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MULTIPLE POINT SELF-TRANSVERSE IMMERSIONS OF CERTAIN MANIFOLDS

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ABSTRACT. In this paper we determine the multiple point manifolds of certain self-transverse immersions in Euclidean spaces. Following the triple points, these immersions have a double point self-intersection set which is the image of an immersion of a smooth 5-dimensional manifold, cobordant to Dold manifold V^5 or a boundary. We show that there is an immersion of $S^7 \times P^2$ in $\mathbb{R}^{13}$ with double point manifold cobordant to Dold manifold V^5, and an immersion of $P^2 \times P^2 \times P^2 \times P^2 \times P^2$ in $\mathbb{R}^{15}$ with double point manifold a boundary and the triple point set with odd number. These are obtained by introducing the product technique and reading off the Stiefel-Whitney numbers of the self-intersection manifolds.

1. Introduction

Given a self-transverse immersion $f : M^n \looparrowright \mathbb{R}^{n+k}$, the r-fold intersection set $I_r(f)$ is defined as follows:

$$I_r(f) = \{y \in \mathbb{R}^{n+k} : |f^{-1}(y)| = r\}.$$

The self-transversality of f implies that this subset of $\mathbb{R}^{n+k}$ is the image of an immersion made self-transverse

$$\theta_r(f) : L^{n-k(r-1)} \looparrowright \mathbb{R}^{n+k}$$

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of a manifold L of dimension $n-k(r-1)$ called an r-fold point manifold of f. It is natural to ask given an immersion how we can find its r-fold point manifold, or, given the manifold L, is there any immersion of a manifold with an r-fold point manifold cobordant to L?

Several authors have considered the problem when the manifold L is 0-dimensional. When the manifold L is 2-dimensional, the problem was considered by A. Szücs in [11] and P. J. Eccles in [4, with correction]. Here we are going to look at the problem when the manifold L is a 5-dimensional Dold manifold V^5. The simplest case is when $r = 2$. Therefore, we are going to ask, is there any immersion $f : M^n \looparrowright \mathbb{R}^{2n-5}$ of some manifold M^n with double point manifold cobordant to V^5? If $n > 10$ and $r > 2$ the r-fold point manifolds are empty, so we consider just the double points. In [2] for even dimensional manifolds and $n > 10$ we have shown that the double point manifolds for such immersions are all boundaries. Here we consider the problem when $n = 8, 9$ and 10. The reasons we bring them in a separate document are; first they have triple points, second we introduce a technique from which we can study the product immersions. If $n = 8$, then the double and triple point manifolds are boundaries, when $n = 9$ there is an immersion of $P^2 \times S^7$ in $\mathbb{R}^{13}$ with double point manifold cobordant to V^5, thanks to the Hopf immersion $S^7 \looparrowright \mathbb{R}^8$ and P. J. Eccles's paper [7]. If $n = 10$ the double point manifolds are boundaries and there is an immersion of $P^2 \times P^2 \times P^2 \times P^2 \times P^2$ in $\mathbb{R}^{15}$ with odd number of triple point set. This can be generalized to the case $M^{2n} \looparrowright \mathbb{R}^{3n}$. The other odd dimensional cases are more complicated, and each dimension has its own difficulties.

We have introduced a method in [3] to determine the multiple point manifolds of an immersion. Here we are going to use that method to solve the above problems. The method is equivalent to detecting the spherical elements in a certain homology group which is generally an outstanding open problem in algebraic topology. We use algebraic topology and, in particular, the correspondence between cobordism groups and homotopy groups of Thom complexes. The un-oriented bordism class of a manifold is determined by its Stiefel-Whitney numbers, and the Stiefel-Whitney numbers of the self-intersection manifolds of an immersion can be read off from certain homological information about the immersion. In the next section we explain briefly what we did in [3]. The main results of this paper are the following.

Theorem 1.1. *Let $f : M^9 \looparrowright \mathbb{R}^{13}$ be an immersion, then there is a 9-dimensional boundary which immerses in $\mathbb{R}^{13}$ with the double point manifold cobordant to Dold manifold V^5.*

Theorem 1.2. *Let $f : M^8 \looparrowright \mathbb{R}^{11}$ be any immersion, then the multiple point manifolds of these immersions are all boundaries.*

Theorem 1.3. *Let $f : M^{10} \looparrowright \mathbb{R}^{15}$ be any immersion, then the double point manifolds are all boundaries and there is an immersion where the triple point set is an odd number.*

