mathcal{K}(\phi)$. The functions $h_{\phi n}$ satisfying

$$h_{\phi n}:=zk'_{\phi n}$$

are examples of functions in $\mathcal{S}^*(\phi)$.

For simplicity, we write

$$k_{\phi 2}=:k_\phi \quad \text{and} \quad h_{\phi 2}=:h_\phi. \tag{2.28}$$

In order to prove our next result, we need the following lemma.

Lemma 2.12. (See [5, 14]) *Let* $\min_{|z|=r}|\phi(z)|=\phi(-r)$, $\max_{|z|=r}|\phi(z)|=\phi(r)$ *with* $|z|=r<1$. *If*

$$f(z)=z+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\cdots\in\mathcal{K}(\phi),$$

then

$$\left(k'_\phi\left(-r^k\right)\right)^{\frac{1}{k}}\le|f'(z)|\le\left(k'_\phi\left(r^k\right)\right)^{\frac{1}{k}},$$

where, k_ϕ *is given by* (2.28)*. This result is sharp.*

Theorem 2.13. *Let* $\min_{|z|=r}|\phi(z)|=\phi(-r)$, $\max_{|z|=r}|\phi(z)|=\phi(r)$ *with* $|z|=r<1$. *If*

$$f\in\mathcal{C}_{sc}^{(k)}(\lambda,\phi)\subset\mathcal{C}^*$$

with $0<\lambda\le1$, *then*

$$\begin{aligned}\frac{1}{\lambda}r^{1-\frac{1}{\lambda}}\int_0^r\int_0^u\frac{1}{s}\int_0^s\phi(-t)\left(k'_\phi\left(-t^k\right)\right)^{\frac{1}{k}}dtdsu^{\frac{1}{\lambda}-2}du\le|f(z)|\\ \le\frac{1}{\lambda}r^{1-\frac{1}{\lambda}}\int_0^r\int_0^u\frac{1}{s}\int_0^s\phi(t)\left(k'_\phi\left(t^k\right)\right)^{\frac{1}{k}}dtdsu^{\frac{1}{\lambda}-2}du,\end{aligned} \tag{2.29}$$

and

(2.30)

$$f(\mathbb{U})\supset\left\{\omega:|\omega|\le\frac{1}{\lambda}r^{1-\frac{1}{\lambda}}\int_0^r\int_0^u\frac{1}{s}\int_0^s\phi(-t)\left(k'_\phi\left(-t^k\right)\right)^{\frac{1}{k}}dtdsu^{\frac{1}{\lambda}-2}du\right\}.$$

Proof. Suppose that $f\in\mathcal{C}_{sc}^{(k)}(\lambda,\phi)$, and $\phi(z)$ is convex and symmetric with respect to the real axis. Since

$$f_{2k}(z)=z+\sum_{l=1}^{\infty}\frac{a_{lk+1}+\overline{a_{lk+1}}}{2}z^{lk+1}=z+\sum_{l=1}^{\infty}\Re(a_{lk+1})z^{lk+1},$$

by Corollary 2.3, it follows that

$$(1-\lambda)f_{2k}(z)+\lambda z f_{2k}'(z)=z+\sum_{l=1}^{\infty}(1+\lambda lk)\Re(a_{lk+1})z^{lk+1}\in\mathcal{K}(\phi).$$

Thus, by Lemma 2.12, we have

$$\left(k_\phi'\left(-r^k\right)\right)^{\frac{1}{k}}\le|\lambda z f_{2k}''(z)+f_{2k}'(z)|\le\left(k_\phi'\left(r^k\right)\right)^{\frac{1}{k}}.$$

Now, for $|z|=r<1$, we have

$$\begin{aligned}\phi(-r)\left(k_\phi'\left(-r^k\right)\right)^{\frac{1}{k}}&\le\left|\left[z\left(\lambda z f''(z)+f'(z)\right)\right]'\right|\\&=\left|\frac{\left[z(\lambda z f''(z)+f'(z))\right]'}{\lambda z f_{2k}''(z)+f_{2k}'(z)}\cdot\left(\lambda z f_{2k}''(z)+f_{2k}'(z)\right)\right|\\&\le\phi(r)\left(k_\phi'\left(r^k\right)\right)^{\frac{1}{k}}.\end{aligned}\tag{2.31}$$

