Bulletin of the Iranian Mathematical Society Vol. 36 *No.* 2 *(2010), pp 239-251.*

A QUASILINEAR PARABOLIC EQUATION WITH INHOMOGENEOUS DENSITY AND ABSORPTION

CH. LIU

Communicated by Henrik Shahgholian

Abstract. We deal with the initial-boundary value problem for a quasilinear degenerate parabolic equation with inhomogeneous density and absorption, which appears in a number of applications to describe the evolution of diffusion processes, in particular non-Newtonian flow in a porous medium. We discuss the extinction of solution and the finite speed of propagation of perturbations.

1. **Introduction**

We consider the following equation,

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q, (x,t) \in Q, \tag{1.1}$$

with the initial-boundary conditions,

$$u(x,0) = u_0(x), \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{1.3}$$

where, $Q = \Omega \times (0,\infty)$, $\Omega \subset R^n$ is a bounded smooth domain, $p > 1$, $q \geq p-1$, $u_0(x) \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is a nonzero nonnegative function and $\rho(x)$ denotes the density. We prefer to consider a typical case of

MSC(2010): Primary: 35B05, 35K55; Secondary: 35K65.

Keywords: Quasilinear parabolic equation, extinction, finite speed of propagation of perturbations.

Received: 3 August 2009, Accepted: 1 October 2009.

$\rho(x)$, that is, $\rho(x) = (1 + |x|)^{-l}, l > 0$ [11]. The equation (1.1) is a prototype of a certain class of degenerate equations and appears to be relevant to the theory of non-Newtonian fluids [1]. For the case $\rho(x) = 1$, there have been many results about the existence, uniqueness and the regularity of the solutions. We refer the readers to the bibliography given in [4, 9, 13] and the references therein. Eidus [5], and Eidus and Kamin [6] considered the following problem:

$$\rho(x)u_t = \Delta G(u), (x,t) \in Q_T = \Omega \times (0,T),$$
$$u(x,0) = u_0(x),$$
$$u|_{\partial\Omega} = 0.$$

They proved that the problem had a unique nonnegative solution, satisfying the condition,

$$\lim_{R\to\infty} R^{1-n} \int_{S(R)} \int_0^T G(u(x,t))dtdx = 0,$$

where, $S(R) = \{x : \ x \in \Omega, |x| = R\}$.

Recently, Tedeev [10] considered the equation

$$\rho(x)\frac{\partial u}{\partial t} = \mathrm{div}(u^{m-1}|Du|^{\lambda-1}Du),$$

where, $\lambda > 0, m + \lambda - 2 > 0$ and with $\rho(x)$ being a positive continuous function. They examined under which conditions on $\rho(x)$, the corresponding nonnegative solutions of the Cauchy problems possessed the finite speed of propagations or the interface blow-up phenomena.

The equation (1.1) is degenerate, if $p > 2$, or singular, if $1 < p < 2$. Therefore, problem (1.1)-(1.3) does not admit classical solutions, in general. So, we study weak solutions in the sense of the following definition.

Definition 1.1. A nonnegative function u is said to be a weak solution of the problem (1.1)-(1.3), if u satisfies following conditions:

$$u \in C(\overline{\Omega} \times (0, +\infty)) \cap L^\infty(0, \infty; W_0^{1,p}(\Omega)),$$

$$\int_0^T \int_\Omega \left(\rho(x)u\frac{\partial\varphi}{\partial t} - |\nabla u|^{p-2}\nabla u\nabla\varphi - u^q\varphi\right) dxdt = 0, \tag{1.4}$$

for all $\varphi \in C_0^\infty(Q_T), andT \in (0, +\infty)$.

The existense proof for problem (1.1)-(1.3) is similar to the one for the case $\rho(x) = 1$. Here, our interest is to investigate the extinction of solutions and the finite speed of propagation of perturbations. As is well known, one important property of solutions of the porous medium equation is the finite speed of propagation of perturbations. So, from the point of view of physical background, it seems to be natural to investigate this property for the equation (1.1). On the other hand, the mathematical description of this property is that if supp u_0 is bounded, then for any $t > 0$, supp $u(\cdot, t)$ is also bounded. So, from a mathematical viewpoint, this problem seems to be quite interesting. The monotonicity of support of weak solutions for the p-Laplacian equation was obtained by Yuan[12]. To prove the extinction of solution, here we use some ideas in [12]. We first construct a supersolution, and then use the comparison principle. Our method is different from the one given in [5] for the proof of the finite speed of propagation. We adopt the Bernis energy approach (see [2], [3]) and the main technical tools are weighted Nirenberg's inequality and Hardy's inequality.