2. The Stiefel-Whitney Numbers of Multiple Points

Let $\mathrm{Imm}(n, k)$ denote the group of bordism classes of immersions $M^n \looparrowright \mathbb{R}^{n+k}$ of compact closed smooth manifolds in Euclidean $(n + k)$-space. By general position every immersion is regularly homotopic, bordant to a self-transverse immersion, and so each element of $\mathrm{Imm}(n, k)$ can be represented by a self-transverse immersion. In the same way, a bordism between self-transverse immersions can be taken to be self-transverse. It is clear that such a bordism will induce a bordism of the immersions of the r-fold point self-intersection map

$$\theta_r : \mathrm{Imm}(n, k) \to \mathrm{Imm}(n - k(r - 1), rk).$$

Let $MO(k)$ denote the Thom complex of the universal $O(k)$-bundle $\gamma^k : EO(k) \to BO(k)$. Using the Pontrjagin-Thom construction, Wells describes an isomorphism

$$\phi : \mathrm{Imm}(n, k) \cong \pi^S_{n+k} MO(k).$$

(For details of cobordism in this setting, see R. Wells [12].) But the stable homotopy group $\pi^S_{n+k} MO(k)$ is known to be isomorphic to homotopy group $\pi_{n+k} QMO(k)$, where the QX stands for the direct limit $\Omega^\infty \Sigma^\infty X = \lim \Omega^n \Sigma^n X$, Σ denotes the reduced suspension functor, and Ω denotes the loop space functor. We consider the $\mathbb{Z}_2$-homology Hurewicz homomorphism

$$h : \pi^S_{n+k} MO(k) \cong \pi_{n+k} QMO(k) \longrightarrow H_{n+k}(QMO(k); \mathbb{Z}_2).$$

The main result of [3] describes how, for a self-transverse immersion $f : M^n \looparrowright \mathbb{R}^{n+k}$ corresponding to $\alpha \in \pi^S_{n+k} MO(k)$, the Hurewicz image $h(\alpha) \in H_{n+k} QMO(k)$ determines the normal Stiefel-Whitney numbers of the self-intersection manifold $\Delta_r(f)$.

Notation: $X^{[n]} = X \wedge X \wedge \cdots \wedge X$ denotes the n times smash product of X.

The r-adic construction on X denoted by D_rX is defined as follows:

$$D_rX = W\Sigma_r \ltimes_{\Sigma_r} X^{[r]} = (W\Sigma_r \times_{\Sigma_r} X^{[r]})/(W\Sigma_r \times_{\Sigma_r} \{*\}).$$

Here Σ_r denotes the permutation group on r elements and $W\Sigma_r$ is a contractible space with a free Σ_r-action. The group Σ_r acts on the smash product $X^{[r]}$ by permuting the factors. There is a natural map $h^r : QX \to QD_rX$ known as the stable James-Hopf map which induces the stable Hopf invariant $h^r_* : \pi^S_{n+k}X \to \pi^S_{n+k}D_rX$ (see [6]). If the self-transverse immersion $f : M^n \looparrowright \mathbb{R}^{n+k}$ corresponds to the element $\alpha \in \pi^S_{n+k}MO(k)$, then the immersion of the r-fold point self-intersection manifold $\theta_r(f) : \Delta_r(f) \looparrowright \mathbb{R}^{n+k}$ corresponds to the element $h^r_*(\alpha) \in \pi^S_{n+k}D_rMO(k)$ given by the stable Hopf invariant (see [8]). The map

$$\xi_* : \pi^S_{n+k}D_rMO(k) \to \pi^S_{n+k}MO(rk),$$

induced by the map of Thom complexes $\xi : D_rMO(k) \to MO(rk)$, makes the following commutative diagram.

$$\begin{array}{ccccc}
\pi^S_{n+k}MO(k) & \xrightarrow{h^r_*} & \pi^S_{n+k}D_rMO(k) & \xrightarrow{\xi_*} & \pi^S_{n+k}MO(rk) \\
\downarrow h & & \downarrow h^S & & \downarrow h^S \\
H_{n+k}QMO(k) & \xrightarrow{h^r_*} & H_{n+k}D_rMO(k) & \xrightarrow{\xi_*} & H_{n+k}MO(rk)
\end{array}$$

Diagram 1

In this diagram the second and third vertical maps are stable Hurewicz homomorphisms defined using the fact that Hurewicz homomorphisms commute with suspension. Notice that the normal Stiefel-Whitney numbers (and so bordism class) of the multiple point self-intersection manifold $\Delta_r(f)$ of an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ corresponding to $\alpha \in \pi^S_nMO(k)$ are determined by (and determine) the Hurewicz image $h^S(\beta)$ of the element $\beta = \xi_*h^r_*(\alpha) \in \pi^S_nMO(rk)$ corresponding to the immersion $\theta_r(f)$. The map ξ_* on the bottom row is determined in ([3], Theorem 3.1 and generally Theorem 3.2). For reference we need its spacial case.

Theorem 2.1. *The homomorphism*

$$\xi_* : H_*D_2MO(k) \longrightarrow H_*MO(2k)$$

is given by

(a) $\quad \xi_*(e_{i_1}e_{i_2}\cdots e_{i_k} \cdot e_{j_1}e_{j_2}\cdots e_{j_k}) = (e_{i_1}e_{i_2}\cdots e_{i_k}e_{j_1}e_{j_2}\cdots e_{j_k});$

$$(b) \quad \xi_*(Q^p e_{i_1} e_{i_2} \cdots e_{i_k}) = \sum_{m_j \geq 0} \prod_{j=1}^{k} \sum_{u=0}^{i_j - 1} \binom{m_j + u - 1}{u} e_{i_j + m_j + u} e_{i_j - u},$$

where $m_1 + m_2 + \cdots + m_k = p - |I|$.