Upon integrating (2.31) from 0 to r two times, we get

$$\begin{aligned}&\int_0^r\frac{1}{s}\int_0^s\phi(-t)\left(k_\phi'\left(-t^k\right)\right)^{\frac{1}{k}}dtds\\&\le|(1-\lambda)f(z)+\lambda z f'(z)|\\&\le\int_0^r\frac{1}{s}\int_0^s\phi(t)\left(k_\phi'\left(t^k\right)\right)^{\frac{1}{k}}dtds.\end{aligned}\tag{2.32}$$

From (2.32), we easily get (2.29). Moreover, (2.30) follows from (2.29). □

Theorem 2.14. *Let* $\min_{|z|=r}|\phi(z)|=\phi(-r)$, $\max_{|z|=r}|\phi(z)|=\phi(r)$ *with* $|z|=r<1$. *If*

$$f\in\mathcal{S}_{sc}^{(k)}(\lambda,\phi)\subset\mathcal{C}$$

with $0<\lambda\le1$, *then*

$$\frac{1}{\lambda}r^{1-\frac{1}{\lambda}}\int_0^r h_\phi(-s)s^{\frac{1}{\lambda}-2}ds\le|f(z)|\le\frac{1}{\lambda}r^{1-\frac{1}{\lambda}}\int_0^r h_\phi(s)s^{\frac{1}{\lambda}-2}ds,$$

and

$$f(\mathbb{U})\supset\left\{\omega:|\omega|\le\frac{1}{\lambda}r^{1-\frac{1}{\lambda}}\int_0^r h_\phi(-s)s^{\frac{1}{\lambda}-2}ds\right\}.$$

Proof. Let $f\in\mathcal{S}_{sc}^{(k)}(\lambda,\phi)\subset\mathcal{C}$ and suppose that

$$\psi(z)=\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f_{2k}(z)+\lambda z f_{2k}'(z)}=\frac{zF'(z)}{F_{2k}(z)}\prec\phi(z),$$

where,

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z)$$

and

$$F_{2k}(z) = (1-\lambda)f_{2k}(z) + \lambda z f'_{2k}(z).$$

By virtue of (2.6), it follows that

$$F_{2k} \in \mathcal{S}^*(\phi).$$

Hence, we have

$$-h_\phi(-r) \leq |F_{2k}(z)| \leq h_\phi(r).$$

Now, for $|z| = r < 1$, we find that

$$\left|\psi(z)\tfrac{F_{2k}(z)}{z}\right| = |F'(z)| \leq \frac{\phi(r)h_\phi(r)}{r} = h'_\phi(r), \tag{2.33}$$

and

$$|F'(z)| \geq \frac{\phi(-r)h_\phi(-r)}{-r} = h'_\phi(-r). \tag{2.34}$$

Upon integrating (2.33) and (2.34) from 0 to r, we can get the first part of Theorem 2.14. The other part follows easily. □

The following results are shown that the classes $\mathcal{S}_{sc}^{(k)}(\lambda,\phi)$ and $\mathcal{C}_{sc}^{(k)}(\lambda,\phi)$ are closed under the following integral operator. The proof of these results are much akin to that of Theorem 2 obtained by Parvatham and Radha [13]. Here, we to omit the details.

Corollary 2.15. *Let* $f \in \mathcal{S}_{sc}^{(k)}(\lambda,\phi)$ *with* $0 < \lambda \leq 1$. *Then*

$$F(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z f(t) t^{\frac{1}{\lambda}-2} dt \in \mathcal{S}_{sc}^{(k)}(\lambda,\phi) \subset \mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Corollary 2.16. *Let* $f \in \mathcal{C}_{sc}^{(k)}(\lambda,\phi)$ *with* $0 < \lambda \leq 1$. *Then*

$$F(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z f(t) t^{\frac{1}{\lambda}-2} dt \in \mathcal{C}_{sc}^{(k)}(\lambda,\phi) \subset \mathcal{C}_{sc}^{(k)}(\phi) \subset \mathcal{C}^* \subset \mathcal{C}.$$

By similarly applying the arguments given in the proof for Theorem 2.1, we easily get the following inclusion relationship for the class $\mathcal{T}_{sc}^{(k)}(\lambda,\phi;g)$, which tells us that $\mathcal{T}_{sc}^{(k)}(\lambda,\phi;g)$ is a subclass of the class of close-to-convex functions.