2. **Comparison principle**

Here, we prove some lemmas.

Lemma 2.1. *For $\varphi \in L^\infty(t_1, t_2; W_0^{1,p}(\Omega))$ with $\varphi_t \in L^2((t_1, t_2) \times \Omega)$, the weak solutions u of the problem (1.1)-(1.3) on Q_T satisfies*

$$\int_\Omega \rho(x)u(x,t_1)\varphi(x,t_1)dx + \int_{t_1}^{t_2}\int_\Omega \left(\rho(x)u\frac{\partial\varphi}{\partial t} - |\nabla u|^{p-2}\nabla u\nabla\varphi\right)dxdt$$
$$= \int_{t_1}^{t_2}\int_\Omega u^q\varphi dxdt + \int_\Omega \rho(x)u(x,t_2)\varphi(x,t_2)dx.$$

In particular, for $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}&\int_\Omega \rho(x)(u(x,t_1) - u(x,t_2))\varphi dx\\ &- \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dxdt - \int_{t_1}^{t_2}\int_\Omega u^q\varphi dxdt = 0\,.\end{aligned} \tag{2.1}$$

Proof. From $\varphi \in L^\infty(t_1, t_2; W_0^{1,p}(\Omega))$ and $\varphi_t \in L^2((t_1, t_2) \times \Omega)$, it follows that there exists a sequence of functions $\{\varphi_k\}$, for fixed $t \in$

$(t_1, t_2), \varphi_k(\cdot, t) \in C_0^\infty(\Omega)$, and as $k \to \infty$,

$$\|\varphi_{kt} - \varphi_t\|_{L^2((t_1,t_2)\times\Omega)} \to 0, \quad \|\varphi_k - \varphi\|_{L^\infty(t_1,t_2;W_0^{1,p}(\Omega))} \to 0.$$

Choose a function $j(s) \in C_0^\infty(R)$ such that $j(s) \geq 0$, for $s \in R$; $j(s) = 0$, $\forall |s| > 1$, and $\int_R j(s)ds = 1$. For $h > 0$, define $j_h(s) = \frac{1}{h}j(\frac{s}{h})$ and

$$\eta_h(t) = \int_{t-t_2+2h}^{t-t_1-2h} j_h(s)ds.$$

Clearly, $\eta_h(t) \in C_0^\infty(t_1, t_2)$, and $\lim_{h\to 0^+} \eta_h(t) = 1$, for all $t \in (t_1, t_2)$. In the definition of weak solutions, choose $\varphi = \varphi_k(x,t)\eta_h(t)$. We have

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega \rho(x)u\varphi_k j_h(t-t_1-2h)dxdt - \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi_k\eta_h\, dxdt \\
&+ \int_{t_1}^{t_2}\int_\Omega \rho(x)u\varphi_{kt}\eta_h dxdt - \int_{t_1}^{t_2}\int_\Omega \rho(x)u\varphi_k j_h(t-t_2+2h)dxdt \\
&= \int_{t_1}^{t_2}\int_\Omega u^q\varphi_k\eta_h\, dxdt.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_\Omega \rho(x)u\varphi_k j_h(t-t_1-2h)dxdt - \int_\Omega (\rho(x)u\varphi_k)|_{t=t_1}dx\Big| \\
&= \Big|\int_{t_1+h}^{t_1+3h}\int_\Omega \rho(x)u\varphi_k j_h(t-t_1-2h)dxdt \\
&- \int_{t_1+h}^{t_1+3h}\int_\Omega (\rho(x)u\varphi_k)|_{t=t_1} j_h(t-t_1-2h)dxdt\Big| \\
&\leq \sup_{t_1+h<t<t_1+3h}\int_\Omega \big|(\rho(x)u\varphi_k)|_t - (\rho(x)u\varphi_k)|_{t_1}\big|dx,
\end{aligned}$$

and $u \in C(Q)$. We see that the right hand side tends to zero, as $h \to 0$. Similarly,

$$\Big|\int_{t_1}^{t_2}\int_\Omega \rho(x)u\varphi_k j_h(t-t_2+2h)dxdt - \int_\Omega (\rho(x)u\varphi_k)|_{t=t_2}dx\Big| \to 0.$$