This theorem enable us to determine the characteristic numbers of the multiple point manifolds. To read off these numbers, we recall the structure of $H_*MO(k)$.

Homology of $MO(k)$ and $QMO(k)$: Let $e_i \in H_iBO(1) \cong \mathbb{Z}_2$ be the non-zero element (for $i \geq 0$). By a counting argument we can show that

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 0 \leq i_1 \leq i_2 \leq \cdots \leq i_k\}$$

is a basis for $H_*BO(k)$. Since the Thom complex $MO(k)$ is homotopy equivalent to the quotient space $BO(k)/BO(k-1)$. It follows that

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k\}$$

is a basis for $\tilde{H}_*MO(k)$.

Dyer and Lashof (see [10]) make use of the Kudo-Araki operations $Q^i : H_mQX \to H_{m+i}QX$ to describe the homology of QX. These operations are trivial for $i < m$ and equal to the Pontrjagin square for $i = m$. If I denotes the sequence $(i_1, i_2, \ldots, i_r)$, then we write $Q^I x = Q^{i_1} Q^{i_2} \cdots Q^{i_r} x$. The sequence I is admissible if $i_j \geq 2i_{j+1}$ for $1 \leq j < r$, and its excess is given by $e(I) = i_1 - i_2 - \cdots - i_r$. With this notation we can give the description of H_*QX as a polynomial algebra: if $\{x_\lambda \mid \lambda \in \Lambda\}$ is a homogeneous basis for $\tilde{H}_*X \subseteq H_*QX$ where X is path-connected space, then

$$H_*QX = \mathbb{Z}_2[Q^I x_\lambda \mid \lambda \in \Lambda, I \text{ admissible of excess } e(I) > \dim x_\lambda].$$

We may define a height function ht on the monomial generators of H_*QX by $ht(x_\lambda) = 1$, $ht(Q^i u) = 2ht(u)$, and $ht(u \cdot v) = ht(u) + ht(v)$ (where $u \cdot v$ represents the Pontrjagin product).

3. Immersions of $M^9 \looparrowright \mathbb{R}^{13}$

According to the above notes, the multiple point manifolds of an immersion are determined by the spherical elements of the homology group $H_*QMO(k)$. Determining the spherical elements, (i.e. the elements in the image of the Hurewicz map in Diagram 1) is a difficult problem, but we can show that every spherical element is primitive and $\mathcal{A}_2$-annihilated. Therefore, first we find the primitive $\mathcal{A}_2$-annihilated submodule of $H_*QMO(k)$. Since in our case the triple points form a

1-dimensional manifold, they are boundaries. On the other hand, the double point manifold is 5-dimensional, so it is cobordant to a boundary or Dold manifold V^5. As a result we need to find those spherical elements in $H_*QMO(k)$ which involve height two elements. Because the group $H_*D_2MO(k) \subseteq H_*QMO(k)$ consists of height two elements, in dimension 13 for $k = 4$ these are the following:

$$Q^9e_1^4, \quad Q^8e_1^3e_2, \quad Q^7e_1^3e_3, \quad Q^7e_1^2e_2^2, \quad e_1^4\cdot e_1^3e_6, \quad e_1^4\cdot e_1^2e_2e_5,$$
$$e_1^4\cdot e_1^2e_3e_4, \quad e_1^4\cdot e_1e_2^2e_4, \quad e_1^4\cdot e_1e_2e_3^2, \quad e_1^4\cdot e_2^3e_3, \quad e_1^3e_2\cdot e_2^4,$$
$$e_1^3e_2\cdot e_1e_2^2e_3, \quad e_1^3e_2\cdot e_1^2e_3^2, \quad e_1^3e_2\cdot e_1^2e_2e_4, \quad e_1^3e_2\cdot e_1^3e_5, \quad e_1^3e_3\cdot e_1e_2^3,$$
$$e_1^3e_3\cdot e_1^2e_2e_3, \quad e_1^3e_3\cdot e_1^3e_4, \quad e_1^2e_2^2\cdot e_1e_2^3, \quad e_1^2e_2^2\cdot e_1^2e_2e_3, \quad e_1^2e_2^2\cdot e_1^3e_4.$$

We can show that every height one element is primitive if and only if it involves e_1, and if a is primitive, then Q^ka is also primitive (see [1] Lemmas [3.1, 3.2, 3.3]). Note that

$$H_{13}(MO(4)) = \langle\{e_{i_1}e_{i_2}e_{i_3}e_{i_4} : 1 \le i_1 \le i_2 \le i_3 \le i_3 \le i_4\}\rangle$$

in which the following height one elements are not primitive:

$$e_2^3e_7, \quad e_2^2e_4e_5, \quad e_2^2e_3e_6, \quad e_2e_3e_4^2, \quad e_2e_3^2e_5, \quad e_3^3e_4.$$

