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ENTROPY OPERATOR FOR CONTINUOUS DYNAMICAL SYSTEMS OF FINITE TOPOLOGICAL ENTROPY

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Abstract. In this paper we introduce the concept of entropy operator for a continuous system of finite topological entropy. It is shown that it generates the Kolmogorov entropy as a special case. If ϕ is invertible then the entropy operator is bounded by the topological entropy of ϕ as its norm.

1. Introduction

Entropy is a numerical invariant measuring the complexity of a dynamical system. It was first introduced into the ergodic theory by Kolmogorov [10] and Sinai [17] via a measure theoretic approach. Adler, Konheim and McAndrew [1] introduced topological entropy as another version of entropy of dynamical systems. It was given in an equivalent way, as a metric approach, by Dinaburg [7] and Bowen [4], which was used to connect the topological entropy and the measure theoretic entropy via a variational principle. Shannon [16], McMillan [12], and Breiman [5] gave local approaches to entropy based on the Theorem of Shannon-McMillan-Briman. A topological version of the Theorem of Shannon-McMillan-Briman was given by Brin and Katok [6]. In case

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of smooth dynamics, Ruelle [15] and Pesin [13] gave local approaches to entropy. One may find other approaches to entropy in [2, 8, 9, 11]. In all of these approaches, entropy is given as a non-negative extended real number assigned to a dynamical system. This paper is an attempt to present a different approach to the entropy of a dynamical system as a linear operator rather than a non-negative number. This *entropy operator* should generate the entropy as a special case and contain the variational principle in its nature. Also, one can use it to define the measure theoretic entropy for complex invariant measures rather than the probability invariant measures. It also results in a local entropy as well.

In section 2, the prerequisities are given briefly. In section 3, we introduce the concept of entropy operator and we state its relationship with the entropy of a dynamical system. Finally, we summarize our results in section 4.

2. **Preliminary facts**

Suppose that $\phi : X \to X$ is a dynamical system on the compact metric space X. Let $M(X)$ be the set of all Borel probability measures on X. The set of all probability invariant measures of ϕ is denoted by $M(X, \phi)$. We also write $E(X, \phi)$ for the set of all ergodic measures of ϕ. $M(X, \phi)$, equipped by the weak*-topology on $M(X)$, is a compact convex metrisable space with extreme points $E(X, \phi)$ [18]. In the following, we recall some preliminary facts which is needed in the rest of the paper. First, we recall Choquet's Theorem and its corollary which results in the ergodic decomposition.

Theorem 2.1. *(Choquet)(Phelps* [14]*) Suppose that Y is a compact convex metrisable subset of a locally convex space E, and that $x_0 \in Y$. Then there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y, i.e., $\Psi(x_0) = \int_Y \Psi d\tau$ for every continuous linear functional Ψ on E, and $\tau(ext(Y)) = 1$.*

Let $\mu \in M(X, \phi)$ and $f : X \to \mathbb{R}$ be a bounded measurable function. Since we know that $E(X, \phi)$ agrees with the set of extreme points of $M(X, \phi)$, applying Choquet's Theorem for $Y = M(X, \phi)$ and $\Psi(\mu) = \int_X f d\mu$, we have the following corollary:

Corollary 2.2. *Suppose that $\phi : X \to X$ is a continuous map on the compact metric space X. Then for each $\mu \in M(X, \phi)$ there is a unique* *measure τ on the Borel subsets of the compact metrisable space $M(X,\phi)$ such that $\tau(E(X,\phi))=1$ and*

$$\int_X f(x)d\mu(x)=\int_{E(X,\phi)}\left(\int_X f(x)dm(x)\right)d\tau(m)$$

for every bounded measurable function $f:X\to\mathbb{R}$.

Under the assumptions of Corollary 2.2 we write $\mu=\int_{E(X,\phi)} m d\tau(m)$. It is called the ergodic decomposition of μ.

Theorem 2.3. *(Jacobs)(Walters [18]) Let $\phi : X\to X$ be a continuous map on a compact metrisable space. If $\mu\in M(X,\phi)$ and $\mu=\int_{E(X,\phi)} m d\tau(m)$ is the ergodic decomposition of μ then we have:*

(i) *If ξ is a finite Borel partition of X then*

$$h_\mu(\phi,\xi)=\int_{E(X,\phi)} h_m(\phi,\xi)d\tau(m).$$

(ii) $h_\mu(\phi)=\int_{E(X,\phi)} h_m(\phi)d\tau(m)$ *(both sides could be ∞).*

The following Theorem shifts the supremum in the definition of Kolmogorov entropy on a countable family of partitions.

