

RINGS WITH ALL FINITELY GENERATED MODULES RETRACTABLE

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ABSTRACT. Several characterizations of a ring R is given for which any non-zero finitely generated module M is *retractable* in the sense that $\text{Hom}_R(M, N)$ is non-zero whenever N is a non-zero submodule of M . Such rings are called *finite retractable*. It is shown that any ring being Morita equivalent to a commutative ring is finite retractable. Also, if the commutative ring is semi-Artinian then any non-zero module is retractable. The class of finite retractable rings is shown to be closed under Morita equivalence and finite direct products. Moreover, for a finite retractable ring R which is a right order in a ring Q , it is shown that Q is also finite retractable.

1. Introduction

The term *retractable* was first used by Khuri [10] for a module M with the property that $\text{Hom}(M, N) \neq 0$ whenever N is a non-zero submodule in M . Retractable modules have been a subject of interest as they are among the essential ingredients in the general study of the endomorphism rings. Often retractability condition combined with another condition such as nonsingularity, or a weak form of projectivity on a module, are enough to characterize the endomorphism ring of the module. For examples of such achievements, see [3], [8], [11], [12]. In [5],

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Faith has vastly studied right CFPF rings. A ring R is a right CFPF ring if for all proper two sided ideals I of R , any finitely generated faithful R/I -module is a generator in $\text{Mod-}R/I$. For such a ring, all finitely generated right modules are retractable. More recently, finitely generated retractable modules over left Noetherian right FBN rings have been characterized by Smith [13]. The purpose of this article is to investigate rings whose non-zero finitely generated modules are all retractable. Such a ring will be called *(right) finite retractable*. We show that all rings which are Morita equivalent to commutative rings, in particular commutative rings, are finite retractable. In fact, being a finite retractable ring will be shown to be a Morita invariant property. Several characterizations of finite retractable rings will then be given; see Theorem 2.6. Finally, we show that the class $\mathcal{C}$ of finite retractable rings is also closed under homomorphic images, and finite direct product. Moreover, the finite retractability of a ring R passes over to a ring Q when R is a right order in Q ; see Theorem 3.3.

2. Finite retractable rings

Throughout the paper, rings will have a non-zero identity element and modules will be unitary. Unless stated otherwise, modules will be right modules. We follow [4] and [7] for the terms not defined here, and for the basic results on module and ring theory that are relevant to this work. We begin by recording a result (with a routine proof) that determines the retractability of a factor module M/N when M is quasi-projective.

Lemma 2.1. *Let M be a quasi-projective right R -module and $N \leq M_R$. $(M/N)_R$ is retractable if and only if for all submodules L that properly contain N , the set $\mathcal{A}_L := \{f \in \text{End}_R(M) \mid f(N) \subseteq N, f(M) \subseteq L \text{ and } f(M) \not\subseteq N\}$ is non-empty.*

Proposition 2.2. *Let I be a proper right ideal in a ring R . The following statements are equivalent.*

- (i) *The cyclic right R -module R/I is retractable.*
- (ii) *For any right ideal J , either $J \subseteq I$ or there exists $x \in J \setminus I$ such that $xI \subseteq I$.*

Proof. (i) $\Rightarrow$ (ii). If $J \subseteq I$, and we are through. Hence, let $a \in J \setminus I$, then consider a non-zero homomorphism $f : R/I \rightarrow (aR + I)/I$ with

$f(1 + I) = ar + I$ for some $r \in R$. We now have $xI \subseteq I$, where $x = ar$.
(ii) $\Rightarrow$ (i). Apply Lemma 2.1 to $M = R$, $N = I$ and use the fact that every element of $\text{End}(R_R)$ is a left multiplication map by some $x \in R$. $\square$

Recall that a subset I of a ring R is said to be *left T-nilpotent* if for any sequence $a_1, a_2, \dots$ in I , there exists an n such that $a_1 \dots a_n = 0$. The next Lemma states an interesting property of left T-nilpotent right ideals that immediately yields the following Corollary.

