Bulletin of the Iranian Mathematical Society Vol. 37 *No. 4* (*2011*), *pp 35-49.*

ON p-Λ-BOUNDED VARIATION

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Communicated by Mohammad Sal Moslehian

ABSTRACT. A characteriation of continuity of the p-Λ-variation function is given and the Helly's selection principle for $\Lambda BV^{(p)}$ functions is established. A characterization of the inclusion of Waterman-Shiba classes into classes of functions with given integral modulus of continuity is given. A useful estimate on modulus of variation of functions of class $\Lambda BV^{(p)}$ is found.

1. **Introduction**

In 1980 M. Shiba [13] introduced the class $\Lambda BV^{(p)}$ $(1 \leq p < \infty)$ expanding a fundamental concept of bounded Λ-variation formulated and usefully applied by D. Waterman in 1972 [18].

The main objective of this note is to find a necessary and sufficient condition for the embedding $\Lambda BV^{(p)} \subset H_1^{\omega}$. However, some estimates on the modulus of variation and on the integral modulus of continuity for functions in the Waterman-Shiba class $\Lambda BV^{(p)}$ are given because they were needed in our proof of the main result. Another part of this note is devoted to a characterization of continuity of p-Λ-variation function. The Helly's selection theorem for $\Lambda BV^{(p)}$ functions is proved also.

MSC(2000): Primary: 26A16; Secondary: 46B04, 46B20.
Keywords: Generalized bounded variation, Helly's theorem, modulus of variation.
Received: 1 November 2009, Accepted: 4 April 2010.
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Let $\Lambda = (\lambda_i)$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f : [a, b] \to \mathbb{R}$ is said to be of bounded p-Λ-variation on a not necessarily closed subinterval $P \subset [a, b]$, if

$$V(f;\, P) \;:=\; \sup \left(\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} \;<\; +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of P and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i. The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p-Λ-variation with domain $[0, 1]$. We will write $V(f)$ instead of $V(f, P)$, if $P = [0, 1]$. Also, we will write $V_\Lambda(f)$ instead of $V(f)$ sometimes when it is important to emphasize for which particular Λ-sequence the p-Λ-variation is computed.

The Waterman-Shiba class $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba in [13] and it clearly is a generalization of the well-known Waterman class ΛBV. Some of the basic properties of functions of class $\Lambda BV^{(p)}$ were discussed by R. G. Vyas in [16] recently. More results concerned with the Waterman-Shiba classes and their applications can be found in [2, 3, 10, 11, 12, 14] and [15]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda, p} := |f(0)| + V(f)$ is a Banach space.

Given a function $f \in \Lambda BV^{(p)}$, we will also consider the p-Λ-variation function of f

$$v(x) \;:=\; V(f;\, [0, x])$$

defined for $x \in [0, , 1]$.

Let $h = (h_n)$ be a concave sequence of positive numbers convergent to 0. The Chanturia class $V[h]$ is defined to be the class of all functions $f : [0, 1] \to \mathbb{R}$ such that $\nu(n, f) = \mathrm{O}(h_n)$ as $n \to \infty$, where the modulus of variation of the function f is given by

$$\nu(n,\, f) \;=\; \nu(n,\, f,\, [0,\, 1]) \;:=\; \sup \sum_{k=1}^{n} |f(I_k)|,$$

the supremum is taken over all n-element families $\{I_k\}$ of nonoverlapping subintervals of $[0, 1]$. The concept was introduced by Z. A. Chanturia in 1974 (see [4]) and many interesting applications of it to Fourier series are well-known. It is easy to see that the modulus of variation is subadditive with respect to the underlying interval, that is, if f is defined on $[a, b]$ and $c \in (a, b)$, then

$$\nu(n,\, f,\, [a,\, b]) \;\; \leq \;\; \nu(n,\, f,\, [a,\, c]) \; + \; \nu(n,\, f,\, [c,\, b])$$

for every positive integer n.

