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COMMON FIXED POINT THEOREMS OF INTEGRAL TYPE IN MODULAR SPACES

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ABSTRACT. Here, some common fixed point theorems for ρ-compatible maps of integral type in modular spaces are presented.

1. **Introduction**

In [3], Jungck defines the notion of compatible self-maps of a metric space (X, d) as a pair of maps $h, T : X \to X$ such that for all sequences $\{x_n\}$ in X with $\lim hx_n = \lim Tx_n = x \in X$ as $n \to \infty$, we have $\lim_n d(hTx_n, Thx_n) = 0$. He then proves a common fixed point theorem for pairs of compatible maps and establishes a further generalization in [4]. The notion of modular space, as a generalization of a metric space, was introduced by Nakano in 1950 and redefined and generalized by Musielak and Orlicz in 1959. Fixed point theorems in modular spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively, for example in [1], [7], [8], [9], [10], [12], etc. Our purpose is to define the notion of ρ-compatible mappings in

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modular spaces and establish some common fixed point theorems in modular spaces.

The remainder of the paper is structured as follows: In Section 2, we recall a basic definition and prove a common fixed point theorem for integral type ρ-compatible maps in modular spaces. In Section 3 and Section 4, two extensions of Theorem 2.2 are presented.

We begin with a brief recollection of concepts and facts of the theory of modular spaces from [2], [4], [5], [6], [10] and [11].

Definition 1.1. Let X be an arbitrary vector space over $K = (\mathbb{R}$ or $\mathbb{C})$.
a) A functional $\rho : X \longrightarrow [0, \infty]$ is called modular if:
(i) $\rho(x) = 0$ iff $x = 0$.
(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$.
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$.
If iii) is replaced by:
(iii)' $\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ for α, $\beta \geq 0$, $\alpha+\beta = 1$, for all $x, y \in X$, then the modular ρ is called convex modular.
b) A modular ρ defines a corresponding modular space; i.e., the space X_ρ given by:

$$X_\rho = \{x \in X \ |\rho(\alpha x) \to 0 \ as \ \alpha \to 0\}.$$

Remark 1.2. Note that ρ is an increasing function. Suppose $0 < a < b$. Then, property (iii) with $y = 0$ shows that $\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx)$.

Definition 1.3. Let X_ρ be a modular space.
a) A sequence $(x_n)_{n\in\mathbb{N}}$ in X_ρ is said to be:
(i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent.
c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_{n\in\mathbb{N}} \subset B$ and $x_n \to x$ we have $x \in B$.
d) A subset $B \subset X_\rho$ is called ρ-bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ-diameter of B.
e) ρ has the Fatou property if:

$$\rho(x - y) \leq \liminf \rho(x_n - y_n),$$

whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
f) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

In the next section, two common fixed point theorems for ρ-compatible mappings satisfying a contractive condition of integral type in modular spaces are proved.

2. A common fixed point theorem for contractive condition maps of integral type

Here, the existence of a common fixed point for ρ-compatible mappings satisfying a contractive condition of integral type in modular spaces is studied. We recall the following definition.

Definition 2.1. Let X_ρ be a modular space, where ρ satisfies the Δ_2-condition. Two self-mappings T and h of X_ρ are called ρ-compatible if $\rho(Thx_n - hTx_n) \to 0$, whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X_ρ such that $hx_n \to z$ and $Tx_n \to z$ for some point $z \in X_\rho$.

Theorem 2.2. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, k, l \in \mathbb{R}^+$, $c > l$ and T, $h : X_\rho \to X_\rho$ are two ρ-compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \leq k \int_0^{\rho(l(hx-hy))} \varphi(t)dt, \tag{2.1}$$

for some $k \in (0,1)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and for all $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{2.2}$$