Lemma 2.3. *Let I be a left T-nilpotent right ideal in a ring R . For any right ideal J of R , either $J \subseteq I$ or there exists $x \in J \setminus I$ with $xI \subseteq I$.*

Proof. If $J \subseteq I$, then we are done. Otherwise, let $b \in J \setminus I$. We are to find $x \in J \setminus I$ with $xI \subseteq I$. If $bI \subseteq I$, then we are through. If not, then there is $a_1 \in I$ such that $ba_1 \notin I$. Again, if $ba_1I \not\subseteq I$, then there exists $a_2 \in I$ such that $ba_1a_2 \notin I$. If this process does not stop, then we get a sequence $a_1, a_2, \dots$ in I with $ba_1 \dots a_k \notin I$, for all k . But this is absurd, by the fact that I is left T-nilpotent. It follows that there is an integer n such that $a_1 \dots a_n = x \in J \setminus I$ and $xI \subseteq I$. The proof is now complete. $\square$

Corollary 2.4. *If I is a left T-nilpotent right ideal in the ring R , then the right R -module R/I is retractable.*

Proof. Use Proposition 2.2 and Lemma 2.3. $\square$

We note that in the definition of the retractable modules, only categorical terms are used, and thus the next useful and evident result is in order.

Proposition 2.5. *Being retractable is a Morita invariant property.*

In part (ii) of the next Theorem, we shall consider elements of the free right R -module R^n as column matrices. Call M_R *essentially retractable* if $\text{Hom}_R(M, N) \neq 0$ for every essential submodule N of M_R ; see [14] for more information about such modules.

Theorem 2.6. *For a ring R , the following statements are equivalent.*

- (i) *R is a finite retractable ring.*
- (ii) *For any $n \geq 1$ and $x \in R^n \setminus K$, where K is a non-zero submodule*

of R^n , there are $r_1, \dots, r_n \in R$ such that $xr_j \notin K$ for some j and $\sum_i xr_i \pi_i(y) \in K$ for all $y \in K$, where $\pi_i : R^n \rightarrow R$ are natural projections.

(iii) For any $n \geq 1$ and any right ideals I, J of $S = \text{Mat}_{n \times n}(R)$, either $J \subseteq I$ or there exists $x \in J \setminus I$ such that $xI \subseteq I$.

(iv) All non-zero finitely generated R -modules are essentially retractable.

(v) $\text{Hom}_R(M, X) = 0 \Leftrightarrow \text{Hom}_R(M, E(X)) = 0$, for any finitely generated M and any $X \in \text{Mod-}R$.

Proof. (i) $\Rightarrow$ (ii). Let $n \geq 1$ and $x \in R^n \setminus K$, for some non-zero $K \leq R^n$. By our assumption, there is a non-zero R -homomorphism $f : R^n/K \rightarrow (xR + K)/K$. If $e_1, \dots, e_n$ are the standard basis of R^n , let $f(e_i + K) = xr_i + K$ for $i = 1, \dots, n$. Clearly, $xr_j \notin K$, for some j . It follows that $\sum_i xr_i \pi_i(y) \in K$, for all $y \in K$.

(ii) $\Rightarrow$ (i). Generally, if $A \in \text{Mat}_{n \times n}(R)$ and $y \in R^n$, then $\sum_i Ae_i \pi_i(y) = Ay$. Suppose now that M is a finitely generated right R -module. We can assume that $M = R^n/K$, for some integer n and right R -submodule K of R^n . Let $0 \neq L/K \leq R^n/K$ and $x \in L \setminus K$. Suppose that $r_1, \dots, r_n$ are the elements corresponding to x . Form the $n \times n$ matrix A whose i th column is xr_i . So, by our assumptions, left multiplication by A defines a non-zero R -homomorphism from R^n/K to L/K .

Under the standard Morita equivalence of R with a matrix ring $S = \text{Mat}_{n \times n}(R)$, n -generated right R -modules correspond to cyclic right S -modules, and conversely. Thus (i) $\Leftrightarrow$ (iii) is a consequence of propositions 2.5 and 2.2.

(i) $\Rightarrow$ (iv). This is clear.