Theorem 2.4. *(Walters [18]) Let $\phi : X\to X$ be a continuous map on the compact metric space X. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X such that diam$(\xi_n)\to 0$ as $n\to\infty$. For every $\mu\in M(X,\phi)$ we have $h_\mu(\phi)=\lim_{n\to\infty} h_\mu(\phi,\xi_n)$.*

3. **Entropy operator**

As in the previous section, let $\phi : X\to X$ be a continuous dynamical system on the compact metric space X. Let $\mathfrak{B}(X)$ be the family of all Borel measurable subsets of X. The set of all complex invariant measures of ϕ is defined as follows:

$$\mathcal{M}(X,\phi):=\{\mu:\mathfrak{B}(X)\to\mathbb{C}:\ \mu(\phi^{-1}(B))=\mu(B)\ \forall B\in\mathfrak{B}(X)\}.$$

$\mathcal{M}(X,\phi)$ is a complex linear space and is equipped with the norm $||\mu||:=|\mu|(X)$. It is easily seen that $\mathcal{M}(X,\phi)$ is closed with respect to this norm. Note that $M(X,\phi),E(X,\phi)\subset\mathcal{M}(X,\phi)$.

To proceed further, choose any sequence $\{\xi_n\}_{n\geq 1}$ of Borel partitions of X such that diam$(\xi_n)\to 0$. We fix it for the remaining of this paper. One may assume that $\xi_n<\xi_{n+1}$, since otherwise one can replace ξ_n by

$\eta_n = \bigvee_{i=1}^n \xi_i$ wherein, the joint of two partitions $\xi = \{A_1, ..., A_n\}$ and $\eta = \{B_1, ..., B_m\}$ is given by

$$\xi \vee \eta = \{A_i \cap B_j : \ i = 1, ..., n, \ j = 1, ..., m\}.$$

Definition 3.1. *Let $x \in X$, $A \in \mathfrak{B}(X)$ and let ξ, η be two Borel partitions of X. The non-negative extended real numbers $\tau_\phi(x, A)$, $S_\phi(x; \xi)$, $S_\phi(x; \xi|\eta)$ and $\bar{S}_\phi(x; \xi)$ are defined as follows:*

$$\tau_\phi(x, A) := \limsup_{n\to\infty} \frac{1}{n} card(\{k \in \{0, 1, ..., n-1\} : \ \phi^k(x) \in A\}),$$

$$S_\phi(x; \xi) := -\sum_{j=1}^{n} \tau_\phi(x, A_j) \log \tau_\phi(x, A_j),$$

$$S_\phi(x; \xi|\eta) := -\sum_{i,j}^{n} \tau_\phi(x, A_i \cap B_j) \log \frac{\tau_\phi(x, A_i \cap B_j)}{\tau_\phi(x, B_j)}$$

and

$$\bar{S}_\phi(x; \xi) := \limsup_{n\to\infty} \frac{1}{n} S_\phi(x; \bigvee_{i=0}^{n-1} \phi^{-i}\xi) \tag{3.1}$$

wherein, $\phi^{-i}\xi$ is the partition given by

$$\phi^{-i}\xi = \{\phi^{-i}(A_1), ..., \phi^{-i}(A_n)\}.$$

Lemma 3.2. *For any $x \in X$, the limit $\lim_{n\to\infty} \bar{S}_\phi(x; \xi_n)$ exists, as a non-negative extended real number.*

Proof. Let $\xi = \{A_i\}$ and $\eta = \{B_j\}$ be two Borel partitions of X and assume, without loss of generality, that all sets have the property that $\tau_\phi(x, A) \neq 0$. (Since if $\xi = \{A_1, ..., A_k\}$ with $\tau_\phi(x, A_i) > 0$ for $1 \leq i \leq r$ and $\tau_\phi(x, A_i) = 0$ for $r < i \leq k$ we can replace ξ by $\{A_1, ..., A_{r-1}, A_r \cup A_{r+1} \cup ... \cup A_k\}$)
By definition

$$S_\phi(x; \xi \vee \eta) = -\sum_{i,j} \tau_\phi(x, A_i \cap B_j) \log \tau_\phi(x, A_i \cap B_j).$$

But we may write

$$\tau_\phi(x, A_i \cap B_j) = \frac{\tau_\phi(x, A_i \cap B_j)}{\tau_\phi(x, A_i)} . \tau_\phi(x, A_i).$$