If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Proof. Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{c}{l}$; i.e., $\frac{l}{c} + \frac{1}{\alpha} = 1$. Let x be an arbitrary point of X_ρ and generate inductively the sequence $(Tx_n)_{n\in\mathbb{N}}$ as follows: $Tx_n = hx_{n+1}$ for each n and $T(X_\rho) \subseteq h(X_\rho)$. For each integer $n \geq 1$, inequality (2.1) shows that

$$\begin{aligned}
\int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq \quad & k \int_0^{\rho(l(hx_{n+1}-hx_n))} \varphi(t)dt \\
\leq \quad & k \int_0^{\rho(c(Tx_n-Tx_{n-1}))} \varphi(t)dt \\
\leq \quad & k^2 \int_0^{\rho(l(hx_n-hx_{n-1}))} \varphi(t)dt.
\end{aligned}$$

By induction,

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))}\varphi(t)dt \le k^n \int_0^{\rho(l(Tx-x))}\varphi(t)dt. \tag{2.3}$$

Taking the limit as $n \to \infty$ yields:

$$\lim_n \int_0^{\rho(c(Tx_{n+1}-Tx_n))}\varphi(t)dt \le 0.$$

Thus, inequality (2.2) implies that

$$\lim_n \rho(c(Tx_{n+1}-Tx_n)) \to 0. \tag{2.4}$$

We now show that $(Tx_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. If not, then, there exists an $\varepsilon > 0$ and two sequences of integers $\{n(s)\}$, $\{m(s)\}$, with $n(s) > m(s) \ge s$, such that

$$d_s = \rho(l(Tx_{n(s)} - Tx_{m(s)})) \ge \varepsilon \text{ for } s = 1, 2, \cdots. \tag{2.5}$$

We can assume that

$$\rho(l(Tx_{n(s)-1} - Tx_{m(s)})) < \varepsilon. \tag{2.6}$$

In order to show this, suppose $n(s)$ is the smallest number exceeding $m(s)$ for which (2.5) holds and

$$\sum_s = \{n \in \mathbb{N} | \exists m(s) \in \mathbb{N}; \rho\ (l(Tx_n - Tx_{m(s)})) \ge \varepsilon \text{ and } n > m(s) \ge s\}.$$

Obviously, $\sum_s \neq \phi$ and since $\sum_s \subset \mathbb{N}$, then by well ordering principle, the minimum element of $\sum_s$ is denoted by $n(s)$, and clearly (2.6) holds. Now,

$$\begin{aligned}\int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))}\varphi(t)dt \le&\quad k\int_0^{\rho(l(hx_{m(s)}-hx_{n(s)}))}\varphi(t)dt\\ =&\quad k\int_0^{\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1}))}\varphi(t)dt.\end{aligned}$$

Moreover,

$$\begin{aligned}\rho(l(Tx_{m(s)-1} - Tx_{n(s)-1})) =&\quad \rho(l(Tx_{m(s)-1} - Tx_{m(s)} +\\ &\qquad Tx_{m(s)} - Tx_{n(s)-1}))\\ =&\quad \rho(\alpha\tfrac{l}{\alpha}(Tx_{m(s)-1} - Tx_{m(s)}) +\\ &\qquad \tfrac{lc}{c}(Tx_{m(s)} - Tx_{n(s)-1}))\\ \le&\quad \rho(\alpha l(Tx_{m(s)-1} - Tx_{m(s)})) +\\ &\qquad \rho(c(Tx_{m(s)} - Tx_{n(s)-1})).\end{aligned}$$

Using the Δ_2-condition and (2.4), then,

$$\lim_{s\to\infty} \rho(\alpha l(Tx_{m(s)-1} - Tx_{m(s)})) = 0.$$

Therefore,

$$\lim_{s} \int_0^{\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1}))} \varphi(t)dt \leq \int_0^{\varepsilon} \varphi(t)dt. \tag{2.7}$$

Also, by the inequality (2.5),

$$\int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))} \varphi(t)dt. \tag{2.8}$$

From inequalities (2.2), (2.4), (2.7) and (2.8), it follows that

$$\begin{aligned} \int_0^{\varepsilon} \varphi(t)dt \leq & \int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))} \varphi(t)dt \\ \leq & \; k\int_0^{\rho(c(Tx_{m(s)-1}-Tx_{n(s)-1}))} \varphi(t)dt \\ \leq & \; k\int_0^{\varepsilon} \varphi(t)dt, \end{aligned} \tag{2.9}$$

which is a contradiction. Therefore, by Δ_2-condition, $(Tx_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. Since X_ρ is ρ-complete, then there exists a $z \in X_\rho$ such that $\rho(c(Tx_n - z)) \to 0$ as $n \to \infty$. If T is continuous, then $T^2x_n \to Tz$ and $Thx_n \to Tz$. Since $\rho(hTx_n - Thx_n) \to 0$, then by ρ-compatibility, $hTx_n \to Tz$.