To see which linear combination of non-primitive elements are primitive, let ψ denote the cup-co-product. Then $\psi(e_n) = \sum_{i+j=n} e_i \otimes e_j$ and the Cartan formula holds. Note that $a \in H_*X$ is primitive if $\psi(a) = a\otimes 1 + 1\otimes a$. The calculations show that the following combinations are primitive:

$$\begin{aligned} A &= e_1^4\cdot e_1^4\cdot e_1^3e_2 + e_1^4\cdot e_2^3e_3 + e_1^3e_2\cdot e_2^4 + e_1^3e_3\cdot e_1e_2^3 + e_1^3e_2\cdot e_1e_2^2e_3 \\ &\quad + \ e_1e_2^3\cdot e_1^2e_2^2 + e_1^2e_2^2\cdot e_1^2e_2e_3 + e_3^3e_4, \\ B &= e_1^4\cdot e_1e_2^2e_4 + e_1^3e_2\cdot e_1e_2^2e_3 + e_1e_2^3\cdot e_1^3e_3 + e_1^2e_2^2\cdot e_1^3e_4 + e_2e_3^2e_5, \\ C &= e_1^4\cdot e_1e_2e_3^2 + e_1e_2^3\cdot e_1^2e_2^2 + e_1^3e_2\cdot e_1^2e_3^2 + e_2e_3e_4^2, \\ D &= e_1^4\cdot e_1^2e_2e_5 + e_1^3e_2\cdot e_1^2e_2e_4 + e_1^3e_3\cdot e_1^2e_2e_3 + e_1^3e_4\cdot e_1^2e_2^2 + e_1^3e_2\cdot e_1^3e_5 \\ &\quad + \ e_2^2e_3e_6, \\ E &= e_1^4\cdot e_1^2e_3e_4 + e_1^3e_2\cdot e_1^2e_3^2 + e_1^3e_3\cdot e_1^2e_2e_3 + e_1^3e_4\cdot e_1^3e_3 + e_1^3e_2\cdot e_1^2e_2e_4 \\ &\quad + \ e_1^2e_2^2\cdot e_1^2e_2e_3 + e_2^2e_4e_5, \\ F &= e_1^4\cdot e_1^3e_6 + e_1^3e_2\cdot e_1^3e_5 + e_1^3e_3\cdot e_1^3e_4 + e_2^3e_7. \end{aligned}$$

So we have the following.

Corollary 3.1. *The primitive submodule of* $H_{13}QMO(4)$ *is generated by the following elements:*

$$A,\quad B,\quad C,\quad D,\quad E,\quad F,\quad \delta,\quad Q^9e_1^4,\quad Q^8e_1^3e_2,\quad Q^7e_1^3e_3,\quad Q^7e_1^2e_2^2.$$

Where δ *runs over all the primitive height one elements.*

The primitive $\mathcal{A}_2$-annihilated submodule of $H_{13}QMO(4)$ can be determined with a long calculation which we omit. Note that $a \in H_*X$ is $\mathcal{A}_2$-annihilated if $Sq_*^i a = 0$ for all $i \geq 1$, where Sq_*^i is dual to the Steenrod operation $Sq^i : H^nX \to H^{n+i}X$. The action of Steenrod squares on the elements of Corollary 3.1 gives us the following lemma.

Lemma 3.2. *The primitive* $\mathcal{A}_2$*-annihilated submodule of* $H_{13}QMO(4)$ *is generated by*

$$Q^7e_1^3e_3.$$

Claim: The element $Q^7e_1^3e_3$ is spherical in $H_{13}QMO(4)$.

In fact we will show that there is an immersion $P^2 \times S^7 \looparrowright \mathbb{R}^{13}$, and if γ denotes this immersion in $\pi_{13}QMO(4)$, then $h(\gamma) = Q^7e_1^3e_3$. We know that there is an immersion $P^2 \looparrowright \mathbb{R}^3$, given by Boy's surface. Let α represents this immersion in $\pi_3QMO(1)$, then since $H_3QMO(1) \cong \langle e_1 \cdot e_1 \cdot e_1,\ e_1 \cdot e_2,\ Q^2e_1,\ e_3\rangle$. Calculations show that the primitive $\mathcal{A}_2$-annihilated submodule is generated by the single element $e_1 \cdot e_1 \cdot e_1 + e_1 \cdot e_2 + Q^2e_1 + e_3$, so necessarily

$$h(\alpha) = e_1 \cdot e_1 \cdot e_1 + e_1 \cdot e_2 + Q^2e_1 + e_3.$$

On the other hand, there is an immersion of $S^7 \looparrowright \mathbb{R}^8$ known as Hopf immersion. In [7] P.J. Eccles has shown that the Hurewicz image of this immersion in $H_8QMO(1)$ is $Q^7e_1 + Q^3e_1 \cdot Q^3e_1$. Now since the suspension kills the products, with a double suspension, if β denotes the Hopf immersion in $\pi_{10}QMO(3)$, then its Hurewicz image is

$$h(\beta) = Q^7e_1^3.$$

Now we describe how to determine the cobordism class of the double point manifold of the product immersion $P^2 \times S^7 \looparrowright \mathbb{R}^3 \times \mathbb{R}^{10} = \mathbb{R}^{13}$ (made self-transverse) from the Hurewicz images $h(\alpha)$ and $h(\beta)$.