Functions in a Waterman-Shiba class $\Lambda BV^{(p)}$ are regulated [15, Theorem 2], hence integrable, and thus it makes sense to consider their integral modulus of continuity

$$\omega_1(\delta,\, f) \; := \; \sup_{0 \leq h \leq \delta} \int_0^{1-h} |f(t+h) - f(t)|\, dt,$$

for $0 \leq \delta \leq 1$. However, if f is defined on $\mathbb{R}$ instead of on $[0, 1]$ and if f is 1-periodic, it is convenient to modify the definition and put

$$\omega_1(\delta,\, f) \; := \; \sup_{0 \leq h \leq \delta} \int_0^{1} |f(t+h) - f(t)|\, dt,$$

since the difference between the two definitions is then nonessential in all applications of the concept. We will use the second definition in our note, and thus the main Theorem 4.1 will actually deal with 1-periodic functions.

A function $\omega :\; [0,\, 1] \to \mathbb{R}$ is said to be a modulus of continuity, if it is nondecreasing, continuous, subadditive and $\omega(0) = 0$. If ω is a modulus of continuity, then H_1^ω denotes the class of functions $f \in L_1[0,\, 1]$ for which $\omega_1(\delta,\, f) = \mathrm{O}(\omega(\delta))$ as $\delta \to 0+$.

2. **Continuity of p-Λ-variation function**

In this section we are going to establish a characterization of continuity of the p-Λ-variation function analogous to [19, Theorem 4] First, we prove a lemma.

Lemma 2.1. *Let f be of class $\Lambda BV^{(p)}$ on I. If $[x, y] \subset I$ and $|f(x) - f(y)| \geq \delta > 0$, then*

$$v(y) - v(x) \geq \left(\frac{\delta}{2}\right)^p \left(\frac{1}{p||f||_{\Lambda,p}^{p-1}}\right) \left(\frac{1}{\lambda_{k_0}}\right);$$

where

$$k_0 = \inf\left\{k : \left(\sum_{i=1}^{k} \frac{1}{\lambda_i}\right)^{1/p} > \left(\frac{2}{\delta}\right) v(x)\right\}.$$

Proof. For given $\eta > 0$, there exist a finite sequence of intervals $I_n,\ n = 1, ..., N$ in $[a, x]$, such that the corresponding numbers $|f(I_n)|$ are non-increasing and

$$v(x) \leq \left(\sum_{n=1}^{N} |f(I_n)|^p/\lambda_n\right)^{1/p} + \eta.$$

Let m $= \inf(\{n : |f(I_n)| < \delta/2\} \cup \{N+1\})$. We shall show that

$$v(y) - v(x) \geq \left(\frac{\delta}{2}\right)^p \left(\frac{1}{\lambda_m}\right) \left(\frac{1}{p||f||_{\Lambda,p}^{p-1}}\right) - \eta. \qquad (*)$$

Put $|f(I_n)|^p = a_n$ and $T = \left(\sum_{n=1}^{N} \frac{a_n}{\lambda_n}\right)^{1/p}$. Let S denote the sum obtained by adjoining δ to the collection $\{a_n\}$ and forming a sum of $N+1$ terms as indicated below:

1^0. If $a_n \geq \delta/2,\ n = 1, ..., k$ but $a_{k+1} < \delta/2$, set

$$S = (a_1/\lambda_1 + ... + a_k/\lambda_k + \delta^p/\lambda_{k+1} + a_{k+2}/\lambda_{k+2} + ... + a_N/\lambda_{N+1})^{1/p};$$

then

$$\begin{aligned} S^p - T^p &> ((\delta^p - a_{k+1})/\lambda_{k+1} + (a_{k+1} - a_{k+2})/\lambda_{k+2} + ... + a_N/\lambda_{N+1}) \\ &> \frac{\delta^p - (\frac{\delta}{2})^p}{\lambda_{k+1}} \end{aligned}$$

Hence,

$$S - T > \frac{S^p - T^p}{pS^{p-1}} > \frac{1}{p||f||_{\Lambda,p}^{p-1}}(S^p - T^p) > \frac{\delta^p - (\frac{\delta}{2})^p}{p||f||_{\Lambda,p}^{p-1}\lambda_{k+1}}.$$