Letting $h \to 0$ and $k \to \infty$, we obtain:

$$\begin{aligned}
&\int_\Omega \rho(x)u(x,t_1)\varphi(x,t_1)dx + \int_{t_1}^{t_2}\int_\Omega \Big(\rho(x)u\frac{\partial\varphi}{\partial t} - |\nabla u|^{p-2}\nabla u\nabla\varphi\Big)dxdt \\
&= \int_{t_1}^{t_2}\int_\Omega u^q\varphi dxdt + \int_\Omega \rho(x)u(x,t_2)\varphi(x,t_2)dx.
\end{aligned}$$

In particular, for $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \rho(x)(u(x,t_1)-u(x,t_2))\varphi dx - \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dxdt - \int_{t_1}^{t_2}\int_\Omega u^q\varphi\, dxdt = 0,$$

which completes the proof. □

For a fixed $\tau \in (0,T)$, set h satisfying $0<\tau<\tau+h<T$. Let $t_1=\tau$, $t_2=\tau+h$, and then multiply (2.1) by $\frac{1}{h}$, for $\varphi \in W_0^{1,p}(\Omega)$, to obtain:

$$\int_\Omega \rho(x)(u_h(x,\tau))_\tau\varphi dx + \int_\Omega (|\nabla u|^{p-2}\nabla u)_h\nabla\varphi dx + \int_\Omega u_h^q\varphi dx = 0, \tag{2.2}$$

where,

$$u_h(x,t) = \begin{cases} \frac{1}{h}\int_t^{t+h} u(\cdot,\tau)d\tau, & t\in(0,T-h),\\ 0, & t>T-h. \end{cases}$$

Lemma 2.2. (**Comparison principle**) *Let u be a weak solution of (1.1)-(1.3). If v satisfies*

$$\rho(x)\frac{\partial v}{\partial t} \geq div\left(|\nabla v|^{p-2}\nabla v\right) - v^q,$$

in the sense of distributions, and

$$v(x,0)\geq u(x,0),$$

$$v(x,t)\geq u(x,t), \quad x\in\partial\Omega,$$

then we have

$$v(x,t)\geq u(x,t), \quad \textit{for all } (x,t)\in Q.$$

Proof. By (2.2), we have for $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \rho(x)(u(x,\tau)-v(x,\tau))_{h\tau}\varphi(x)dx + \int_\Omega (u^q-v^q)_h(x,\tau)\varphi dx + \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)_h(x,\tau)\nabla\varphi dx \leq 0.$$

For a fixed τ, we take $\varphi(x) = [(u-v)_h]_+$. By the property of the Steklov mean value and noting that $v(x,t)\geq u(x,t), x\in\partial\Omega$, we see that $\varphi(x) = [(u-v)_h]_+ \in W_0^{1,p}(\Omega)$. Substituting this function into the above integral equality, we obtain:

$$\begin{aligned} &\int_\Omega \rho(x)(u(x,\tau)-v(x,\tau))_{h\tau}[(u-v)_h]_+ dx \\ \leq\ & -\int_\Omega [(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)_h](x,\tau)\nabla[(u-v)_h]_+ dx \\ & -\int_\Omega [(u^q - v^q)_h](x,\tau)[(u-v)_h]_+ dx. \end{aligned}$$

Integrating the above equality with respect to τ over $(0,t)$, we have

$$\begin{aligned} &\int_\Omega \rho(x)[(u-v)_h]_+^2(x,t)dx - \int_\Omega \rho(x)[(u-v)_h]_+^2(x,0)dx \\ \leq & -\int_\Omega [(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)_h](x,\tau)\nabla[(u-v)_h]_+ dx \\ & -\int_\Omega [(u^q - v^q)_h](x,\tau)[(u-v)_h]_+ dx, \end{aligned}$$

It is easily seen that

$$\lim_{h\to 0}\int_\Omega \rho(x)[(u-v)_h]_+(x,0)dx = 0.$$

Letting $h \to 0$, we have

$$\int_\Omega \rho(x)|(u-v)_+|^2(x,t)dx \leq 0,$$

that is, $\int_\Omega |(u-v)_+|^2 dx = 0$. Therefore, $v \geq u$, and the proof is complete. □

3. **Extinction and monotonicity of support**

We now to prove the following theorem.