Let $m : MO(1) \wedge MO(3) \to MO(4)$ be the map of Thom complexes arising from the Whitney sum map $BO(1) \times BO(3) \to BO(4)$. Then the product immersion corresponds to the element

$$\alpha \cdot \beta = m_*(\alpha \wedge \beta) \in \pi_{13}MO(4).$$

If α is represented by a map $f : S^3 \wedge S^n \to MO(1) \wedge S^n$ and β is represented by a map $g : S^{10} \wedge S^m \to MO(3) \wedge S^m$, then since $\dim P^2 < \dim S^7 + 1 + 3$, $\alpha \wedge \beta$ is represented by the map

$$\begin{aligned} S^{3+n} \wedge S^{10+m} &\xrightarrow{f\wedge 1} MO(1) \wedge S^{10+m} \xrightarrow{1\wedge g} MO(1) \wedge QMO(3) \\ &\xrightarrow{l} Q(MO(1) \wedge MO(3)) \xrightarrow{Qm} QMO(4), \end{aligned}$$

where the map $l : MO(1) \wedge QMO(3) \to Q(MO(1) \wedge MO(3))$ is defined by $l[x, z](t) = [x, z(t)]$ for $x \in MO(1)$, $z \in QMO(3)$.

Now recall that the cobordism class of the double point manifold of the product immersion is determined by $\xi_* h^2_* h(\alpha \cdot \beta)$, where

$$h^2_* h(\alpha \cdot \beta) \in H_{13} D_2 MO(4).$$

To evaluate this, observe that from [5] the map $l : X \wedge QY \longrightarrow Q(X \wedge Y)$ corresponds to a combinatorial map $l : X \wedge \Gamma^+ Y \longrightarrow \Gamma^+(X \wedge Y)$ representing the filtration of Γ^+ and thus inducing a map $l : X \wedge D_2 Y \longrightarrow D_2(X \wedge Y)$. This is given by $l(x, [w, y_1, y_2]) = [w, (x, y_1), (x, y_2)]$. Furthermore, from [6] the James-Hopf maps h^2 commute with l:

$$\begin{array}{ccc} X \wedge QY & \xrightarrow{\quad l \quad} & Q(X \wedge Y) \\ \downarrow{\scriptstyle 1 \wedge h^2} & & \downarrow{\scriptstyle h^2} \\ X \wedge QD_2 Y & \xrightarrow{\quad l \quad} & QD_2(X \wedge Y) \end{array}$$

so the following diagram commutes.

$$\begin{array}{ccccc} \tilde{H}_* MO(1) \otimes \tilde{H}_* QMO(3) & \xrightarrow{l_*} & \tilde{H}_* Q(MO(1) \wedge MO(3)) & \xrightarrow{Qm_*} & \tilde{H}_* QMO(4) \\ \downarrow{\scriptstyle 1 \otimes h^2_*} & & \downarrow{\scriptstyle h^2_*} & & \downarrow{\scriptstyle h^2_*} \\ \tilde{H}_* MO(1) \otimes \tilde{H}_* D_2 MO(3) & \xrightarrow{l_*} & \tilde{H}_* D_2(MO(1) \wedge MO(3)) & \xrightarrow{Qm_*} & \tilde{H}_* D_2 MO(4) \end{array}$$

Thus $h^2_* h(\alpha \cdot \beta) = Qm_* l_*(h^S(\alpha) \otimes h(\beta))$, where $h^S(\alpha)$ denotes the stable Hurewicz image of α. But in calculations it is necessary to describe the homomorphisms

$$l_* : \tilde{H}_*(MO(1) \wedge D_2 MO(3)) \longrightarrow \tilde{H}_* D_2(MO(1) \wedge MO(3))$$

and

$$Qm_* : \tilde{H}_* D_2(MO(1) \wedge MO(3)) \to \tilde{H}_* D_2 MO(4).$$

The calculation of the map Qm_* is easy. In fact we have

$$Qm_*(a \otimes b \cdot c \otimes d) = ab \cdot cd, \qquad Qm_*(Q^p(a \otimes b)) = Q^p ab.$$

To evaluate l_*, recall that $\tilde{H}_*(X \wedge Y)$ is generated by homology classes of the form $a \otimes b$ where $a \in \tilde{H}_*(X)$, $b \in \tilde{H}_*(Y)$, and $\tilde{H}_* D_2(X \wedge Y)$ is generated by height two elements in $\tilde{H}_* Q(X \wedge Y)$. Note that the generators of the homology group $\tilde{H}_*(X \wedge D_2 Y)$ have the forms $a \otimes b \cdot c$ and $a \otimes Q^n b$, where $a \in \tilde{H}_*(X)$ and $b, c \in \tilde{H}_*(Y)$. Although we do not need this lemma, but for reference in the future works we bring it here.