2^0. If $a_n \geq \delta/2$, for all n, set

$$S = (a_1/\lambda_1 + .. + a_N/\lambda_N + \delta^p/\lambda_{N+1})^{1/p};$$

then

$$S - T > \frac{S^p - T^p}{pS^{p-1}} > \frac{1}{p||f||_{\Lambda,p}^{p-1}}(S^p - T^p) > \frac{\delta^p}{p||f||_{\Lambda,p}^{p-1}\lambda_{N+1}}.$$

3^0. If $\delta/2 > a_n$, for all n, set

$$S = (\delta^p/\lambda_1 + a_1/\lambda_2 + ... + a_N/\lambda_{N+1})^{1/p};$$

then

$$S - T > \frac{S^p - T^p}{pS^{p-1}} > \frac{1}{p||f||_{\Lambda,p}^{p-1}}(S^p - T^p) > \frac{\delta^p - (\frac{\delta}{2})^p}{p||f||_{\Lambda,p}^{p-1}\lambda_1}.$$

Now, $v(y) \geq S$. Hence,

$$v(y) - v(x) \geq v(y) - T - \eta \geq S - T - \eta$$

and so

$$v(y) - v(x) \geq \begin{cases} \left(\frac{\delta}{2}\right)^p \left(\frac{1}{\lambda_{k+1}}\right) \left(\frac{1}{p||f||_{\Lambda,p}^{p-1}}\right) - \eta \quad If\ a_n \geq \delta/2 \\ \qquad\qquad for\ n \leq k,\ a_{k+1} < \delta/2, \\ \left(\frac{\delta}{2}\right)^p \left(\frac{1}{\lambda_{N+1}}\right) \left(\frac{1}{p||f||_{\Lambda,p}^{p-1}}\right) - \eta \ \ If\ a_n \geq \delta/2\ \ for\ all\ n, \\ \left(\frac{\delta}{2}\right)^p \left(\frac{1}{\lambda_1}\right) \left(\frac{1}{p||f||_{\Lambda,p}^{p-1}}\right) - \eta \qquad If\ a_n < \delta/2\ \ for\ all\ n, \end{cases}$$

which is (*).

Now, $k_0 \geq m$ since

a) if m=1, therefore $k_0 \geq 1$;

b) if m=N+1, then

$$v(x) \geq \left(\sum_{n=1}^{N} |f(I_n)|^p/\lambda_n\right)^{1/p} \geq \left(\frac{\delta}{2}\right) \left(\sum_{n=1}^{N} 1/\lambda_n\right)^{1/p}$$

where for $I_n = [a_n, b_n],\ f(I_n) = f(b_n) - f(a_n)$. Therefore, $k_0 \geq N + 1$;

c) if $1 < m < N + 1$, then

$$v(x) \ \geq \left(\frac{\delta}{2}\right) \left(\sum_{n=1}^{m-1} 1/\lambda_n\right)^{1/p};$$

therefore $k_0 \geq m$.

From (*) we have

$$v(y) - v(x) \geq \left(\frac{\delta}{2}\right)^p \frac{1}{p||f||_{\Lambda,p}^{p-1}}.\frac{1}{\lambda_{k_0}} - \eta;$$

which implies the desired result since k_0 is independent of η . □

Let us recall that all $\Lambda BV^{(p)}$ functions are regulated [15, Theorem 2], that is, the left-side limit $f(x-)$ and the right-side limit $f(x+)$ exist and are finite at every point.

Lemma 2.2. *If f is of bounded p-Λ-variation on $[a, b]$, then*

(i) $\lim_{\epsilon\to 0^+} V(f;\, (a,\, a+\epsilon]) \;=\; 0 \;=\; \lim_{\epsilon\to 0^+} V(f;\, [b-\epsilon,\, b))$;

(ii) $\lim_{\epsilon\to 0^+} V(f;\, [a,\, a+\epsilon]) \;=\; \dfrac{|f(a)-f(a+)|}{\lambda_1^{\frac{1}{p}}}$;

(iii) $\lim_{\epsilon\to 0^+} V(f;\, [b-\epsilon,\, b]) \;=\; \dfrac{|f(b)-f(b-)|}{\lambda_1^{\frac{1}{p}}}$.