(iv) $\Rightarrow$ (i). Let M be any non-zero finitely generated R -module and $0 \neq N \leq M_R$. Clearly, there exists a non-zero R -module homomorphism $f : M \rightarrow E(N)$. Now, since by our assumption, $f(M)$ is essentially retractable, $\text{Hom}_R(f(M), N \cap f(M))$ is non-zero. It follows that $\text{Hom}_R(M, N) \neq 0$. This proves (i).

(iv) $\Rightarrow$ (v). Note that $f(M)$ is essentially retractable for any non-zero $f \in \text{Hom}_R(M, E(X))$.

(v) $\Rightarrow$ (iv). Let M be a non-zero finitely generated R -module and $N \leq_{\text{ess}} M$. Then, $E(N) = E(M)$. Clearly, $\text{Hom}_R(M, E(M)) \neq 0$. Hence, $\text{Hom}_R(M, N) \neq 0$ by (v). $\square$

It is well known that if R is a commutative ring and $x \in S = \text{Mat}_{n \times n}(R)$, then there exists $s \in S$ such that $xs = \det(x)I_{(n \times n)}$ is a

central element, and hence $(xs)J \subseteq J$, for any right ideal J of S . In the following result, we show that commutative rings are finite retractable and consequently, matrix rings over commutative rings satisfy the more informative condition (iii) of Theorem 2.6.

Theorem 2.7. *If R is a ring Morita equivalent to a commutative ring, then R is a finite retractable ring.*

Proof. By Proposition 2.5, we can suppose that R is commutative. We shall prove that the condition (v) of Theorem 2.6 holds. Let $0 \neq f \in \text{Hom}_R(M, E(X))$ when M_R is finitely generated. Let $q_1R + \dots + q_nR = \text{Im } f$, for some $n \geq 1$. Since X^n is an essential submodule of $E(X)^n$, then there is $r \in R$ such that $rq_i \in X$, for all i and $rq_j \neq 0$, for some j . Multiplication by r defines a non-zero R -homomorphism from $\text{Im } f$ to X . Consequently, $\text{Hom}_R(M, X) \neq 0$. $\square$

A ring R is said to be *right semi-Artinian* if every non-zero right R -module contains a simple R -submodule. In [6], it is proved that a ring R is right semi-Artinian if and only if each essential submodule of an R -module M is *strongly essential* in M (N is a strongly essential submodule of M_R if for any non-zero non-empty subset $X \subseteq M$, there exists $r \in R$ with $0 \neq Xr \subseteq N$). This interesting property of semi-Artinian rings shows that adding the semi-Artinian hypothesis in Theorem 2.7, we obtain *all* non-zero modules retractable.

Theorem 2.8. *Over a ring Morita equivalent to a commutative semi-Artinian ring R , every non-zero module is retractable.*

Proof. It suffices to show that every non-zero R -module M is retractable. Let $0 \neq N \leq M_R$. Then, clearly there exists a non-zero R -module homomorphism $f : M \rightarrow E(N)$. Because N is a strongly essential submodule of $E(N)$, by [6, Theorem 4.5], there exists $r \in R$ such that $0 \neq f(M)r \subseteq N$. Thus, $\text{Hom}_R(M, N)$ contains the non-zero element $g : M \rightarrow N$, where $g(m) = f(m)r$ for all $m \in M$. $\square$

Remark 2.9. Camillo and Fuller [2] have shown that commutative semi-Artinian rings are Max-rings (i.e., every non-zero module has a maximal submodule). We now observe that this property of commutative semi-Artinian rings is an immediate consequence of Theorem 2.8.

An R -module M is called *finitely annihilated* if there exist a positive integer n and elements $m_i \in M$ ($1 \leq i \leq n$) such that $A := \text{ann}_R(M) = \{r \in R \mid m_i r = 0 \text{ for all } 1 \leq i \leq n\}$, or equivalently there exists an embedding $\theta : R/A \rightarrow M^{(n)}$. It is well known that a ring R is right Artinian if and only if every right R -module is finitely annihilated [1]. In the following result, we consider finitely annihilated modules over finite retractable rings. Let M be an R -module, and denote the full subcategory of $\text{Mod-}R$, whose objects are submodules of M -generated modules by $\sigma[M_R]$.