Therefore

$$
\begin{aligned}
S_\phi(x;\xi\vee\eta) =& -\sum_{i,j}\tau_\phi(x,A_i\cap B_j)\log\frac{\tau_\phi(x,A_i\cap B_j)}{\tau_\phi(x,A_i)}\\
& -\sum_{i,j}\tau_\phi(x,A_i\cap B_j)\log\tau_\phi(x,A_i) \qquad (3.2)\\
=& S_\phi(x;\eta|\xi)-\sum_{i,j}\tau_\phi(x,A_i\cap B_j)\log\tau_\phi(x,A_i).
\end{aligned}
$$

On the other hand, one may easily see that

$$\sum_j \tau_\phi(x,A_i\cap B_j)\geq\tau_\phi(x,A_i)$$

for all $i\in\mathbb{N}$. Multiplying both sides by $-\log\tau_\phi(x,A_i)$ and summing over i one obtain

$$-\sum_{i,j}\tau_\phi(x,A_i\cap B_j)\log\tau_\phi(x,A_i)\geq-\sum_i\tau_\phi(x,A_i)\log\tau_\phi(x,A_i). \tag{3.3}$$

Combining (3.2) and (3.3) we have

$$S_\phi(x;\xi\vee\eta)\geq S_\phi(x;\eta|\xi)+S_\phi(x;\xi). \tag{3.4}$$

Now, let $\xi<\eta$; then $\xi\vee\eta=\eta$ and since $S_\phi(x;\eta|\xi)\geq 0$, by (3.4) one obtains

$$S_\phi(x;\eta)\geq S_\phi(x;\eta|\xi)+S_\phi(x;\xi)\geq S_\phi(x;\xi). \tag{3.5}$$

Since $\xi<\eta$ then $\vee_{i=0}^{n-1}\phi^{-i}\xi<\vee_{i=0}^{n-1}\phi^{-i}\eta$ for all $n\in\mathbb{N}$. It follows from (3.1) and (3.5) that $\bar{S}_\phi(x;\xi)\leq\bar{S}_\phi(x;\eta)$. Thus, for any $x\in X$, $\{\bar{S}_\phi(x;\xi_n)\}_{n\geq1}$ is an increasing sequence of non-negative extended real numbers. Therefore $\lim_{n\to\infty}\bar{S}_\phi(x;\xi_n)$ exists as a non-negative extended real number. □

Definition 3.3. *Under the previous conditions, the* entropy kernel *of ϕ is defined as follows:*

$$\mathcal{J}_\phi(x):=\lim_{n\to\infty}\bar{S}_\phi(x;\xi_n).$$

Now we are in a position to introduce the *entropy operator* of a dynamical system ϕ.

Definition 3.4. *The entropy operator of ϕ, $\mathcal{L}_\phi:\mathcal{M}(X,\phi)\to\mathfrak{L}(C(X);\mathbb{C})$, given by $\mu\mapsto\mathcal{L}_\phi(\mu)$ is defined as follows:*

$$\left(\mathcal{L}_\phi(\mu)\right)(f):=\int_X f\mathcal{J}_\phi d\mu$$

for all $f \in C(X)$, where $\mathfrak{L}(C(X);\mathbb{C})$ is the space of linear functionals on $C(X)$.

Clearly, $\mathcal{L}_\phi$ is a linear operator between the vector spaces $\mathcal{M}(X,\phi)$ and $\mathfrak{L}(C(X);\mathbb{C})$. The norm of $\mathcal{L}_\phi$ is given by

$$||\mathcal{L}_\phi|| := \sup_{||\mu||=1} ||\mathcal{L}_\phi(\mu)||$$

wherein

$$||\mathcal{L}_\phi(\mu)|| = \sup_{||f||_\infty=1} \left| \int_X f \mathcal{J}_\phi d\mu \right|.$$

The operator $\mathcal{L}_\phi$ is an entropy generator operator, in the sense that, one can obtain the entropies of ϕ from $\mathcal{L}_\phi$. (See Theorem 3.6)

Definition 3.5. *For $f \in C(X)$, the evaluation map $\hat{f} : \mathfrak{L}(C(X);\mathbb{C}) \to \mathbb{C}$ is defined by $\hat{f}(L) := L(f)$ for all $L \in \mathfrak{L}(C(X);\mathbb{C})$.*

The following theorem is our main result which shows that how one can extract entropies of ϕ from $\mathcal{L}_\phi$.