Theorem 3.1. *Let u be a nonnegative weak solution of the problem (1.1)-(1.3), and $p > 2$. Then,*

$$supp u(\cdot,s) \subset supp u(\cdot,t),$$

for all s,t with $0 < s < t$.

Lemma 3.2. *Let u be a nonnegative weak solution of the problem (1.1)-(1.3) If $p > 2$, then*

$$\frac{\partial u}{\partial t} \geq -\frac{u}{(p-2)t},$$

in the sense of distributions.

Proof. Denote:

$$u_r(x,t) = ru(x, r^{p-2}t), \text{ for all } \quad (x,t) \in Q, \quad r > 1.$$

By $p - 1 \leq q$ and $r > 1$, we have

$$\rho(x)\frac{\partial u_r}{\partial t} \geq \operatorname{div}(|\nabla u_r|^{p-2}\nabla u_r) - u_r^q,$$

$$u_r(x,0) = ru_0(x), \tag{3.1}$$

$$u_r(x,t) = 0, \quad x \in \partial\Omega. \tag{3.2}$$

Noting that $r > 1$, and using (1.2), (3.1), and (3.2), we get

$$u_r(x,0) \geq u_0(x), \tag{3.3}$$

$$u_r(x,t) = u(x,t), \quad x \in \partial\Omega. \tag{3.4}$$

Applying the comparison principle, we have

$$u_r(x,t) \geq u(x,t). \tag{3.5}$$

For $p > 2$, by (3.5), we obtain:

$$\frac{[u(x,\lambda t)]^{p-2} - [u(x,t)]^{p-2}}{\lambda t - t} \geq \frac{(1/\lambda - 1)[u(x,t)]^{p-2}}{\lambda t - t},$$

where, $\lambda = r^{p-2}$. Letting $\lambda \to 1^+$, we get

$$\frac{\partial}{\partial t}[u(x,t)]^{p-2} \geq -\frac{1}{t}[u(x,t)]^{p-2},$$

in the distribution, which implies that lemma holds. Thus the proof is now complete. □

Proof of Theorem 3.1. For $p > 2$, from Lemma 3.2 we obtain:

$$\frac{\partial(t^{1/(p-2)}u)}{\partial t} \geq 0.$$

□

Theorem 3.3. *Let u be a weak solution of (1.1)-(1.3). If $1 < p < 2$, then there exists a time T such that*

$$u(x,t) = 0,$$

for all $(x,t) \in \Omega \times (T,\infty)$.

Proof. Denote $s_+ = \max\{s,0\}$, for all $s \in (-\infty,+\infty)$.

Define an auxiliary function,

$$v(x,t) = k(T-t)_+^{1/(2-p)} \ln(m + x_1 + \cdots + x_n), \tag{3.6}$$

where,

$$k = \left[\frac{(p-1)(2-p)n^{p/2}}{(2m)^p \ln(2m)}\right]^{1/(2-p)}, \quad T = \left(\frac{\max \|u_0\|_\infty}{k \ln 2}\right)^{2-p}, \tag{3.7}$$

and

$$m = \sup_{x\in\Omega}\{|x_1| + \cdots + |x_n|\} + 2. \tag{3.8}$$

Compute

$$\frac{\partial v}{\partial t} = -\frac{k}{2-p}(T-t)_+^{(p-1)/(2-p)} \ln(m + x_1 + \cdots + x_n), \tag{3.9}$$

$$\begin{aligned}
&\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)\\
=&\operatorname{div}\{k^{p-1}(T-t)_+^{(p-1)/(2-p)} n^{(p-2)/2}\\
&((m + x_1 + \cdots + x_n)^{1-p}, \cdots, (m + x_1 + \cdots + x_n)^{1-p})\}\\
=&-(p-1)k^{p-1}(T-t)_+^{(p-1)/(2-p)}\cdot\\
&\cdot n^{(p-2)/2}(m + x_1 + \cdots + x_n)^{-p},
\end{aligned} \tag{3.10}$$

and

$$v^q = k^q (T-t)_+^{q/(2-p)} \ln^q(m + x_1 + \cdots + x_n).$$

By $\rho(x) = (1+|x|)^{-l}, l > 0$, and using (3.6)-(3.10), we get

$$\rho(x)\frac{\partial v}{\partial t} \geq \frac{\partial}{\partial x}\left(\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right) - v^q,$$

and

$$\begin{gathered}
v(x,0) \geq u(x,0),\\
v(x,t) \geq u(x,t) = 0, \quad x \in \partial\Omega.
\end{gathered}$$

