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FULLY IDEMPOTENT AND COIDEMPOTENT MODULES

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Abstract. In this paper, the notion of fully idempotent modules is defined and it is shown that this notion inherits most of the essential properties of the usual notion of von Neumann's regular rings. Furthermore, we introduce the dual notion of fully idempotent modules (that is, fully coidempotent modules) and investigate some properties of this class of modules.

1. Introduction

Throughout this paper R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Also for a submodule N of an R-module M, $Ann^k_R(N)$ and $E(M)$ will denote $(Ann_R(N))^k$ and the injective hall of M, respectively.

An R-module M is said to be a *multiplication module* [12] if for every submodule N of M, there exists an ideal I of R such that $N = IM$. It is easy to see that M is a multiplication module if and only if $N = (N :_R M)M$ for every submodule N of M.

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$.

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It also follows that M is a comultiplication module if and only if $N = (0 :_M Ann_R(N))$ for every submodule N of M [5].

A submodule N of M is said to be *pure* if $IN = N \cap IM$ for every ideal I of R [4]. Moreover, N is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [8].

Let N and K be two submodules of M. The *product* of N and K is defined by $(N :_R M)(K :_R M)M$ and it is denoted by NK. Also, the *coproduct* of N and K is defined by $(0 :_M Ann_R(N)Ann_R(K))$ and it is denoted by $C(NK)$ [6].

In this paper, we introduce the notions of fully idempotent, fully coidempotent, fully pure, and fully copure modules and provide some useful information concerning these new classes of modules.

A submodule N of M is said to be *idempotent* (respectively, *coidempotent*) if $N = N^2$ (respectively, $N = C(N^2)$). Moreover, M is said to be *fully idempotent* (respectively, *fully coidempotent*) if every submodule of M is idempotent (respectively, coidempotent) (Definitions 2.1, 2.4, 3.1, and 3.2).

A module M is said to be *fully pure* (respectively, *fully copure*) if every submodule of M is pure (respectively, copure) (Definitions 2.14 and 3.10).

In Section 2 of this paper, among other results, we prove that if M is a multiplication and comultiplication module such that M does not have any non-zero nilpotent submodule, then M is fully idempotent (Theorem 2.10). Also, it is shown that if M is a fully idempotent module, then M is cosemisimple and every prime (respectively, finitely generated) submodule of M is maximal (respectively, cyclic) (Theorems 2.12, 2.10, and 2.19).

In Section 3, it is shown that if M is a semisimple comultiplication module, then M is fully coidempotent (Theorem 3.8). In Theorem 3.12, we provide some useful characterizations for copure submodules of a comultiplication module. In Corollary 3.16, we investigate the relation between fully idempotent (respectively, fully pure) modules with fully coidempotent (respectively, fully copure) modules. Finally, it is proved that if M is a finitely generated fully coidempotent module, then M is semisimple.

We refer the reader to [4] and [20] for all concepts and basic properties of modules not defined here.

2. **Fully idempotent modules**

Below, we recall the concept of idempotent submodules which is introduced and investigated by some authors (see [2, 3, 14], and [26]).

In [14], a submodule N of an R-module M is called *idempotent* provided $N = Hom(M, N)N = \sum\{\varphi(N) : \varphi : M \to N\}$.

In [2], a submodule N of an R-module M is called *idempotent* if $N = (N :_R M)N$.

Definition 2.1. We say that a submodule N of an R-module M is *idempotent* if $N = N^2$.

The following lemma and Example 2.3 show the relation between the above various concepts of idempotent submodules.

Lemma 2.2. Let N be a submodule of an R-module M. Consider the following statements.

(a) $N = N^2$.

(b) $N = (N : M)N$.

(c) $N = Hom_R(M, N)N = \sum\{\varphi(N) : \varphi : M \to N\}$.

Then $(a) \Leftrightarrow (b)$ and $(b) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$. We have $N = (N :_R M)^2M \subseteq (N :_R M)N \subseteq N$. Thus $N = (N : M)N$.

$(b) \Rightarrow (a)$. We have $N = (N :_R M)N = (N :_R M)(N :_R M)N \subseteq (N :_R M)^2M \subseteq (N :_R M)N = N$. Hence $N = (N :_R M)^2M$.

$(b) \Rightarrow (c)$. Let $x \in N$. Then by assumption, there exists $r \in (N :_R M)$ and $y \in N$ such that $x = ry$. Now consider the homomorphism $f : M \to N$ defined by $f(m) = rm$. Then $x = f(y) \in \sum\{\varphi(N) : \varphi : M \to N\} = Hom_R(M, N)N$. Therefore, $N = Hom_R(M, N)N$ because the reverse inclusion is clear $\square$

Example 2.3. For any prime number p, the submodule $N = \mathbb{Z}_p \oplus 0$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ is not idempotent but $N = Hom_{\mathbb{Z}}(M, N)N$.

Definition 2.4. An R-module M is said to be *fully idempotent* if every submodule of M is idempotent.