Theorem 3.6. *Suppose that $\phi : X \to X$ is a continuous dynamical system of finite topological entropy on the compact metric space X. Then*

(i) $\left(\hat{1} o \mathcal{L}_\phi\right)(\mu) = h_\mu(\phi)$ *for all* $\mu \in M(X,\phi)$.
(ii) $||\mathcal{L}_\phi|| \geq h_{\text{top}}(\phi)$.
(iii) *Moreover, if ϕ is invertible then $||\mathcal{L}_\phi|| = h_{\text{top}}(\phi)$.*

Proof. (i) First, let $m \in E(X,\phi)$. For any Borel set $A \subseteq X$ and $x \in X$, applying Birkhoff's ergodic Theorem one may obtain $\tau_\phi(x,A) = m(A)$ for almost all $x \in X$. Hence if ξ is a Borel partition of X then $S_\phi(x;\xi) = H_m(\xi)$ for almost all $x \in X$ where $H_m(\xi)$ is the entropy of the partition ξ. Thus, for each $n \in \mathbb{N}$ one can find a Borel set Y_n with $m(Y_n) = 1$ such that

$$\limsup_{l\to\infty} \frac{1}{l} S_\phi(x; \bigvee_{i=0}^{l-1} \phi^{-i}\xi_n) = h_m(\phi,\xi_n)$$

for all $x \in Y_n$. Put $X_1 := \cap_{n=1}^\infty Y_n$, then $m(X_1) = 1$ and for $x \in X_1$

$$\limsup_{l\to\infty} \frac{1}{l} S_\phi(x; \bigvee_{i=0}^{l-1} \phi^{-i}\xi_n) = h_m(\phi,\xi_n)$$

for all $n \geq 1$ or equivalently,

$$\bar{S}_\phi(x;\xi_n) = h_m(\phi,\xi_n)$$

for all $n \geq 1$. If $n \to \infty$ one can obtain $\mathcal{J}_\phi(x) = h_m(\phi)$ for almost all $x \in X_1$. This easily results in

$$\int_X \mathcal{J}_\phi dm = h_m(\phi).$$

Now let $\mu \in M(X,\phi)$ and let $\mu = \int_{E(X,\phi)} m d\tau(m)$ be the ergodic decomposition of μ. For $n \in \mathbb{N}$ put $\mathcal{J}_n := \min\{\mathcal{J}_\phi, n\}$. Then $\{\mathcal{J}_n\}_{n\geq 1}$ is an increasing sequence of bounded measurable maps on X such that $\mathcal{J}_n \nearrow \mathcal{J}_\phi$. Applying Corollary 2.2, Theorem 2.3 and Monotone Convergence Theorem one obtains

$$\begin{aligned}
\left(\hat{1} o \mathcal{L}_\phi\right)(\mu) &= \left(\mathcal{L}_\phi(\mu)\right)(1) \\
&= \int_X \mathcal{J}_\phi d\mu \\
&= \lim_{n\to\infty} \int_X \mathcal{J}_n d\mu \\
&= \lim_{n\to\infty} \int_{E(X,\phi)} \left(\int_X \mathcal{J}_n dm \right) d\tau(m) \\
&= \int_{E(X,\phi)} \left(\int_X \mathcal{J}_\phi dm \right) d\tau(m) \\
&= \int_{E(X,\phi)} h_m(\phi) d\tau(m) \\
&= h_\mu(\phi).
\end{aligned}$$

(ii) If $\mu \in M(X,\phi)$ then

$$||\mathcal{L}_\phi(\mu)|| = \sup_{||f||_\infty = 1} \left| \int_X f \mathcal{J}_\phi d\mu \right| \geq \int_X \mathcal{J}_\phi d\mu = h_\mu(\phi).$$

Therefore

$$||\mathcal{L}_\phi|| \geq \sup_{\mu \in M(X,\phi)} ||\mathcal{L}_\phi(\mu)|| \geq \sup_{\mu \in M(X,\phi)} h_\mu(\phi) = h_{\text{top}}(\phi).$$