Applying Lemma 2.2, we obtain:

$$u(x,t) \le v(x,t),$$

for all $(x,t) \in Q$. By the definition of $v(x,t)$, we have

$$u(x,t) \le v(x,t) = 0,$$

for all $(x,t) \in \Omega \times (T, +\infty)$. Thus, the proof is complete. $\square$

4. **Finite speed of propagation of perturbations**

Here, we study the property of finite speed of perturbations.

Theorem 4.1. *Assume $p > 2$, $|\xi_n(0)| \le a$, $|\sigma_n(0)| \le b$, and u is the weak solution of the problem (1.1)-(1.3). Then, for any fixed $t > 0$, we have*

$$\sigma_n(t) - \sigma_n(0) \le C_1 t^{\mu} \left(\int_0^T \int_\Omega |\nabla u|^p dxdt \right)^{\beta},$$

$$\xi_n(t) - \xi_n(0) \ge -C_2 t^{\mu} \left(\int_0^T \int_\Omega |\nabla u|^p dxdt \right)^{\beta},$$

where, C_1, C_2 are constants depending on $n, p, a, b, u_0(x)$, $\sigma_n(t) = \sup\{z; x \in suppu(\cdot,t)\}$, $\xi_n(t) = \inf\{z; x \in suppu(\cdot,t)\}$, $z = x_n$, $\beta > 0, \mu > 0$, and $a, b > 0$ are constants independent of t.

Proof. First, we discuss the following Dirichlet problem,

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}((|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u) - u^q, \tag{4.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{4.2}$$

$$u(x,0) = u_{0\varepsilon}(x). \tag{4.3}$$

It is well known that (4.1)-(4.3) has a nonnegative classical solution u_ε [8].

Multiplying (4.1) by $(z-y)_+^s u_\varepsilon(x)$, $s \ge 2p$, $y \ge \sigma_n(0)$, and letting $\varepsilon \to 0$, we have,

$$\begin{aligned} I =& \frac{1}{2}\int_\Omega \rho(x)(z-y)_+^s |u(x,t)|^2 dx + \int_0^t \int_\Omega (z-y)_+^s u^{q+1} dxd\tau \\ =& -\int_0^t \int_\Omega |\nabla u|^{p-2} \nabla u \nabla((z-y)_+^s u) dxd\tau. \end{aligned}$$

Then, we have,

$$
\begin{aligned}
I =& -\int_0^t \int_\Omega |\nabla u|^{p-2}\nabla u \nabla((z-y)_+^s u)]dxd\tau \\
=& -\int_0^t \int_\Omega (z-y)_+^s |\nabla u|^p dxd\tau \\
& -\int_0^t \int_\Omega \nabla[(z-y)_+^s]u|\nabla u|^{p-2}\nabla u dxd\tau.
\end{aligned}
$$

Using Hölder inequality,

$$
\begin{aligned}
I \leq& -\int_0^t \int_\Omega (z-y)_+^s |\nabla u|^p dxd\tau + \frac{1}{2}\int_0^t \int_\Omega (x-y)_+^s |\nabla u|^p dxd\tau \\
& + C_1 \int_0^t \int_\Omega (z-y)_+^{s-p} |u|^p dxd\tau \\
\leq& -\frac{1}{2}\int_0^t \int_\Omega (x-y)_+^s |\nabla u|^p dxd\tau + C_1 \int_0^t \int_\Omega (z-y)_+^{s-p} |u|^p dxd\tau.
\end{aligned}
$$

Applying Hardy inequality [7], we obtain:

$$
\int_\Omega (z-y)_+^{s-p}|u|^p dx \leq \left(\frac{p}{s-p+1}\right)^p \int_\Omega (z-y)_+^s |D_z u|^p dx.
$$