Example 2.5. For each prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_p$ is fully idempotent. Moreover, $E(\mathbb{Z}_p) = \mathbb{Z}_{p^\infty}$ is not a fully idempotent $\mathbb{Z}$-module.

Let M be an R-module. A proper submodule P of M is said to be *naturally semi-prime* if for a submodule N of M, the relation $N^2 \subseteq P$

implies that $N \subseteq P$ (see [6]). An element x of M is said to be *idempotent* if there exists $t \in (Rx :_R M)$ such that $x = tx$ (see [3]).

In the following proposition, we characterize the fully idempotent R-modules.

Proposition 2.6. Let M be an R-module. Then the following statements are equivalent.

(a) M is a fully idempotent module.
(b) Every cyclic submodule of M is idempotent.
(c) Every element of M is idempotent.
(d) Every proper submodule of M is naturally semi-prime.
(e) For all submodules N and K of M, we have $N \cap K = NK$.

Proof. $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(a) \Rightarrow (d)$. These are clear.

$(c) \Rightarrow (a)$. Let N be a submodule of M and $x \in N$. Then by hypothesis, there exists $t \in (Rx :_R M)$ such that $x = tx$. It follows that $Rx = (Rx :_R M)^2M$. Thus $N \subseteq N^2$. Since the reverse inclusion is clear, N is idempotent.

$(d) \Rightarrow (a)$. Suppose that N is a proper submodule of M. Since N^2 is naturally semi-prime, $N^2 \subseteq N^2$ implies $N \subseteq N^2$. Hence M is fully idempotent because the reverse inclusion is clear.

$(a) \Rightarrow (e)$. Let N and K be two submodules of M. Then

$$N \cap K = (N \cap K :_R M)^2M \subseteq (N :_R M)(K :_R M)M = NK,$$

as required.

$(e) \Rightarrow (a)$. For a submodule N of M, we have $N = N \cap N = NN = N^2$. □

Proposition 2.7. Let M be a fully idempotent R-module.

(a) M is a multiplication module.
(b) Every submodule and every homomorphic image of M is fully idempotent.
(c) If M is a finitely generated faithful R-module, then R is a von Neumann regular ring.
(d) If S is a multiplicatively closed subset of R, then $S^{-1}M$ is a fully idempotent $S^{-1}R$-module.
(e) M is co-Hopfian.

Proof. The statements (a) and (d) are straightforward.

(b) It it is easy to see that every homomorphic image of M is fully idempotent. Now suppose that N is a submodule of M and K is a submodule of N. By part (a), M is a multiplication R-module. Hence $K = (K :_R M)^2 M$ implies that $K = (K :_R M)^3 M$. Thus

$$K = (K :_R M)^3 M \subseteq (K :_R N)^2 (N :_R M)M \subseteq (K :_R N)^2 N.$$

Therefore, $K = (K :_R N)^2 N$. Thus N is fully idempotent.

(c) This follows from [16, 3.1] and part (a).

(e) Let $f : M \to M$ be a monomorphism and $x \in M$. Since M is fully idempotent, $f(x) = tf(x)$ for some $t \in (f(x)R :_R M)$. Hence $tx = f(x)r$ for some $r \in R$. Therefore, $f(x) = f(tx) = f(f(x)r)$. Since f is monomorphism, it follows that $x = f(xr)$, as desired. □

The following example shows that the converse of part (a) of the above proposition is not true in general.

Example 2.8. $\mathbb{Z}_4$ is a multiplication $\mathbb{Z}$-module which is not fully idempotent.

Let M be an R-module and let N be a submodule of M. The following example shows that if N and M/N are fully idempotent modules, then M is not necessarily a fully idempotent module.

Example 2.9. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z}$ and set $N = 2\mathbb{Z}/4\mathbb{Z}$. Then N and M/N are fully idempotent $\mathbb{Z}$-modules, while M is not fully idempotent.

Let M be an R-module. A submodule N of M is said to be *nilpotent* if there exists a positive integer k such that $N^k = 0$, where N^k means the product of N, k times (see [6]).

A proper submodule N of M is said to be *prime* if for any $r \in R$ and any $m \in M$ with $rm \in N$, we have $m \in N$ or $r \in (N :_R M)$.

Theorem 2.10. *Let M be an R-module. Then the following hold.*

(a) *If M is a multiplication and comultiplication module such that M does not have any non-zero nilpotent submodule, then M is fully idempotent.*

(b) *If M is a fully idempotent module, then every element of M with zero annihilator generates M.*

(c) *If M is a fully idempotent module, then every prime submodule of M is maximal.*

Proof. (a) Let N be a submodule of M with $N^2 \neq N$. Then there exists $x \in N$ such that $x \notin N^2$. Since M is a comultiplication module, $Ann_R(N^2)x \neq 0$. Thus there exists $a \in Ann_R(N^2)$ such that $ax \neq 0$.