Lemma 3.3. $l_*(a \otimes b \cdot c) = \sum (a' \otimes b) \cdot (a'' \otimes c) \in H_* D_2(MO(1) \wedge MO(3))$ *where*

$$\psi(a) = \sum a' \otimes a'' \in \tilde{H}_*(MO(1) \wedge MO(1)) = \tilde{H}_* MO(1) \otimes \tilde{H}_* MO(1)$$

which is the vector space dual of the cup product

$$H^* MO(1) \otimes H^* MO(1) \to H^* MO(1).$$

Proof. It is easy to see that the following diagram is commutative.

$$\begin{array}{ccccc} X \wedge (Y \wedge Y) & \xrightarrow{\triangle \wedge 1 \wedge 1} & (X \wedge X) \wedge (Y \wedge Y) & \xrightarrow{1 \wedge \tau \wedge 1} & (X \wedge Y) \wedge (X \wedge Y) \\ \downarrow & & & & \downarrow \\ X \wedge D_2 Y & & \xrightarrow{\quad l \quad} & & D_2(X \wedge Y) \end{array}$$

Now the Lemma is immediate from the related homology diagram, where $X = MO(1)$ and $Y = MO(3)$. ☐

Lemma 3.4. *If $a \in \tilde{H}_*(X)$ is $\mathcal{A}_2$-annihilated, then $l_*(a \otimes Q^n d) = Q^n(a \otimes d)$.*

Proof. We use the argument on page 56 of Madsen and Milgram [9]. As explained there, a class $x \in H^m(X)$ induces a cohomology class $e^0 \otimes x \otimes x \in H^{2m}(W\Sigma_2 \times_{\Sigma_2} X^2)$. Now using the definition of the Steenrod squares given by Steenrod and Epstain it follows that

$$\begin{aligned}(1 \times \triangle)^*(e^0 \otimes x \otimes x) &= \sum_{i=0}^{m} e^i \otimes Sq^{m-i} x \in H^{2m}(W\Sigma_2 \times_{\Sigma_2} X) \\ &= H^{2m}(B\Sigma_2 \times X).\end{aligned}$$

Hence

$$(1\times\triangle)^*(e^{k-m}\otimes x\otimes x)=\sum_{i=0}^{m}e^{k-m+i}\otimes Sq^{m-i}x$$
$$=e^k\otimes x+\sum_{i=0}^{m-1}e^{k-m+i}\otimes Sq^{m-i}x.$$

Therefore, by vector space duality, if $a\in H_m(X)$ is $\mathcal{A}_2$-annihilated

$$1\times\triangle: B\Sigma_2\times X=W\Sigma_2\times_{\Sigma_2}X\longrightarrow W\Sigma_2\times_{\Sigma_2}X^2$$

induces $(1\times\triangle)_*(e_k\otimes a)=e_{k-m}\otimes a\otimes a$ ($=0$ for $k<m$). It follows that the map

$$l{:}X\times(W\Sigma_2\times_{\Sigma_2}Y^2)=W\Sigma_2\times_{\Sigma_2}X\times Y^2\overset{1\times\triangle\times 1^2}{\longrightarrow}W\Sigma_2\times_{\Sigma_2}X^2\times Y^2$$
$$=W\Sigma_2\times(X\times Y)^2$$

induces $l_*(a\otimes e_k\otimes b\otimes b)=e_{k-m}\otimes(a\otimes b)\otimes(a\otimes b)$, ($=0$ for $k<m$). In other words $l_*(a\otimes Q^{n+k}b)=Q^{n+k}(a\otimes b)$, where $b\in\tilde{H}_n(Y)$, proving the lemma. □

Proof of the Claim: Recall that since

$$h(\beta)=Q^7e_1^3,\quad h(\alpha)=e_1\cdot e_1\cdot e_1+e_1\cdot e_2+Q^2e_1+e_3,$$

so $h^S(\alpha)=e_3$. Therefore, from Lemma 3.4 we have

$$h(\alpha\cdot\beta)=Qm_*l_*(e_3\otimes Q^7e_1^3)=Qm_*(Q^7(e_3\otimes e_1^3))=Q^7e_1^3e_3.$$

This proves our Claim. □

Now by Theorem 2.1,

$$\xi_*Q^7e_1^3e_3=e_1^6e_3e_4+e_1^6e_2e_5+e_1^7e_6+e_1^5e_2e_3^2.$$

On the other hand, from Diagram (1), the double point self-intersection manifold of an immersion $M^9\looparrowright\mathbb{R}^{13}$ may be identified, up to a bordism, by using the stable Hurewicz homomorphism

$$h^S:\pi_{13}^S(MO(8))\to H_{13}(MO(8)).$$

To determine these, note that the homology group $H_{13}(MO(8))$ has a basis

$$e_1^3e_2^5,\quad e_1^4e_2^3e_3,\quad e_1^5e_2^2e_4,\quad e_1^5e_2e_3^2,\quad e_1^6e_2e_5,\quad e_1^6e_3e_4,\quad e_1^7e_6.$$