Proposition 2.10. *Let R be a finite retractable ring. If M is a finitely annihilated R -module, then $\text{Hom}_R(M, N) \neq 0$, for any non-zero $N \in \sigma[M_R]$*

Proof. Let N be a non-zero submodule in $\sigma[M_R]$ and $A := \text{ann}_R(M)$. By the hypothesis, there exists an embedding $\theta : R/A \rightarrow M^{(n)}$, for some $n \geq 1$. Since $NA = 0$, then there exists an R -module homomorphism from R/A to N . It follows that $\text{Hom}_R(M^{(n)}, E(N)) \neq 0$. Now, $\text{Hom}_R(M, N) \neq 0$, by Theorem 2.6(v).

It is known that a commutative nonsingular ring with finitely generated essential socle is a semisimple Artinian ring (in fact, a more general result holds, namely, any semiprime ring with finitely generated essential right socle is semisimple; see also [9, Theorem 2.18] for a result even much stronger than the latter one). In the following result, the aforementioned fact is extended to the class of finite retractable nonsingular rings which is larger than the class of commutative nonsingular rings by Theorem 2.7.

Proposition 2.11. *A ring R is semisimple Artinian if and only if it is a right nonsingular finite retractable ring such that $\text{Soc}(R_R)$ is a finitely generated essential right ideal.*

Proof. ($\Rightarrow$). This is clear.

($\Leftarrow$). Let S be a minimal right ideal in R and $0 \neq x \in E$, where E is the injective hull of S_R . Then, by our assumption xR is a retractable R -module and hence there exists a non-zero $f : xR \rightarrow S$. If $\text{Ker } f$ is non-zero, then it must be an essential submodule of xR . It follows that S_R is singular, contradiction. Hence, $x \in \text{Soc}(E_R) = S$. Consequently, every

minimal right ideal of R is injective. This shows that $R = \text{Soc}(R_R)$, as desired. $\square$

Remark 2.12. In contrast to Theorem 2.8 and considering the previous result, we observe that there are right Artinian right non-singular rings which are not finite retractable. On such example is $\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$; see the discussion in Examples 3.1. This ring shows also that the commutativity is an indispensable condition for the validity of theorems 2.7 and 2.8.

3. Ring extensions of finite retractable rings

Throughout this section, $\mathcal{C}$ denotes the class of all finite retractable rings. It is easily seen that $\mathcal{C}$ is closed under homomorphic images and by Propositions 2.5 it is closed under Morita equivalence.

The following shows that a formal triangular matrix ring never belongs to $\mathcal{C}$ unless it is a direct sum of suitable rings.

Example 3.1. Suppose that A and B are rings and ${}_A M_B$ is a non-zero bimodule. Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, $I = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ lies in $eR \cap \ell.\text{ann}_R(e)$. Now, if $f \in \text{Hom}_R(eR, I)$, then $f(eR) = f(e)eR \subseteq IeR = 0$. It follows that R is never a finite retractable ring.

Let R_1 and R_2 be rings and $T = R_1 \oplus R_2$. It is well known that any T -module M has the form $M_1 \oplus M_2$, for some R_i -modules $M_i (i = 1, 2)$. Hence, whenever each $M_i (i = 1, 2)$, is a retractable R_i -module, then $M = M_1 \oplus M_2$ is a retractable T -module. Moreover, since $\mathcal{C}$ is closed under homomorphic images, then we have the following result.

Proposition 3.2. *Let $R_i (1 \leq i \leq n)$ be rings. Then, $\prod_{i=1}^n R_i \in \mathcal{C}$ if and only if $R_i \in \mathcal{C}$, for all $i = 1, \dots, n$.*

Next, we consider certain extensions of a ring in $\mathcal{C}$.