Hence,

$$
\begin{aligned}
& \frac{1}{2}\int_\Omega \rho(x)(z-y)_+^s |u|^2 dx + \frac{1}{2}\int_0^t \int_\Omega (z-y)_+^s |\nabla u|^p dxd\tau \\
\leq& C \int_0^t \int_\Omega (z-y)_+^{s-p} |u|^p dxd\tau.
\end{aligned}
$$

Thus,

$$
\sup_{0<\tau\leq t} \int_\Omega \rho(x)(z-y)_+^s |u|^2 dx \leq C \iint_{Q_t} (z-y)_+^{s-p}|u|^p dxd\tau, \tag{4.4}
$$

and

$$
\iint_{Q_t} (z-y)_+^s |\nabla u|^p dxd\tau \leq C \iint_{Q_t} (z-y)_+^{s-p}|u|^p dxd\tau. \tag{4.5}
$$

For (4.4), using Hardy inequality, again we have

$$
\sup_{0<\tau\leq t} \int_\Omega \rho(x)(z-y)_+^s |u|^2 dx \leq C \iint_{Q_t} (z-y)_+^s |\nabla u|^p dxd\tau. \tag{4.6}
$$

Set

$$f_s(y)=\iint_{Q_t}(z-y)_+^s|\nabla u|^p dxd\tau,\ f_0(y)=\int_0^t\int_{\{x\in\Omega,x_n>y\}}|\nabla u|^p dxd\tau.$$

From (4.5) and weighted Nirenberg inequality, we have

$$\begin{aligned}
&f_{2p+1}(y)\\
\le&C_1\iint_{Q_t}(z-y)_+^{p+1}|u|^p dxd\tau\\
\le&C\int_0^t\left(\int_\Omega(z-y)_+^{p+1}|\nabla u|^p dx\right)^a\left(\int_\Omega(z-y)_+^{p+1}|u|^2dx\right)^{\frac{(1-a)p}{2}}d\tau,
\end{aligned}$$

where, $\frac{1}{p}=a(\frac{1}{p}-\frac{1}{n+p+1})+(1-a)\frac{1}{2}$. Therefore,

$$a=\frac{\frac{1}{p}-\frac{1}{2}}{\frac{1}{p}-\frac{1}{n+p+1}-\frac{1}{2}}<1.$$

Using (4.6), we obtain:

$$\begin{aligned}
&f_{2p+1}(y)\\
\le&C\left(\iint_{Q_t}(z-y)_+^{p+1}|\nabla u|^p dxd\tau\right)^{\frac{(1-a)p}{2}}\int_0^t\left(\int_\Omega(z-y)_+^{p+1}|\nabla u|^p dx\right)^a d\tau\\
\le&C[f_{p+1}(y)]^{(1-a)p/2}\left(\iint_{Q_t}(z-y)_+^{p+1}|\nabla u|^p dxd\tau\right)^a t^{1-a}\\
\le&Cf_{p+1}(y)^{(1-a)p/2+a}t^{1-a}.
\end{aligned}$$

Denote $\lambda=1-a$ and $\mu=a+(1-a)p/2$. Then, $\lambda>0$ and $1<\mu$. Applying Hölder's inequality, we have

$$\begin{aligned}
&f_{2p+1}(y)\\
\le&Ct^\lambda\left[\iint_{Q_t}(z-y)_+^{p+1}|\nabla u|^p dxds\right]^\mu\\
\le&Ct^\lambda\left[\iint_{Q_t}(z-y)_+^{2p+1}|\nabla u|^p dxds\right]^{\frac{(p+1)\mu}{2p+1}}\left[\int_0^t\int_\Omega|\nabla u|^p dxds\right]^{\frac{p\mu}{2p+1}}\\
\le&Ct^\lambda[f_{2p+1}(y)]^{\frac{(p+1)\mu}{2p+1}}[f_0(y)]^{\frac{p\mu}{2p+1}}.
\end{aligned}$$

Therefore,

$$f_{2p+1}(y) \le Ct^{\frac{\lambda}{\sigma}}[f_0(y)]^{\frac{p\mu}{(2p+1)\sigma}}, \qquad \sigma = 1 - \frac{p+1}{2p+1}\mu > 0.$$