Corollary 2.17. *Let*

$$\phi \in \mathcal{P} \quad \text{and} \quad 0 \leq \lambda \leq 1.$$

Then,

$$\mathcal{T}_{sc}^{(k)}(\lambda,\phi;g) \subset \mathcal{T}_{sc}^{(k)}(0,\phi;g) \subset \mathcal{C} \subset \mathcal{S}.$$

By similarly applying the arguments given in the proof for Theorem 5 obtained in [13], we get the following structured formula for the class $\mathcal{T}_{sc}^{(k)}(\lambda, \phi; g)$.

Corollary 2.18. *$f \in \mathcal{T}_{sc}^{(k)}(\lambda, \phi; g)$ if and only if there exist a function G in $\mathbb{U}$ with $G(0) = 0$ such that*

$$\frac{zG'(z)}{G(z)} \prec \phi(z),$$

and an analytic function p with $p(0) = 1$ and

$$p(z) \prec \phi(z), \qquad (z \in \mathbb{U})$$

such that

$$f'(z) = \begin{cases} \frac{p(z)G(z)}{z}, & (\lambda = 0), \\ \\ \frac{1}{\lambda z^{1/\lambda}} \int_0^z p(t)G(t)t^{\frac{1}{\lambda}-2}dt, & (0 < \lambda \leq 1). \end{cases}$$

3. Properties of the classes $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ and $\mathcal{H}_{sc}^{(k)}(\alpha, \phi; g)$

Here, we give some inclusion relationships for the classes $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ and $\mathcal{H}_{sc}^{(k)}(\alpha, \phi; g)$, which tell us that these classes are subclasses of the class $\mathcal{C}$.

Theorem 3.1. *Let*

$$\alpha \geq 0 \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then,

$$\mathcal{K}_{sc}^{(k)}(\alpha, \phi) \subset \mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Proof. Suppose that $f \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi)$. It follows from (1.6) that

$$(1-\alpha)\frac{zf'(z)}{f_{2k}(z)} + \alpha\frac{f'(z) + zf''(z)}{f'_{2k}(z)} \prec \phi(z). \tag{3.1}$$

Upon substituting z by $\varepsilon^\mu z$ $(\mu = 0, 1, 2, \ldots, k-1)$ in (3.1), then (3.1) also holds true, that is,

$$(1-\alpha)\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)}{f_{2k}(\varepsilon^\mu z)} + \alpha\frac{f'(\varepsilon^\mu z) + \varepsilon^\mu zf''(\varepsilon^\mu z)}{f'_{2k}(\varepsilon^\mu z)} \prec \phi(z). \tag{3.2}$$

From (3.2), we have

$$(1-\alpha)\frac{\overline{\varepsilon^\mu \overline{z}}\ \overline{f'(\varepsilon^\mu \overline{z})}}{\overline{f_{2k}(\varepsilon^\mu \overline{z})}} + \alpha\frac{\overline{f'(\varepsilon^\mu \overline{z})} + \overline{\varepsilon^\mu \overline{z}}\ \overline{f''(\varepsilon^\mu \overline{z})}}{\overline{f'_{2k}(\varepsilon^\mu \overline{z})}} \prec \phi(z). \tag{3.3}$$

Upon summing (3.2) and (3.3) and applying the similar arguments given in the proof for Theorem 2.1, we have

$$(1-\alpha)\frac{zf_{2k}'(z)}{f_{2k}(z)}+\alpha\frac{(zf_{2k}'(z))'}{f_{2k}'(z)}\prec\phi(z). \tag{3.4}$$

We now let

$$q(z)=\frac{zf_{2k}'(z)}{f_{2k}(z)}.$$

Then, (3.4) can be written as follows:

$$(1-\alpha)\frac{zf_{2k}'(z)}{f_{2k}(z)}+\alpha\frac{(zf_{2k}'(z))'}{f_{2k}'(z)}=q(z)+\alpha\frac{zq'(z)}{q(z)}\prec\phi(z).$$