We show that $(Rax)^2 = 0$. Let $y \in (Rax)^2$. Then there exist $r, s \in (Rax :_R M)$ and $m \in M$ such that $y = rsm$. But $sm \in Rax$ implies that $sm = tax$ for some $t \in R$. Therefore, $y = rtax = artx$. As M is a multiplication module, $rtx \in N^2$. Thus, $y = 0$. Hence $(Rax)^2 = 0$. Now by hypothesis, $Rax = 0$. This implies that $ax = 0$, which is a contradiction.

(b) Let x be an element of M with $Ann_R(x) = 0$. Since M is fully idempotent, there exists $t \in (Rx :_R M)$ such that $x = tx$. This in turn implies that $t = 1$. Hence $(Rx :_R M) = R$. Therefore, $Rx = M$ as required.

(c) Let P be a prime submodule of M and let $x \in M \setminus P$. Since M is fully idempotent, there exists $t \in (Rx :_R M)$ such that $x = tx$. Thus $(1 - t) \in (P :_R M)$ because P is a prime submodule. Therefore, $R = (Rx :_R M) + (P :_R M)$. This implies that $M = Rx + P$ because by Proposition 2.7 (a), M is a multiplication R-module. It follows that, P is a maximal submodule of M. □

Remark 2.11. By Theorem 2.15, every multiplication von Neumann regular module is fully idempotent. Hence part (c) of the above theorem extends [3, Prop. 10].

It is known (see [4, 18.23]) that a commutative ring R is cosemisimple if and only if R is a von Neumann regular ring. The following result is a generalization of this fact.

Theorem 2.12. *Let M be a fully idempotent R-module. Then M is a cosemisimple module. The converse holds when M is a multiplication R-module.*

Proof. Let N be a submodule of M and $x \in M \setminus N$. Since M is a fully idempotent module, there exist $t \in (Rx :_R M)$ such that $x = tx$. Set

$$\Omega = \{H \leq M \mid N \leq H, x \notin H\}.$$

Since $N \in \Omega$, we have $\Omega \neq \phi$. By Zorn's Lemma, Ω has a maximal member, say K. We show that K is a prime submodule of M. Suppose that $sm \in K$, where $s \in R$ and $m \in M$ such that $m \notin K$ and $s \notin (K :_R M)$. Since K is a maximal element of Ω, we have $x \in Rm + K$ and $x \in sM + K$. Hence $x = am + y$ and $x = sm' + y'$, where $y, y' \in K$, $a \in R$ and $m' \in M$. Thus $x = tx = t(sm' + y') = tsm' + ty'$. Now since $tm' \in Rx$, we have $tm' = bx$ for some $b \in R$. So

$$x = sbx + ty' = sb(am + y) + ty' = sbam + sby + ty' \in K.$$

This contradiction shows that K is a prime submodule of M. Now by Theorem 2.10 (c), K is a maximal submodule of M. This in turn implies that N is an intersection of maximal submodules, as desired. Conversely, suppose that N is a submodule of M and $N \not\subseteq N^2$. Hence there exists $x \in N$ such that $x \notin N^2$. Since M is cosemisimple, there exists a maximal submodule K of M such that $N^2 \subseteq K$ and $x \notin K$. Thus $Rx + K = M$ so that $N + K = M$. Therefore, $N(N + K) = NM$. Now since M is a multiplication module, by [6, 3.6], $N^2 + NK = N$. Hence $N \subseteq K + NK = K$, which is a contradiction. Thus $N \subseteq N^2$, as required. □

An R-module M is said to be *von Neumann regular* if every cyclic submodule of M is a direct summand of M (see [19]).

Though the proofs of parts (a), (b), and (c) of the following proposition can be deduced from [3], however, we prefer to provide their direct proofs.

Proposition 2.13. Let M be an R-module. Then the following hold.

(a) If M is a fully idempotent module, then M is a von Neumann regular module.
(b) If M is a multiplication von Neumann regular module, then M is fully idempotent.
(c) If M is a fully idempotent R-module, then M is a locally simple module.
(d) If M is a locally simple multiplication module, then M is a fully idempotent module.

Proof. (a) Let $x \in M$. Since M is a fully idempotent module, there exists $t \in (Rx :_R M)$ such that $x = tx$. We claim that $M = Rx + (1-t)M$ (d.s.). Let $m \in M$. Since $tm \in Rx$, we have $m = (1-t)m + tm \in (1-t)M + Rx$. Now assume that $y \in Rx \cap (1-t)M$. Then $y = sx = (1-t)m$, where $s \in R$ and $m \in M$. Since $tm \in Rx$, there exists $u \in R$ such that $tm = ux$. Hence $sx + ux = m$ so that $(s+u)x = m$. This implies that $y = (1-t)(s+u)x = (s+u)0 = 0$, as required.