We now prove that z is a fixed point of T. We have,

$$\int_0^{\rho(c(T^2x_n-Tx_n))} \varphi(t)dt \leq k\int_0^{\rho(l(hTx_n-hx_n))} \varphi(t)dt. \tag{2.10}$$

Taking the limit as $n \to \infty$ yields:

$$\int_0^{\rho(c(Tz-z)} \varphi(t)dt \leq k\int_0^{\rho(l(Tz-z)} \varphi(t)dt,$$

which implies that

$$\int_0^{\rho(c(Tz-z))} \varphi(t)dt \leq 0.$$

Using inequality (2.2), $\rho(c(Tz - z)) = 0$ or $Tz = z$.

Moreover, $T(X_\rho) \subseteq h(X_\rho)$, and thus there exists a point z_1 such that $z = Tz = hz_1$. The inequality,

$$\int_0^{\rho(c(T^2x_n-Tz_1))} \varphi(t)dt \leq k\int_0^{\rho(l(hTx_n-hz_1))} \varphi(t)dt,$$

as $n \to \infty$, yields:

$$\int_0^{\rho(c(Tz-Tz_1))} \varphi(t)dt \leq k\int_0^{\rho(l(Tz-hz_1))} \varphi(t)dt,$$

and thus,

$$\begin{aligned}\int_0^{\rho(c(z-Tz_1))}\varphi(t)dt \leq & \quad k\int_0^{\rho(l(z-hz_1))}\varphi(t)dt\\ \leq & \quad k\int_0^{\rho(l(z-z))}\varphi(t)dt,\end{aligned}$$

resulting in $z = Tz_1 = hz_1$ and also $hz = hTz_1 = Thz_1 = Tz = z$ (see [5]). In addition, if one considers h to be continuous (instead of T), then by a similar argument (as above), one can prove $hz = Tz = z$.
Finally, suppose that z and w are two arbitrary common fixed points of T and h. Then, we have,

$$\begin{aligned}\int_0^{\rho(c(z-w))}\varphi(t)dt = & \quad \int_0^{\rho(c(Tz-Tw))}\varphi(t)dt\\ \leq & \quad k\int_0^{\rho(l(hz-hw))}\varphi(t)dt\\ \leq & \quad k\int_0^{\rho(c(z-w))}\varphi(t)dt,\end{aligned}$$

Which implies that $\rho(c(z-w)) = 0$, and hence $z = w$. □

Remark 2.3. If $c = l$ or $c = l = 1$, then Theorem 2.2 is not valid.

The following theorem is another version of Theorem 2.2 when $l = c$, by adding the restrictions that T, $h : B \to B$, where B is a ρ-closed and ρ-bounded subset of X_ρ.

Theorem 2.4. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition and B is a ρ-closed and ρ-bounded subset of X_ρ. Suppose T, $h : B \to B$ are two ρ-compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))}\varphi(t)dt \leq k\int_0^{\rho(c(hx-hy))}\varphi(t)dt,$$

for all $x, y \in B$, where $c, k \in \mathbb{R}^+$ with $k \in (0,1)$, and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and

$$\int_0^{\epsilon}\varphi(t)dt > 0, \text{ for all } \epsilon > 0.$$

If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Proof. Let $x \in B,\ m, n \in \mathbb{N}$. Then,

$$\begin{aligned} \int_0^{\rho(c(Tx_{n+m}-Tx_m))} \varphi(t)dt \leq & \int_0^{\rho(c(hx_{n+m}-hx_m))} \varphi(t)dt \\ \leq & \ k \int_0^{\rho(c(Tx_{n+m-1}-Tx_{m-1}))} \varphi(t)dt \\ \leq & \ k^2 \int_0^{\rho(c(Tx_{n+m-2}-Tx_{m-2}))} \varphi(t)dt \\ & \cdots \quad \cdots \\ \leq & \ k^m \int_0^{\rho(c(Tx_n-x))} \varphi(t)dt \\ \leq & \ k^m \int_0^{\delta_\rho(B)} \varphi(t)dt. \end{aligned}$$