The calculations show that the primitive $\mathcal{A}_2$-annihilated submodule is generated by the single element:

$$e_1^7e_6+e_1^6e_2e_5+e_1^6e_3e_4+e_1^5e_2e_3^2.$$

But there is an immersion of $V^5 \looparrowright \mathbb{R}^{13}$. If γ denotes this immersion in $\pi_{13}MO(8)$, then necessarily

$$h^S(\gamma) = e_1^7 e_6 + e_1^6 e_2 e_5 + e_1^6 e_3 e_4 + e_1^5 e_2 e_3^2.$$

Note that V^5 is the only nontrivial manifold in 5-dimensional cobordism group and its Hurewicz image is stable because it embeds in $\mathbb{R}^{13}$. Now we observe that $\xi_* Q^7 e_1^3 e_3 = h^S(\gamma)$, showing that the double point manifold of the immersion $P^2 \times S^7 \looparrowright \mathbb{R}^{13}$ is cobordant to V^5. This proves Theorem 1.1.

4. Immersions of $M^8 \looparrowright \mathbb{R}^{11}$

By a similar calculations as above we will have the following. The homology group of $H_{11}QMO(3)$ is generated by the following elements:

$$\begin{gathered} e_1^2e_9,\quad e_1e_2e_8,\quad e_1e_3e_7,\quad e_1e_4e_6,\quad e_1e_5^2,\quad e_2^2e_7,\quad e_2e_3e_6,\quad e_2e_4e_5,\quad e_3^2e_5,\\ e_3e_4^2,\quad e_1^3\cdot e_1^2e_6,\quad e_1^3\cdot e_1e_2e_5,\quad e_1^3\cdot e_1e_3e_4,\quad e_1^3\cdot e_2^2e_4,\quad e_1^3\cdot e_2e_3^2,\\ e_1^2e_2\cdot e_1^2e_5,\quad e_1^2e_2\cdot e_1e_2e_4,\quad e_1^2e_2\cdot e_1e_3^2,\quad e_1^2e_2\cdot e_2^2e_3,\quad e_1^2e_3\cdot e_1^2e_4,\\ e_1^2e_3\cdot e_1e_2e_3,\quad e_1^2e_3\cdot e_2^3,\quad e_1e_2^2\cdot e_1^2e_4,\quad e_1e_2^2\cdot e_1e_2e_3,\quad e_1e_2^2\cdot e_2^3,\\ Q^8e_1^3,\quad Q^7e_1^2e_2,\quad Q^6e_1^2e_3,\quad Q^6e_1e_2^2,\\ e_1^3\cdot e_1^3\cdot e_1^2e_3,\quad e_1^3\cdot e_1^3\cdot e_1e_2^2,\quad e_1^2e_2\cdot Q^4e_1^3,\quad e_1^3\cdot Q^5e_1^3,\quad e_1^3\cdot e_1^2e_2\cdot e_1^2e_2. \end{gathered}$$

Corollary 4.1. *The primitive submodule of $H_{11}QMO(3)$ is generated by the following elements:*

$$\begin{aligned} &e_1^2e_9,\quad e_1e_2e_8,\quad e_1e_3e_7,\quad e_1e_4e_6,\quad e_1e_5^2,\quad Q^8e_1^3,\quad Q^7e_1^2e_2,\quad Q^6e_1^2e_3,\\ &Q^6e_1e_2^2,\quad A = e_1^3\cdot e_1^2e_6 + e_1^2e_2\cdot e_1^2e_5 + e_1^2e_3\cdot e_1^2e_4 + e_2^2e_7,\\ &B = e_1^3\cdot e_1e_2e_5 + e_1^2e_2\cdot e_1e_2e_4 + e_1^2e_3\cdot e_1e_2e_3 + e_1e_2^2\cdot e_1^2e_4 + e_1^2e_2\cdot e_1^2e_5\\ &\quad + e_2e_3e_6,\\ &C = e_1^3\cdot e_1e_3e_4 + e_1^2e_2\cdot e_1e_2e_4 + e_1^2e_3\cdot e_1^2e_4 + e_1^2e_2\cdot e_1e_3^2 + e_1e_2^2\cdot e_1e_2e_3\\ &\quad + e_1^2e_3\cdot e_1e_2e_3 + e_2e_4e_5,\\ &D = e_1^3\cdot e_1^2e_2\cdot e_1^2e_2 + e_1^3\cdot e_1^3\cdot e_1^2e_3 + e_1^2e_3\cdot e_2^3 + e_1^2e_3\cdot e_1^2e_4 + e_1^2e_2\cdot e_2^2e_3\\ &\quad + e_1^3\cdot e_2^2e_4 + e_3^2e_5,\\ &E = e_1^3\cdot e_1^3\cdot e_1e_2^2 + e_1e_2^2\cdot e_2^3 + e_1^2e_2\cdot e_1e_3^2 + e_1^3\cdot e_2e_3^2 + e_3e_4^2. \end{aligned}$$