(b) By Proposition 2.6, it is enough to show that every cyclic submodule of M is idempotent. Let $x \in M$. By hypothesis, $M = Rx + K$ (d.s.), where K is a submodule of M. Thus

$$(Rx :_R M)M = (Rx :_R M)Rx + (Rx :_R M)K.$$

Since $(Rx :_R M)K = 0$ and M is a multiplication module, $Rx = (Rx :_R M)^2M$.

(c) Let M be a fully idempotent R-module. Since M_P is a fully idempotent R_P-module for every prime ideal P of R by Proposition 2.7 (d), we may assume that R is a local ring. Hence by [16, 2.5], M contains exactly one maximal submodule. It turn out by Theorem 2.12, that this maximal submodule is zero. Therefore, M is locally simple.

(d) Let M be a multiplication locally simple R-module and let N be a submodule of M. Since M_P is simple for each prime ideal P of R and M is a multiplication R-module, we have $(N^2)_P = N_P$. This implies that $N = N^2$ and the proof is completed. □

Let M be an R-module. A submodule N of M is said to be *Cohen-pure* if for every R-module F, the natural homomorphism $F \otimes N \longrightarrow F \otimes M$ is injective (see [15]). M is said to be *Fieldhouse regular* if every submodule of M is Cohen-pure (see [17]).

Definition 2.14. We say that an R-module M is *fully pure* if every submodule of M is pure.

Theorem 2.15. *Let M be an R-module. Then*

$$M \text{ is fully idempotent } \Rightarrow M \text{ is von Neumann regular } \Rightarrow$$

$$M \text{ is Fieldhouse regular } \Rightarrow M \text{ is fully pure.}$$

These concepts are equivalent if M is a multiplication module.

Proof. The first implication is proved in Proposition 2.13. The second and third implications follow respectively from [11, p. 244] and [20, p. 157]. Now suppose that M is a multiplication R-module. It is straightforward to see that every pure submodule of M is idempotent. □

Example 2.16. Set $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then M as a $\mathbb{Z}$-module is von Neumann regular but it is not fully idempotent.

An R-module M is said to be a *weak multiplication module* if M does not have any prime submodule or for every prime submodule N of M there exists an ideal I of R such that $N = IM$ (see [1]).

Theorem 2.17. *Let M be an R-module. Then we have the following.*

(a) *If every prime submodule of M is idempotent, then M is a weak multiplication module.*
(b) *Every prime submodule of M is idempotent if and only if every prime submodule of the R_P-module M_P is idempotent for every prime (or maximal) ideal P of R.*

(c) *If every prime submodule of M is idempotent and M is Noetherian, then M is a fully idempotent module.*

Proof. (a) Let P be a prime submodule of M. By assumption, $P = (P :_R M)^2 M \subseteq (P :_R M)M$, as requited.

(b) Let N be a prime submodule of M_P, where P is a prime ideal of R. By [21, Proposition 1], we know that $N \cap M$ is a prime submodule of M. Hence $N \cap M = (N \cap M :_R M)^2 M$. Therefore, $N = (N \cap M)_P = (N \cap M :_R M)^2_P M_P$. Conversely, let N be a prime submodule of M. It is enough to show that $((N/(N :_R M)^2 M)_P = 0$ for every maximal ideal P of R. If $(N :_R M) \subseteq P$, then by [21, Proposition 1], N_P is a prime submodule. Thus $N_P = (N_P :_{R_P} M_P)^2 M_P$, and by [21, Corollary 1], $(N_P :_{R_P} M_P) = (N :_R M)_P$. Hence

$$(N/(N :_R M)^2 M)_P \cong N_P/(N :_R M)^2_P M_P = N_P/(N_P :_{R_P} M_P)^2 M_P = 0.$$

If $(N :_R M) \not\subseteq P$, then clearly $N_P = M_P$ and $(N :_R M)_P = R_P$. Therefore,

$$(N/(N :_R M)^2 M)_P \cong M_P/M_P = 0.$$

(c) Suppose that M is Noetherian and every prime submodule of M is idempotent. By part (a), M is a weak multiplication R-module. Thus by [10, 2.7], M is a multiplication R-module. We claim that M is locally simple. Then the result follows from Proposition 2.13 (d). To see this, assume that (R, m) is a local ring. By [16, 2.5], mM is the only maximal submodule of M. By assumptions, $mM = m^2 M$. As M is Noetherian, mM is finitely generated. Thus by Nakayama' Lemma, $mM = 0$. Therefore, M is a simple R-module, as desired. □

The following example shows that in part (c) of the above theorem, the condition M is Noetherian can not be omitted.

Example 2.18. The only prime submodule of $\mathbb{Q}$ as a $\mathbb{Z}$-module is (0) and $\mathbb{Q}$ is clearly idempotent. But $\mathbb{Q}$ as a $\mathbb{Z}$-module is not fully idempotent.

Let M be an R-module. If N is a submodule of M, then $cl(N)$ denotes the set of all maximal submodules of M that contain N. M is called a *spectral module* if $cl(K) \cup cl(N) = cl(K \cup N)$ for every submodules K and N of M. If M is a spectral module, then the set $Max(M)$ of all maximal submodules of M becomes a topological space by taking sets $\{cl(N) : N \textit{ is a submodule of } M\}$ as closed sets (see [25]).