Since B is ρ-bounded, then,

$$\lim_{n,m\to\infty} \int_0^{\rho(c(Tx_{m+n}-Tx_m))} \varphi(t)dt \leq 0,$$

which implies that $\lim_{n,m\to\infty} \rho(c(Tx_{n+m} - Tx_m)) = 0$. Therefore, by Δ_2-condition, $(Tx_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. Since X_ρ is ρ-complete and B is ρ-closed, there exists a $z \in B$ such that $\lim_{n\to\infty} \rho(c(Tx_n - z)) = 0$. If T is continuous, then $T^2x_n \to Tz$ and $Thx_n \to Tz$. Since $\rho(hTx_n - Thx_n) \to 0$, then by ρ-compatibility, $hTx_n \to Tz$.
We now prove that z is a fixed point of T. We have,

(2.11)
$$\int_0^{\rho(c(T^2x_n-Tx_n))} \varphi(t)dt \leq k \int_0^{\rho(c(hTx_n-hx_n))} \varphi(t)dt.$$

Taking the limit as $n \to \infty$ yields:

$$\int_0^{\rho(c(Tz-z)} \varphi(t)dt \leq k \int_0^{\rho(c(Tz-z)} \varphi(t)dt,$$

which implies that

$$\int_0^{\rho(c(Tz-z))} \varphi(t)dt \leq 0.$$

Using inequality (2.2), $\rho(c(Tz - z)) = 0$ or $Tz = z$.
Since $T(X_\rho) \subseteq h(X_\rho)$, then there exists a point z_1 such that $z = Tz = hz_1$, and

$$\int_0^{\rho(c(T^2x_n-Tz_1))} \varphi(t)dt \leq k \int_0^{\rho(c(hTx_n-hz_1))} \varphi(t)dt,$$

as $n \to \infty$ yields:

$$\int_0^{\rho(c(z-Tz_1))} \varphi(t)dt \leq k \int_0^{\rho(c(z-z))} \varphi(t)dt,$$

resulting in $z = Tz_1 = hz_1$ and also $hz = hTz_1 = Thz_1 = Tz = z$ (see [5]). In addition, if one considers h to be continuous (instead of T), then by a similar argument (as above), one can prove $hz = Tz = z$.
Finally, suppose that z and w are two arbitrary common fixed points of T and h. Then,

$$\begin{aligned} \int_0^{\rho(c(z-w))} \varphi(t)dt = & \int_0^{\rho(c(Tz-Tw))} \varphi(t)dt \\ \leq & \ k \int_0^{\rho(c(z-w))} \varphi(t)dt, \end{aligned}$$

which implies that $\rho(c(z-w)) = 0$, and hence $z = w$. □

In the next section, the existence of a common fixed point for a quasi-contraction map of integral type in modular spaces is presented.

3. A common fixed point theorem for quasi-contraction maps of integral type

The purpose of this section is to study Theorem 2.2 for quasi-contraction maps of integral type. We present the following Definition.

Definition 3.1. Two self-mappings $T,\ h : X_\rho \longrightarrow X_\rho$ of a modular space X_ρ are (c, l, q)-generalized contraction of integral type, if there exists $0 < q < 1$ and $c, l \in \mathbb{R}^+$ with $c > l$, such that

(3.1)
$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \leq q \int_0^{m(x,y)} \varphi(t)dt, \quad \textit{for all } x, y \in X_\rho,$$

where, $m(x, y) = \max\{\rho(l(hx-hy)), \rho(l(hx-Tx)), \rho(l(hy-Ty)), [\rho(l(hx-Ty)) + \rho(l(hy-Tx))]/2\}$, and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and

$$\int_0^{\epsilon} \varphi(t)dt > 0, \quad \text{for all } \epsilon > 0.$$

We now present the main theorem of this section.