(iii) Let ϕ be invertible. First, we show that if $\mu \in \mathcal{M}(X,\phi)$ then $|\mu|$ is an invariant measure of ϕ. Let $\mu \in \mathcal{M}(X,\phi)$ and let B be a Borel set. Then $\mu(\phi^{-1}(E)) = \mu(E)$ for every Borel set E. We know that

$$|\mu|(B) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$$

where the supremum is taken over all partitions $\{E_i\}$ of B. Therefore,

$$\begin{aligned} |\mu|(B) &= \sup \sum_{i=1}^{\infty} |\mu(E_i)| \\ &= \sup \sum_{i=1}^{\infty} |\mu(\phi^{-1}(E_i))| \\ &\leq \sup \sum_{i=1}^{\infty} |\mu(C_i)| \\ &= |\mu|(\phi^{-1}(B)) \end{aligned}$$

where in the first and second summations the supremum is taken over all partitions $\{E_i\}$ of B while in the last summation the supremum is taken over all partitions $\{C_i\}$ of $\phi^{-1}(B)$. So, we have shown that $|\mu|(B) \leq |\mu|(\phi^{-1}(B))$ for every Borel set B. Since ϕ is invertible then by symmetry, $|\mu|(\phi^{-1}((B)) \leq |\mu|(B)$ which means that $|\mu|$ is an invariant measure of ϕ.

For $\mu \in \mathcal{M}(X, \phi)$ with $||\mu|| = 1$ we have

$$\begin{aligned} ||\mathcal{L}_\phi(\mu)|| &= \sup_{||f||_\infty = 1} |\left(\mathcal{L}_\phi(\mu)\right)(f)| \\ &= \sup_{||f||_\infty = 1} \left| \int_X f \mathcal{J}_\phi d\mu \right| \\ &\leq \sup_{||f||_\infty = 1} \int_X ||f||_\infty \mathcal{J}_\phi d|\mu| \\ &= \int_X \mathcal{J}_\phi d|\mu| \\ &= h_{|\mu|}(\phi). \end{aligned}$$

Since $|\mu|$ is an invariant probability measure, by the variational principle

$$||\mathcal{L}_\phi|| = \sup_{||\mu||=1} ||\mathcal{L}_\phi(\mu)|| \leq \sup_{||\mu||=1} h_{|\mu|}(\phi) \leq h_{\text{top}}(\phi).$$

This completes the proof. □

4. **Summary and discussion**

In this paper, the concept of entropy operator for a continuous dynamical system $\phi : X \to X$ of finite topological entropy is introduced. In this case, the entropy operator of ϕ, $\mathcal{L}_\phi$, is a linear operator between

$\mathcal{M}(X,\phi)$ and $\mathfrak{L}(C(X);\mathbb{R})$. We summarize the properties of $\mathcal{L}_\phi$ as follows:

(i) Theorem 3.6 shows that $\mathcal{L}_\phi$ is indeed an entropy generator operator, in the sense that, the restriction of $\mathcal{L}_\phi$ to $M(X,\phi)$ composed with $\hat{1}$ equals the entropy map $\mu \mapsto h_\mu(\phi)$. On the other hand, when ϕ is invertible, the norm of $\mathcal{L}_\phi$, as a bounded linear operator, equals the topological entropy of ϕ, i.e., $||\mathcal{L}_\phi|| = h_{\text{top}}(\phi)$.

(ii) Theorem 3.6 (i) motivates us to define the entropy of ϕ for complex invariant measures. One can do this by considering the combination $\hat{1}o\mathcal{L}_\phi : \mathcal{M}(X,\phi) \to \mathbb{C}$. In other words, for any complex invariant measure μ, the entropy of ϕ may be defined as $\left(\hat{1}o\mathcal{L}_\phi\right)(\mu)$, because the restriction of $\hat{1}o\mathcal{L}_\phi$ to $M(X,\phi)$ is the usual entropy map.

(iii) The map $\mathcal{J}_\phi : X \to [0,\infty]$ is indeed a local entropy, because by Theorem 3.6 (i)

$$\int_X \mathcal{J}_\phi d\mu = \left(\mathcal{L}_\phi(\mu)\right)(1) = \left(\hat{1}o\mathcal{L}_\phi\right)(\mu) = h_\mu(\phi).$$

for all $\mu \in M(X,\phi)$. On the other hand, unlike the classical local approaches to the measure theoretic entropy, [3, 5, 10, 13], the local entropy $\mathcal{J}_\phi$ is universal, in the sense that, it does not depend on any measure and its integral with respect to any invariant measure equals the entropy of the dynamical system with respect to that measure.

(iv) If f is a distribution function then the value $\left(\mathcal{L}_\phi(\mu)\right)(f) = \int_X f\mathcal{J}_\phi d\mu$ is indeed a weightened entropy of ϕ. This value equals the classical Kolmogorov entropy when there is no weight in the middle, i.e., $f \equiv 1$.

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