We recall that an element m of M is said to be *singular* if $Ann_R(m)$ is a large submodule of R. Moreover, M is said to be a *nonsingular module* if M has no non-zero singular element (see [20]).

Theorem 2.19. *Let M be a fully idempotent R-module. Then we have the following.*

(a) *M is spectral.*
(b) *Every finitely generated submodule of M is cyclic.*
(c) *If M is faithful, then $E(R) \cong E(M)$.*

Proof. (a) By Proposition 2.13 (a) and [25, 2.32], it is enough to show that M is a distributive R-module. To see this, let N, K, and H be submodules of M. By Proposition 2.7 (a), M is a multiplication module. So by using [16, 1.7], we have

$$(N\cap K)+(N\cap H) = ((N :_R M)\cap(K :_R M))M+((N :_R M)\cap(H :_R M))M$$
$$\supseteq ((K :_R M) + (H :_R M))N.$$

This implies that

$$(N \cap K) + (N \cap H) = ((K :_R M) + (H :_R M))M \cap N = (K + H) \cap N$$

because by Theorem 2.15, every submodule of M is pure.

(b) This follows from part (a) and [22, 11.13].

(c) First we show that M is a nonsingular R-module. Let x be a non-zero element of M such that $Ann_R(x)$ is a large ideal of R. Since M is fully idempotent, there exists $t \in (Rx :_R M)$ such that $x = tx$. As $Ann_R(x)$ is a large ideal of R, we have $Rt \cap Ann_R(x) \neq 0$. Thus there exists $r \in R$ such that $0 \neq rt \in Ann_R(x)$. Hence $rtx = 0$ so that $rx = 0$. This implies that $rt \in Ann_R(M)$, which is a contradiction because M is faithful. Therefore, M is a nonsingular R-module. Now the result follows from [24, 1.4] because M is a multiplication R-module by Proposition 2.7 (a). □

3. **Fully coidempotent modules**

Definition 3.1. We say that a submodule N of an R-module M is *coidempotent* if $N = C(N^2)$.

Definition 3.2. An R-module M is said to be *fully coidempotent* if every submodule of M is coidempotent.

Example 3.3. For each prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_p$ is fully coidempotent. Moreover, $E(\mathbb{Z}_P) = \mathbb{Z}_{p^\infty}$ is not a fully coidempotent $\mathbb{Z}$-module.

Let M be an R-module. A submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of R-submodules of M, then $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M (see [18]).

A non-zero submodule S of M is said to be *naturally semi-coprime* if for a submodule N of M, the relation $S \subseteq C(N^2)$ implies that $S \subseteq N$ (see [6]).

In the following proposition, we characterize the fully coidempotent R-modules.

Proposition 3.4. Let M be an R-module. Then the following statements are equivalent.

(a) M is a fully coidempotent module.
(b) Every completely irreducible submodule of M is coidempotent.
(c) Every non-zero submodule of M is naturally semi-coprime.
(d) For all submodules N and K of M, we have $N + K = C(NK)$.

Proof. $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$. These are clear.

$(b) \Rightarrow (a)$. Suppose that N is a submodule of M and L is a completely irreducible submodule of M such that $N \subseteq L$. Then $C(N^2) \subseteq C(L^2) = L$. This implies that $C(N^2) \subseteq N$. Hence $C(N^2) = N$ because the reverse inclusion is clear.

$(c) \Rightarrow (a)$. Suppose that N is a submodule of M. Then by hypothesis, $C(N^2) \subseteq C(N^2)$ implies that $C(N^2) \subseteq N$, as required.

$(a) \Rightarrow (d)$. Let N and K be two submodules of M. Then

$$N + K = (0 :_M Ann_R^2(N + K)) \supseteq (0 :_M Ann_R(N)Ann_R(K)) = C(NK).$$

Thus $N + K = C(NK)$ because the reverse inclusion is clear.

$(d) \Rightarrow (a)$. For a submodule N of M, we have $N = N + N = C(N^2)$. □

Proposition 3.5. Let M be a fully coidempotent R-module.

(a) M is a comultiplication module.
(b) Every submodule and every homomorphic image of M is fully coidempotent.
(c) If M is a finitely generated module, then M is a multiplication module.

(d) If R is a Noetherian ring and M is an injective R-module, then every submodule of M is also an injective R-module.

Proof. (a) This is clear.

(b) It is easy to see that every submodule of M is fully coidempotent. Now let N be a submodule of M and let K/N be a submodule of M/N. By part (a), M is a comultiplication R-module. Hence $K = (0 :_M Ann^2_R(K))$ implies that $K = (0 :_M Ann^3(K))$. Thus

$$(0 :_{M/N} Ann^2(K/N)) = (0 :_M Ann_R(N)Ann^2_R(K/N))/N$$
$$\subseteq (0 :_M Ann^3_R(K))/N = K/N.$$

Therefore, $K/N = (0 :_{M/N} Ann^2_R(K/N))$.