Theorem 3.2. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose T and h are (c, l, q)-generalized contraction of integral type selfmaps of X_ρ and $T(X_\rho) \subseteq h(X_\rho)$. If one of h or T is continuous, then there exists a unique common fixed point of h and T.*

Proof. Choose $c > 2l$ and let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{c}{l}$; i.e., $\frac{l}{c} + \frac{1}{\alpha} = 1$. Then, $c > 2l$ implies that $\alpha l < c$.

Let x be an arbitrary point of X_ρ and generate inductively the sequence $(Tx_n)_{n\in\mathbb{N}}$ as follows: $Tx_n = hx_{n+1}$ and $T(X_\rho) \subseteq h(X_\rho)$. Thus, we have,

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq q \int_0^{m(x_{n+1},x_n)} \varphi(t)dt,$$

where,

$$\begin{aligned} m(x_{n+1},x_n) = \quad & \max\{\rho(l(hx_{n+1}-hx_n)), \rho(l(Tx_n - hx_n)), \\ & \rho(l(hx_{n+1}-Tx_{n+1})), \\ & [\rho(l(hx_{n+1}-Tx_n)) + \rho(l(hx_n - Tx_{n+1}))]/2\}. \end{aligned}$$

Then,

$$\begin{aligned} m(x_{n+1},x_n) = \quad & \max\{\rho(l(hx_{n+1}-hx_n)), \rho(l(Tx_n - Tx_{n+1})), \\ & [0 + \rho(l(hx_n - Tx_{n+1}))]/2\}. \end{aligned}$$

Moreover, by $\alpha l < c$,

$$\begin{aligned} \rho(l(hx_n - Tx_{n+1})) = \quad & \rho(l(Tx_{n-1} - Tx_{n+1}) \\ = \quad & \rho(\alpha \tfrac{l}{\alpha}(Tx_{n+1} - Tx_n) + \tfrac{lc}{c}(Tx_n - Tx_{n-1})) \\ \leq \quad & \rho(\alpha l(Tx_{n+1} - Tx_n)) + \rho(c(Tx_n - Tx_{n-1})) \\ \leq \quad & \rho(c(Tx_{n+1} - Tx_n)) + \rho(c(Tx_n - Tx_{n-1})). \end{aligned}$$

Then,

$$m(x_{n+1},x_n) \leq \rho(c(Tx_n - Tx_{n-1})),$$

and

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq q \int_0^{\rho(c(Tx_n - Tx_{n-1}))} \varphi(t)dt.$$

Continuing this process, we have,

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq q^n \int_0^{\rho(c(Tx-x))} \varphi(t)dt.$$

Taking the limit as $n \to \infty$ results in $\lim_n \rho(c(Tx_n - Tx_{n+1})) = 0$.
Now, suppose $l < c' < 2l$. Since ρ is an increasing function, then one may write $\rho(c'(Tx_n - Tx_{n+1})) \leq \rho(c(Tx_n - Tx_{n+1}))$, whenever $c' < 2l \leq c$. Taking the limit from both sides of this inequality shows that $\lim_n \rho(c'(Tx_n - Tx_{n+1})) = 0$, for $l < c' < 2l$. Thus, we have $\lim_n \rho(c(Tx_n - Tx_{n+1})) = 0$ for any $c > l$.
We now show that $(Tx_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. If not, then using the same argument as in the proof of Theorem 2.2 there exists an $\varepsilon > 0$ and two subsequences $\{m(s)\}$ and $\{n(s)\}$ and $n(s) > m(s) \geq s$ such that

$$\rho(c(Tx_{m(s)} - Tx_{n(s)})) \geq \varepsilon,$$

and we can assume,

$$\rho(c(Tx_{m(s)} - Tx_{n(s)-1})) < \varepsilon.$$

Then,

$$\int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))} \varphi(t)dt \leq q \int_0^{m(x_{m(s)},x_{n(s)})} \varphi(t)dt,$$