Theorem 3.3. *If a ring $R \in \mathcal{C}$ and R is a right order in a ring Q , then $Q \in \mathcal{C}$.*

Proof. We shall show that Q satisfies condition (iii) of Theorem 2.6. Thus, assume that $n \geq 1$ and I, J are two right ideals of $S = \text{Mat}_{n \times n}(Q)$. We may assume that $J \not\subseteq I$. Let $T = \text{Mat}_{n \times n}(R)$ and note that S is trivially a right quotient ring of T , with respect to the set $D = \{A \in T \mid A \text{ is a diagonal matrix over the set of regular elements of } R\}$. Clearly, $J \not\subseteq I$ implies that $(J \cap T) \not\subseteq (I \cap T)$. Now, since R satisfies condition (iii) of Theorem 2.6, then there exists $x \in (J \cap T) \setminus (I \cap T)$ with $x(I \cap T) \subseteq (I \cap T)$. But $I = (I \cap T)S$, and hence $x \in J \setminus I$ with $xI \subseteq I$, as desired. $\square$

Proposition 3.4. *Let $R \subseteq T$ be rings such that ${}_R T$ is free. If $T \in \mathcal{C}$, then $R \in \mathcal{C}$.*

Proof. Let $T \in \mathcal{C}$ and M_R be a non-zero finitely generated R -module. Then, $M \otimes_R T$ is a finitely generated T -module which has a non-zero submodule isomorphic to $N \otimes_R T$. Thus, by the finite retractable condition on T we have:

$$\begin{aligned} 0 \neq \text{Hom}_T(M \otimes_R T, N \otimes_R T) &\simeq \text{Hom}_R(M, \text{Hom}_T(T, N \otimes_R T)) \\ &\simeq \text{Hom}_R(M, N \otimes_R T). \end{aligned}$$

On the other hand, T being free over R , there exists a set A with ${}_R T \simeq R^{(A)}$. Hence, the following holds:

$$\text{Hom}_R(M, N \otimes_R T) \hookrightarrow \prod_A \text{Hom}_R(M, N \otimes_R R) \simeq \prod_A \text{Hom}_R(M, N).$$

Consequently, $\text{Hom}_R(M, N) \neq 0$. It follows that $R \in \mathcal{C}$. $\square$

We remark that for $T = R[x, \alpha, \delta]$, where α is a ring homomorphism and δ is an α -derivation on R , clearly ${}_R T$ is free. The diligent readers may then be interested in the following project.

Project: Let R be a finite retractable ring. Investigate the finite retractability of the ring $R[x, \alpha, \delta]$.

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REFERENCES

- [1] J.A. Beachy, On quasi-Artinian rings, *J. London Math. Soc.* **3**(2) (1971) 449-452.
- [2] V.P. Camillo and K.R. Fuller, On Loewy length of rings, *Pacific J. Math.* **53** (1974) 347-354.
- [3] A.W. Chatters and S.M. Khuri, Endomorphism rings of modules over non-singular CS-rings, *J. London Math. Soc.* **2**(21) (1980) 434-444.
- [4] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending Modules*, Longman: Burnt Mill, 1994.
- [5] C. Faith and S. Page, *FPF ring theory, faithful modules and generators of mod- R* , London Math. Soc. Lecture Note 88, Cambridge Univ. Press, 1984.
- [6] M. Ghirati and O.A.S. Karamzadeh, On strongly essential submodules, *Comm. Algebra* **36**(2) (2008) 564-580.
- [7] K.R. Goodearl and R.B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Second Edition, Volume 61, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2004.
- [8] A. Haghany and M.R. Vedadi, Study of semi-projective retractable modules, *Algebra Colloquium* **14** (2007) 489-496.
- [9] O.A.S. Karamzadeh and Sh. Rahimpour, On λ -finitely embedded modules. *Algebra Colloq.* **12**(2) (2005) 281-292.
- [10] S.M. Khuri, Endomorphism rings and lattice isomorphisms, *J. Algebra* **59** (1979) 401-408.
- [11] S.M. Khuri, The endomorphism ring of a non-singular retractable module, *East-West Journal Math.* **2** (2000) 161-170.
- [12] S.T. Rizvi and C.S. Roman, Baer and Quasi-Baer Modules, *Comm. Algebra* **32**(1) (2004) 103-123.
- [13] P.F. Smith, Modules with many homomorphisms, *Journal of Pure and Appl. Algebra* **197**(1-3) (2005) 305-321.
- [14] M.R. Vedadi, Essentially retractable modules, *J. Sci. Islam. Repub. Iran* **18**(4) (2007) 355-369.

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