(c) Let N be a submodule of M. Since M is fully coidempotent, $N = C(N^2)$. Thus $(0 :_{M/N} Ann_R(N)) = 0$. By part (a) and (b), M/N is a comultiplication R-module. Now since M/N is finitely generated, $R = Ann_R(N) + (N :_R M)$ by [7, 3.5]. This implies that $N = (N :_R M)M$, as required.

(d) Set $I = Ann_R(N)$. Since M is fully coidempotent, $N = (0 :_M I^2)$. Thus $\Gamma_I(M) = N$, where $\Gamma_I(M) = \cup_{n\in\mathbb{N}}(0 :_M I^n)$. But since M is an injective R-module, $\Gamma_I(M)$ is an injective R-module by [13, 2.1.4]. □

The following example shows that the converse of part (a) of the above proposition is not true in general.

Example 3.6. $\mathbb{Z}_4$ is a comultiplication $\mathbb{Z}$-module which is not fully coidempotent.

Let M be an R-module and let N be a submodule of M. The following example shows that if N and M/N are fully coidempotent modules, then M is not necessarily a fully coidempotent module.

Example 3.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z}$ and set $N = 2\mathbb{Z}/4\mathbb{Z}$. Then N and M/N are fully coidempotent $\mathbb{Z}$-modules, while M is not fully coidempotent.

Theorem 3.8. *Let M be an R-module. Then we have the following.*

(a) *If M is a Noetherian fully idempotent module, then M is a fully coidempotent module.*

(c) *If R is a von Neumann regular ring and M is a comultiplication R-module, then M is a fully coidempotent R-module.*

(b) *If M is a comultiplication module such that every completely irreducible submodule of M is a direct summand of M, then M is a fully coidempotent module.*

(d) *If M is a semisimple comultiplication module, then M is a fully coidempotent module.*

Proof. (a) Let N be a submodule of M. Then since M is fully idempotent, $N = (N :_R M)N$. As N is finitely generated by Nakayama' Lemma, $R = (N :_R M) + Ann_R(N)$. Hence

$$(0 :_M Ann_R(N)) = (N :_R M)(0 :_M Ann_R(N)).$$

Thus $(0 :_M Ann_R(N)) \subseteq N$. This implies that M is a comultiplication R-module. Since M is Noetherian, M is a semisimple R-module by Theorem 2.19 (b) and Proposition 2.13. Therefore, the result follows from part (a).

(b) This is clear.

(c) By Proposition 3.4, it is enough to show that every completely irreducible submodule of M is coidempotent. Let L be a completely irreducible submodule of M. By hypothesis, $M = L + K$ (d.s.), where K is a submodule of M. Thus

$$\begin{aligned}(L :_M Ann_R(L)) &= (L :_L Ann_R(L)) + (L :_K Ann_R(L))\\ &= L + (0 :_K Ann_R(L)).\end{aligned}$$

Since M is a comultiplication R-module, it follows that $L = (0 :_M Ann^2_R(L))$, as desired.

(d) This follows from part (c). □

A non-zero submodule N of an R-module M is said to be *second* if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero (see [27]).

Theorem 3.9. *Let M be a fully coidempotent R-module. Then the following hold.*

(a) *M is Hopfian.*

(b) *If R is a domain and M is a faithful R-module, then M is simple.*

(c) *Every second submodule of M is a minimal submodule of M.*

Proof. (a) Let $f : M \to M$ be an epimorphism. Then by assumption and Proposition 3.5 (a), $Ker(f) = (0 :_M I) = (0 :_M I^2)$, where $I = Ann_R(ker(f))$. If $y \in Ker(f)$, then $y \in (0 :_{f(M)} I)$ because f is an epimorphism. Thus $y = f(x)$ for some $x \in M$ and $f(x)I = 0$. Hence $xI^2 = 0$. It follows that $xI = 0$. Therefore, $y = 0$, as required.

(b) By [9, 3.3], every non-zero endomorphism of M is an epimorphism. Thus by part (a) every non-zero endomorphism of M is an isomorphism. Now let $0 \neq x \in M$ and $a \in Ann_R(x)$. Then $ax = 0$. Therefore, $a = 0$.

Hence $Ann_R(x) = 0$. Thus as M is a comultiplication R-module, we have $Rx = M$, as desired.

(c) Let S be a second submodule of M and let K be a submodule of S. If $Ann_R(K) \subseteq Ann_R(S)$, then $S \subseteq K$ because M is a comultiplication R-module by Proposition 3.5 (a). If $Ann_R(K) \not\subseteq Ann_R(S)$, then there exists $r \in Ann_R(K) \setminus Ann_R(S)$. Since S is second, $rS = S$. By Proposition 3.5 (b), S is fully coidempotent. Hence by part (a), S is Hopfian. It follows that the epimorphism $r : S \to S$ is an isomorphism. Hence $rK = 0$ implies that $K = 0$, as required. □

Definition 3.10. We say that an R-module M is *fully copure* if every submodule of M is copure.