where,

$$\begin{aligned} m(x_{m(s)},x_{n(s)}) = \ & \max\{\rho(l(hx_{m(s)}-hx_{n(s)})), \rho(l(Tx_{n(s)}-hx_{n(s)})), \\ & \rho(l(Tx_{m(s)}-hx_{m(s)})), \\ & [\rho(l(Tx_{n(s)}-hx_{m(s)}))+\rho(l(hx_{n(s)}-Tx_{m(s)}))]/2\}. \end{aligned}$$

Note that

$$\begin{aligned} \rho(l(Tx_{m(s)-1}-Tx_{n(s)-1})) = \ & \rho(l(Tx_{m(s)-1}-Tx_{m(s)}+ \\ & Tx_{m(s)}-Tx_{n(s)-1})) \\ \leq \ & \rho(\alpha\tfrac{l}{\alpha}(Tx_{m(s)-1}-Tx_{m(s)})+ \\ & \tfrac{lc}{c}(Tx_{m(s)}-Tx_{n(s)-1})) \\ \leq \ & \rho(\alpha l(Tx_{m(s)-1}-Tx_{m(s)}))+ \\ & \rho(c(Tx_{m(s)}-Tx_{n(s)-1})). \end{aligned}$$

Using Δ_2-condition, as $s \to \infty$, we get $\rho(\alpha l(Tx_{m(s)-1}-Tx_{m(s)})) \to 0$ and $\rho(l(Tx_{n(s)-1}-Tx_{n(s)})) \to 0$. Therefore, as $s \longrightarrow \infty$,

$$\int_0^{m(x_{m(s)},x_{n(s)})} \varphi(t)dt \leq \int_0^{\varepsilon} \varphi(t)dt.$$

On the other hand, as $s \to \infty$,

$$\int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))} \varphi(t)dt.$$

Therefore,

$$\begin{aligned} \textstyle\int_0^{\varepsilon} \varphi(t)dt \leq \ & \textstyle\int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))} \varphi(t)dt \\ \leq \ & q \textstyle\int_0^{m(x_{m(s)},x_{n(s)})} \varphi(t)dt \\ \leq \ & q \textstyle\int_0^{\varepsilon} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Therefore, by Δ_2-condition, $(Tx_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. Since X_ρ is ρ-complete, then there exists a $z \in X_\rho$ such that $\rho(c(Tx_n - z)) \to 0$ as $n \to \infty$.

We now prove that z is a fixed point of T. If T is continuous, then $T^2x_n \to Tz$ and $Thx_n \to Tz$. Since $\rho(hTx_n - Thx_n) \to 0$, then by ρ-compatibility, $hTx_n \to Tz$. Note that

$$\int_0^{\rho(c(Tx_n-T^2x_n))} \varphi(t)dt \leq q \int_0^{m(x_n,Tx_n)} \varphi(t)dt,$$

where,

$$m(x_n, Tx_n) = \max\{\rho(l(hx_n - hTx_n)), \rho(l(hx_n - Tx_n)), \rho(l(hTx_n - TTx_n)), [\frac{\rho(l(hx_n - TTx_n)) + \rho(l(Tx_n - hTx_n))}{2}]\}.$$

Taking the limit as $n \longrightarrow \infty$, then,

$$\int_0^{\rho(c(z-Tz))} \varphi(t)dt \leq q \int_0^{\rho(c(z-Tz))} \varphi(t)dt,$$

and so $Tz = z$. Since $T(X_\rho) \subseteq h(X_\rho)$, then there exists a point z_1 such that $z = Tz = hz_1$. We have,

$$\int_0^{\rho(c(T^2x_n - Tz_1))} \varphi(t)dt \leq q \int_0^{m(Tx_n, z_1)} \varphi(t)dt,$$

and

$$m(Tx_n, z_1) = \max\{\rho(l(hTx_n - z)), \rho(l(hTx_n - T^2x_n)), \rho(l(z - Tz_1)), [\frac{\rho(l(hTx_n - Tz_1)) + \rho(l(z - T^2x_n))}{2}]\}.$$

Taking the limit as $n \longrightarrow \infty$, we have,

$$\int_0^{\rho(c(z-Tz_1))} \varphi(t)dt \leq q \int_0^{\rho(c(z-Tz_1))} \varphi(t)dt,$$

resulting in $z = Tz_1 = hz_1$ and also $hz = hTz_1 = Thz_1 = Tz = z$ (see [5]).