Using Hölder's inequality again, we get

$$f_1(y) \le [f_0(y)]^{\frac{2p}{2p+1}}[f_{2p+1}(y)]^{\frac{1}{2p+1}} \le Ct^{\gamma}[f_0(y)]^{1+\theta},$$

where,

$$\gamma = \frac{\lambda}{(2p+1)\sigma}, \quad \theta = \frac{p\mu}{(2p+1)^2\sigma} - \frac{1}{2p+1} > 0.$$

Noticing that $f_1'(y) = -f_0(y)$, we obtain:

$$f_1'(y) \le -Ct^{-\gamma/(\theta+1)}[f_1(y)]^{1/(\theta+1)}.$$

If $f_1(\sigma_n(0)) = 0$, then $\sigma_n(t) \le b$. If $f_1(\sigma_n(0)) > 0$, then there exists a maximal interval $(\sigma_n(0), x^*)$, in which $f_1(y) > 0$, and

$$\left[f_1(y)^{\theta/(\theta+1)}\right]' = \frac{\theta}{\theta+1}\frac{f_1'(y)}{[f_1(y)]^{1/(\theta+1)}} \le -Ct^{-\gamma/(\theta+1)}.$$

Integrating the above inequality over $(\sigma_n(0), x^*)$, we have

$$f_1(x^*)^{\theta/(\theta+1)} - f_1(\sigma_n(0))^{\theta/(\theta+1)} \le -Ct^{-\gamma/(\theta+1)}(x^* - \sigma_n(0)),$$

which implies that

$$x^* \le \sigma_n(0) + Ct^{\gamma}(f_0(y))^{\theta},$$

noticing that $f_0(y)$ can be controlled by a constant C independent of y.

Similarly, we have

$$\xi_n(t) \ge \xi_n(0) - C_2 t^{\mu}\left(\int_0^T \int_\Omega |\nabla u|^p dxdt\right)^{\beta}.$$

The proof is now complete. □

Acknowledgments

The author thanks Professor Eiji Yanagida and Professor Kazuhiro Ishige for careful reading of and valuable suggestions for the my manuscript.

References

[1] N. D. Alikakos and L. C. Evans, Continuity of the gradient for weak solutions of degenerate parabolic equation, *J. Math. Pures Appl.* **62** (1983) 253-268.

[2] F. Bernis, Finite speed of propagation and asymptotic rates for some nonlinear higher order parabolic equations with absorption, *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986) 1-19.

[3] F. Bernis, Qualitative properties for some nonlinear higher order degenerate parabolic equations, *Houston J. Math.* **14** (1988) 319-352.

[4] E. Dibenedetto, Degenerate Parabolic Equations, *Springer-Verlag, New York*, 1993.

[5] D. Eidus, The Cauchy problem for the nonlinear filtration equation in an inhomogeneous medium, *J. Differential Equations* **84** (1990) 309-318.

[6] D. Eidus and S. Kamin, The filtration equation in a class of functions decreasing at infinity, *Proc. Amer. Math. Soc.* **120** (1994) 825-830.

[7] G. H. Hardy, J. E. Littlewood and G. P'olya, Inequalities, *Cambridge University Press, Cambridge*, 1952.

[8] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, *American Mathematical Society, Providence*, RI, 1968.

[9] L. A. Peletier and J. Wang, A very singular solution of a quasilinear degenerate diffusion equation with absorption, *Trans. Amer. Math. Soc.* **307** (1988) 813–826.

[10] A. F. Tedeev, The interface blow-up phenomenon and local estimates for doubly degenerate parabolic equations, *Appl. Anal.* **86** (2007) 755-782.

[11] A. F. Tedeev, Conditions for the time global existence and nonexistence of a compact support of solutions to the Cauchy problem for quasilinear degenerate parabolic equations, *Siberian Math. J.* **45** (2004) 155-164.

[12] H. Yuan, Extinction and positivity for the evolution P-Laplacian equation, *J. Math. Anal. Appl.* **196** (1995) 754-763.

[13] J. Zhao, Source type solutions of a quasiliear degenerate parabolic equation with absorption, *Chin. Ann. Math.* **15B** (1994) 89-104.

Changchun Liu
Department of Mathematics, Jilin University, Changchun 130012, P. R. China and Mathematical Institute, Tohoku University, Sendai 980-8578, Japan.
Email: `liucc@jlu.edu.cn`