Lemma 3.11. Let M be a semisimple R-module. Then M is fully copure.

Proof. Let N be a submodule of M. Then there exists a submodule K of M such that $M = K + N$ (d.s.). Now for every ideal I of R,

$$(N :_M I) = (N :_K I) + (N :_N I) = (0 :_K I) + N \subseteq (0 :_M I) + N.$$

This completes the proof because the reverse inclusion is clear. □

Theorem 3.12. *Let M be a comultiplication R-module and let N be a submodule of M. Then the following statements are equivalent.*

(a) *N is a copure submodule of M.*
(b) *M/N is a comultiplication R-module and N is a coidempotent submodule of M.*
(c) *M/N is a comultiplication R-module and $K = (N :_M Ann_R(K))$, where K is a submodule of M with $N \subseteq K$.*
(d) *M/N is a comultiplication R-module and $(N :_M Ann_R(K)) = (N :_M (N :_R K))$, where K is a submodule of M.*

Proof. $(a) \Rightarrow (b)$. Let K/N be a submodule of M/N. Then since N is copure,

$$(0 :_{M/N} Ann_R(K/N)) = (N + (0 :_M Ann_R(K/N)))/N \subseteq$$
$$(N + (0 :_M Ann_R(K))/N = (N + K)/N = K/N.$$

This implies that M/N is a comultiplication R-module. Now as N is copure,

$$(N :_M Ann_R(N)) = N + (0 :_M Ann_R(N)).$$

Thus $(0 :_M Ann^2_R(N)) = N$, and hence N is coidempotent.

$(b) \Rightarrow (c)$. Let K be a submodule of M with $N \subseteq K$. Since M/N is a comultiplication R-module, we have $K/N = (0 :_{M/N} Ann_R(K/N))$. Thus as N is coidempotent,

$$K/N = (N :_M Ann_R(K/N))/N = (N :_M Ann_R(N)Ann_R(K/N))/N$$
$$\supseteq (N :_M Ann_R(K))/N.$$

It follows that $(N :_M Ann_R(K)) = K$ because the reverse inclusion is clear.

$(c) \Rightarrow (a)$. Let I be an ideal of R. Since $N \subseteq (0 :_M I) + N$, we have

$$N+(0 :_M I) = (N :_M Ann_R(N+(0 :_M I)) = (N+(0 :_M I) :_M Ann_R(N))$$
$$\supseteq ((0 :_M I) :_M Ann_R(N)) = (N :_M I).$$

This implies that N is a copure submodule of M.

$(b) \Rightarrow (d)$. Let K be a submodule of M. Since N is coidempotent,

$$(N :_M (N :_R K)) = ((N :_M Ann_R(N)) :_M (N :_R K)) =$$
$$(N :_M Ann_R(N)(N :_R K)) \supseteq (N :_M Ann_R(K)).$$

This implies that $(N :_M Ann_R(K)) = (N :_M (N :_R K))$ because the reverse inclusion is clear.

$(d) \Rightarrow (b)$. Take $K = N$. □

Corollary 3.13. Let M be an R-module. Then we have the following.

(a) If M is a fully coidempotent module, then M is fully copure.
(b) If M is a comultiplication fully copure module, then M is fully coidempotent.

Proof. (a) By Proposition 3.5 (b) and (a), every homomorphic image of M is a comultiplication R-module. Now the result follows from Theorem 3.12 $(b) \Rightarrow (a)$.

(b) This follows from Theorem 3.12 $(a) \Rightarrow (b)$. □

The following example shows that in part (b) of the above corollary, the condition M is a comultiplication module can not be omitted.

Example 3.14. Set $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then M as a $\mathbb{Z}$-module is fully copure, while M is not fully coidempotent.

Proposition 3.15. Let M be an R-module and let N be a submodule of M.

(a) If M is a multiplication module and N is a copure submodule of M, then N is idempotent.

(b) If M is a comultiplication module and N is a pure submodule of M, then N is coidempotent.

Proof. (a) Suppose that M is a multiplication module and N is a copure submodule of M. Then we have $(N :_M (N :_R M)) = N + (0 :_M (N :_R M))$. This in turn implies that $M = N + (0 :_M (N :_R M))$. It follows that $(N :_R M)M = (N :_R M)N$. Hence as M is a multiplication module, we have $N = (N :_R M)^2M$.

(b) Suppose that M is a comultiplication module and N is a pure submodule of M. Then we have $Ann_R(N)N = N \cap Ann_R(N)M$. Hence

$$N = (0 :_M Ann_R(N)) = (N \cap Ann_R(N)M :_M Ann_R(N)) =$$
$$(N :_M Ann(N)) = (0 :_M Ann^2(N)).$$

□

Corollary 3.16. Let M be an R-module. Then we have the following.