Moreover, if h is continuous instead of T, by a similar proof as above, $hz = Tz = z$. Now, for uniqueness, let z and w be two arbitrary fixed points of T and h. Then,

$$\begin{aligned} m(z, w) &= \max\{\rho(l(z - w)), 0, 0, [\frac{\rho(l(z-w)) + \rho(l(w-z))}{2}]\} \\ &= \rho(l(z - w)). \end{aligned}$$

Therefore,

$$\int_0^{\rho(c(z-w))} \varphi(t)dt \leq q \int_0^{\rho(l(z-w))} \varphi(t)dt,$$

which implies that $z = w$. □

4. Generalizations

Here, we extend the results of the last section. We need a general contractive inequality of integral type. Let $\mathbb{R}^+$ be a set of nonnegative real numbers and consider,

($\star$) $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as a nondecreasing and right continuous function

such that $\phi(t) < t$ for any $t > 0$.
We now recall the following lemma (see [11]).

Lemma 4.1. *Let $t > 0$. $\phi(t) < t$ if only if $\lim_k \phi^k(t) = 0$, where ϕ^k denotes the k-times repeated composition of ϕ with itself.*

Therefore, we can now study a new version of Theorem 2.2.

Theorem 4.2. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in \mathbb{R}^+$, $c > l$ and T, $h : X_\rho \to X_\rho$ are two ρ-compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \leq \phi(\int_0^{\rho(l(hx-hy))} \varphi(t)dt),$$

where ϕ is a function satisfying the property $(\star)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative and, for all $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0.$$

If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Proof. Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{c}{l}$; i.e., $\frac{l}{c} + \frac{1}{\alpha} = 1$. Let x be an arbitrary point of X_ρ and generate inductively the sequence $(Tx_n)_{n\in\mathbb{N}}$ as follows: $Tx_n = hx_{n+1}$ and $T(X_\rho) \subseteq h(X_\rho)$. For each integer $n \geq 1$,

$$\begin{aligned} \textstyle\int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq\ & \phi(\textstyle\int_0^{\rho(l(hx_{n+1}-hx_n))} \varphi(t)dt) \\ \leq\ & \phi(\textstyle\int_0^{\rho(c(Tx_n-Tx_{n-1}))} \varphi(t)dt) \\ \leq\ & \phi^2(\textstyle\int_0^{\rho(l(hx_n-hx_{n-1}))} \varphi(t)dt). \end{aligned}$$

By induction,

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq \phi^n(\int_0^{\rho(l(Tx-x))} \varphi(t)dt).$$

Taking the limit as $n \to \infty$, we obtain yields by Lemma 4.1,

$$\lim_n \int_0^{\rho(c(Tx_{n+1}-Tx_n))} \varphi(t)dt \leq 0.$$

Using the same method as in the proof of Theorem 2.2, T and h have a unique common fixed point. □

A new version of Theorem 3.2 follows next.

Theorem 4.3. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in \mathbb{R}^+$, $c > l$ and $T, h : X_\rho \to X_\rho$ such that $T(X_\rho) \subseteq h(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \leq \phi(\int_0^{\rho(m(x,y))} \varphi(t)dt),$$

where $m(x,y) = \max\{\rho(l(hx-hy)), \rho(l(hx-Tx)), \rho(l(hy-Ty)), [\rho(l(hx-Ty)) + \rho(l(hy-Tx))]/2\}$ and ϕ is a function satisfying the property $(\star)$. If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Proof. See the proof of Theorem 3.2. □

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