Corollary 4.2. *The primitive $\mathcal{A}_2$-annihilated submodule of $H_{11}QMO(3)$ is generated by the single element:*

$$e_1e_3e_7.$$

The above element is spherical. In fact there is an immersion of $P^2 \times P^6 \looparrowright \mathbb{R}^{11}$. If α represents this immersion in $\pi_{11}QMO(3)$, then necessarily $h(\alpha) = e_1e_3e_7$. Therefore, the multiple point manifolds are boundaries. The other manifolds and immersions fullfil the same result. This proves Theorem 1.2.

5. Immersions of $M^{10} \looparrowright \mathbb{R}^{15}$

By a similar calculations we have the following lemma.

Lemma 5.1. *The primitive submodule of $H_{15}(QMO(5))$ is generated by the following elements:*

$$Q^{10}e_1^5, \quad Q^9e_1^4e_2, \quad Q^8e_1^3e_2^2, \quad Q^8e_1^4e_3, \quad \delta$$

$$A = e_1^5 \cdot e_1^4e_6 + e_1^4e_2 \cdot e_1^4e_5 + e_1^4e_3 \cdot e_1^4e_4 + e_2^4e_7,$$

$$B = e_1^5 \cdot e_1^3e_2e_5 + e_1^4e_2 \cdot e_1^3e_2e_4 + e_1^4e_3 \cdot e_1^3e_2e_3 + e_1^4e_4 \cdot e_1^3e_2^2 + e_1^4e_5 \cdot e_1^4e_2 + e_2^3e_3e_6,$$

$$C = e_1^5 \cdot e_1^3e_3e_4 + e_1^4e_2 \cdot e_1^3e_3^2 + e_1^4e_3 \cdot e_1^3e_2e_3 + e_1^4e_4 \cdot e_1^4e_3 + e_1^4e_2 \cdot e_1^3e_2e_4 + e_1^3e_2^2 \cdot e_1^3e_2e_3 + e_2^3e_4e_5,$$

$$D = e_1^5 \cdot e_1^2e_2^2e_4 + e_1^4e_2 \cdot e_1^2e_2^2e_3 + e_1^4e_3 \cdot e_1^2e_2^3 + e_1^4e_4 \cdot e_1^3e_2^2 + e_2^2e_3^2e_5,$$

$$E = e_1^5 \cdot e_1^2e_2e_3^2 + e_1^3e_2^2 \cdot e_1^2e_2^3 + e_1^3e_3^2 \cdot e_1^4e_2 + e_2^2e_3e_4^2,$$

$$F = e_1^5 \cdot e_1e_2^3e_3 + e_1^4e_2 \cdot e_1e_2^4 + e_1^4e_3 \cdot e_1^2e_2^3 + e_2e_3^3e_4 + e_1^4e_2 \cdot e_1^2e_2^2e_3 + e_1^3e_2^2 \cdot e_1^2e_2^3 + e_1^3e_2^2 \cdot e_1^3e_2e_3,$$

$$G = e_1^5 \cdot e_1^5 \cdot e_1^5 + e_1^5 \cdot e_2^5 + e_1^4e_2 \cdot e_1e_2^4 + e_3^5.$$

Here δ runs over a basis of primitive height one elements.

Now the action of the Steenrod squares to elements of 5.1 gives the following Corollary.

Corollary 5.2. *The primitive $\mathcal{A}_2$-annihilated submodule of $H_{15}QMO(5)$ is generated by the following elements:*

$$Q^8e_1^4e_3 + A + \delta, \quad G, \quad \delta.$$

Here δ denotes a primitive combination of the height one elements.

If the element $Q^8e_1^4e_3 + A + \delta$ is spherical, by a simple calculation we can show that the double point manifolds are boundaries. The element G is spherical. Because there is an immersion

$$P^2 \times P^2 \times P^2 \times P^2 \times P^2 \looparrowright \mathbb{R}^{15}$$

with Hurewicz image G. Let η denotes its representation in $\pi_{15}QMO(5)$. Thus,

$$h(\eta) = e_1^5 \cdot e_1^5 \cdot e_1^5 + e_1^5 \cdot e_2^5 + e_1^4e_2 \cdot e_1e_2^4 + e_3^5.$$

Now as we see from above $h_*^2(h(\eta)) = e_1^5 \cdot e_2^5 + e_1^4e_2 \cdot e_1e_2^4$ and $h_*^3(h(\eta)) = e_1^5 \cdot e_1^5 \cdot e_1^5$. Therefore by Theorem 2.1 we see

$$\xi_*h_*^2(h(\eta)) = 0, \quad \xi_*h_*^3(h(\eta)) = e_1^{15}.$$

These show that the double point manifold of this immersion is a boundary and the triple point set is an odd number. This proves Theorem 1.3.

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