(a) If M is a multiplication fully copure module, then M is fully pure.
(b) If M is a comultiplication fully pure module, then M is fully copure.
(c) If M is a multiplication fully coidempotent module, then M is fully idempotent.
(d) If M is a comultiplication fully idempotent module, then M is fully coidempotent.

Proof. (a) By Proposition 3.15 (a), every submodule of M is idempotent. Hence the result follows from Theorem 2.15.

(b) By Proposition 3.15 (b), every submodule of M is coidempotent. Hence the result follows from Corollary 3.13 (a).

(c) This follows from Corollary 3.13 (a) and Proposition 3.15 (a).

(d) By Theorem 2.15, M is fully pure. Thus by part (b), M is fully copure. So the result follows from Corollary 3.13 (b). □

The following example shows that in part (d) of the above corollary, the condition M is a comultiplication module can not be omitted.

Example 3.17. Let

$$R = \{(a_n) \in \prod_{i=1}^{\infty} \mathbb{Z}_2 : a_n \textit{ is eventually constant}\},$$

and let

$$P = \{(a_n) \in R : a_n \textit{ is eventually } 0\}.$$

Then R is a Boolean ring and P is a maximal ideal of R. Moreover, $Ann_R(P) = 0$. Hence P is an idempotent submodule of R but it is not a coidempotent submodule of R. Thus R is a fully idempotent R-module but it is not a fully coidempotent R-module.

Example 3.18. By [23, p. 117]), $\mathbb{Z}_n$ (as a $\mathbb{Z}_n$-module) is semisimple if and only if n is square free. Also $\mathbb{Z}_n$ is a multiplication and comultiplication $\mathbb{Z}_n$-module. Therefore, $\mathbb{Z}_n$ is a fully idempotent and fully coidempotent $\mathbb{Z}_n$-module if and only if n is square free.

Theorem 3.19. *Let M be a fully coidempotent R-module.*

(a) *For every submodule K of M and every collection $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of M, $\cap_{\lambda\in\Lambda}(N_\lambda + K) = \cap_{\lambda\in\Lambda}N_\lambda + K$.*

(b) *If M is a finitely generated R-module, then M is a semisimple R-module.*

Proof. (a) Let K be a submodule of M and let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a collection of submodules of M. Clearly

$$\cap_{\lambda\in\Lambda}N_\lambda + K \subseteq \cap_{\lambda\in\Lambda}(N_\lambda + K).$$

By Proposition 3.5 (a), M is a comultiplication R-module. Hence

$$\cap_{\lambda\in\Lambda}(N_\lambda + K) = \cap_{\lambda\in\Lambda}(0 :_M Ann_R(N_\lambda) \cap Ann_R(K)) \subseteq$$

$$\cap_{\lambda\in\Lambda}(0 :_M Ann_R(N_\lambda)Ann_R(K)) = (\cap_{\lambda\in\Lambda}N_\lambda :_M Ann_R(K)).$$

Now since $\cap_{\lambda\in\Lambda}N_\lambda$ is a copure submodule of M by Corollary 3.13 (a), we have

$$\cap_{\lambda\in\Lambda}(N_\lambda + K) \subseteq \cap_{\lambda\in\Lambda}N_\lambda + K.$$

(b) Let N be a submodule of M. By part (b) and (a) of Proposition 3.5, M/N is a comultiplication R-module. Thus by [7, 3.4], M/N is a finitely cogenerated R-module because M/N is a finitely generated R-module. Hence there exist completely irreducible submodules L_1, $L_2, \cdots, L_n$ of M such that $N = \cap_{i=1}^n L_i$. Now we use induction on n to show that N is a direct summand of M. To see this, first suppose that L is a completely irreducible submodule of M. Then

$$L = (0 :_M Ann_R(L)) = (0 :_M \sum_{t_i\in Ann_R(L)} Rt_i) = \cap_{t_i\in Ann_R(L)}(0 :_M t_i).$$

So there exists $t \in Ann_R(L)$ such that $L = (0 :_M t)$. Thus $L = (0 :_M t) = (0 :_M t^2)$. It follows that $tM = t^2M$ because M is a comultiplication R-module. Now by Nakayama's Lemma, there exists $r \in R$ such that $(1 - rt)tM = 0$. It follows that $M = ((0 :_M t) :_M 1 - rt)$. Since by

Corollary 3.13 (a), every submodule of M is copure, we have $M = (0 :_M t) + (0 :_M 1 - rt)$. Clearly, $(0 :_M t) \cap (0 :_M 1 - rt) = 0$. Thus L is a direct summand of M. Now suppose that L_1 and L_2 are completely irreducible submodules of M. Then $M = L_1 + T_1$ (d.s.) and $M = L_2 + T_2$ (d.s.) for some submodules T_1 and T_2 of M. By part (a), $M = L_1 \cap L_2 + (T_1 \cap L_2 + T_2)$ (d.s.). Therefore, by induction on n, it follows that N is a direct summand of M, as desired. □

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