Thus, by Lemma 1.6, we have

$$q(z)=\frac{zf_{2k}'(z)}{f_{2k}(z)}\prec\phi(z).$$

At the same time, if we let

$$p(z)=\frac{zf'(z)}{f_{2k}(z)},$$

then (1.6) can be written as follows:

$$(1-\alpha)\frac{zf'(z)}{f_{2k}(z)}+\alpha\frac{(zf'(z))'}{f_{2k}'(z)}=p(z)+\alpha\frac{zp'(z)}{q(z)}\prec\phi(z).$$

Since

$$q(z)\prec\phi(z),$$

by Lemma 1.7, we know that

$$p(z)=\frac{zf'(z)}{f_{2k}(z)}\prec\phi(z),$$

which implies that

$$\mathcal{K}_{sc}^{(k)}(\alpha,\phi)\subset\mathcal{S}_{sc}^{(k)}(\phi).$$

Furthermore, by Lemma 1.8 we have

$$\mathcal{K}_{sc}^{(k)}(\alpha,\phi)\subset\mathcal{S}_{sc}^{(k)}(\phi)\subset\mathcal{C}\subset\mathcal{S}.$$

□

Indeed, from (3.4) we know that if $f\in\mathcal{K}_{sc}^{(k)}(\alpha,\phi)$, then $f_{2k}\in\mathcal{C}(\alpha,\phi)$. Thus, by Lemma 1.10 we easily get the following result.

Corollary 3.2. *Let* $f \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi)$. *Then,*

$$F(z) = f_{2k}(z)\left(\frac{zf'_{2k}(z)}{f_{2k}(z)}\right)^{\alpha} \in \mathcal{S}^*(\phi),$$

where, f_{2k} *is defined by* (1.2).

By similarly applying the arguments given in the proof for Theorem 3.1, we easily get the following result.

Corollary 3.3. *Let*

$$\alpha \geq 0 \quad \text{and} \quad \phi \in \mathcal{P}.$$

Then,

$$\mathcal{H}_{sc}^{(k)}(\alpha, \phi; g) \subset \mathcal{H}_{sc}^{(k)}(0, \phi; g) \subset \mathcal{C} \subset \mathcal{S}.$$

For the class $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$, by similarly applying the arguments given in the proof for Theorem 2.8, we get the following integral representation for functions belonging to this class.

Corollary 3.4. *Let* $f \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi)$. *Then,*

$$f(z) = \int_0^z \exp\left(\int_0^{\zeta} \frac{\phi(\omega_1(t)) - 1}{t} dt\right) \cdot \phi(\omega_2(\zeta)) d\zeta,$$

where, ω_j $(j = 1, 2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_j(0) = 0 \quad \text{and} \quad |\omega_j(z)| < 1, \quad (j = 1, 2;\ z \in \mathbb{U}).$$

We now give some inclusion relationships for the classes $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ and $\mathcal{H}_{sc}^{(k)}(\alpha, \phi; g)$. The proof of the following result is much akin to that of Theorem 3 obtained by Yuan and Liu [21]. We thus choose to omit the analogous details.

Corollary 3.5. *Let* $\alpha_2 \geq \alpha_1 \geq 0$. *Then,*

$$\mathcal{K}_{sc}^{(k)}(\alpha_2, \phi) \subset \mathcal{K}_{sc}^{(k)}(\alpha_1, \phi),$$

and

$$\mathcal{H}_{sc}^{(k)}(\alpha_2, \phi; g) \subset \mathcal{H}_{sc}^{(k)}(\alpha_1, \phi; g).$$

From the proof of theorems 2.10 and 3.1, we easily get the following convolution property for the function class $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$.

Corollary 3.6. *Let* $f \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi)$. *Then,*

$$\frac{1}{z}\left[f * \left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})}{2} h\right)(z) - \frac{\phi(e^{i\theta})}{2} \cdot \overline{(f * h)(\overline{z})}\right] \neq 0,$$

for all $z \in \mathbb{U}$ *and* $0 \leq \theta < 2\pi$, *where* h *is given by* (2.27).

By similarly applying the arguments given in the proof for Theorem 4 obtained in [21], we know that the class $\mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ is also closed under the following integral operator.

Corollary 3.7. *Let* $f \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi)$ *with* $\alpha > 0$. *Then*

$$F(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z f(t) t^{\frac{1}{\alpha}-2} dt \in \mathcal{K}_{sc}^{(k)}(\alpha, \phi) \subset \mathcal{S}_{sc}^{(k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

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