Proof. We are going to prove (i) first. We may assume without loss of generality that $\lambda_1 \geq 1$. Indeed, it suffices to replace Λ by the Λ-sequence $\Lambda' := \frac{1}{\lambda_1}\cdot\Lambda$, if necessary.

Clearly, $V(f;\, (a,\, a+\epsilon]) \geq 0$, for every $\epsilon \in (0,\, b-a)$. Suppose that (i) does not hold. Then, $\limsup_{\epsilon\to 0^+} V(f;\, (a,\, a+\epsilon]) > g$, for some $g > 0$. Because $V(f;\, (a,\, a+\epsilon])$ is non-decreasing a function of ϵ, we conclude that $V(f;\, (a,\, a+\epsilon]) > g$, for every $\epsilon > 0$.

Given a positive integer n, the sequence $(\lambda_i)_{i=n+1}^{\infty}$ is again Λ-sequence and it is denoted by $\Lambda_{(n)}$ usually. We claim now that for every positive integer n there is a number $\gamma_n > 0$ such that

$$V^p_{\Lambda_{(n)}}(f;\, (a,\, a+\epsilon]) \;>\; \frac{g^p}{2}$$

for every $\epsilon \in (0,\, \gamma_n)$. In fact, it suffices to take γ_n small enough to satisfy

$$|f(I)|^p \;<\; \frac{\lambda_1}{n}\cdot\frac{g^p}{2} \tag{2.1}$$

for all subintervals $I \subset (a,\, a+\gamma_n)$. Such a choice is feasible by the existence of the right-hand side limit of f at the point a. If $\gamma > 0$ is smaller than γ_n, then the inequality $V^p_\Lambda(f;\, (a,\, a+\gamma]) > g^p$ implies that there is a family $\{I_i\}_{i=1}^k$ of non-overlapping subintervals of $(a,\, a+\gamma]$ such that

$$\sum_{i=1}^{k}\frac{|f(I_i)|^p}{\lambda_i} \;>\; g^p.$$

Because of (2.1) it must be $k > n$ and thus

$$\sum_{i=n+1}^{k}\frac{|f(I_i)|^p}{\lambda_i} \;>\; g^p \;-\; \sum_{i=1}^{n}\frac{|f(I_i)|^p}{\lambda_i} \;\stackrel{(2.1)}{>}\; \frac{g^p}{2}$$

which proves our claim. Therefore, we can construct an increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers and a sequence $\left(\{I_i^{(n)}\}_{i=1+k_n}^{k_{n+1}}\right)_{n=1}^{\infty}$ of finite families of subintervals inductively so that

$$\max_{1+k_{n+1}\le i\le k_{n+2}} \left(\sup I_i^{(n+1)}\right) \;\le\; \min_{1+k_n\le i\le k_{n+1}} \left(\inf I_i^{(n)}\right)$$

and

$$\sum_{i=1+k_n}^{k_{n+1}} \frac{|f(I_i^{(n)})|^p}{\lambda_i} \;>\; \frac{g^p}{2}$$

for all n. It follows that $f \notin \Lambda BV^{(p)}$, a contradiction. Thus, it must be $V(f;\,(a,\,a+\epsilon]) \to 0$ as $\epsilon \to 0^+$.

Replacing $f(x)$ by $f(b-a-x)$, we see that the second equality in (i) follows from the already proven one.

We will now prove that (ii) holds. Given any family $\{I_i\}_{i=1}^n$ of $[a,\,a+\epsilon]$, one has

$$\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} \;\le\; \sup_{a<x\le a+\epsilon} \frac{|f(a)-f(x)|^p}{\lambda_1} \;+\; \sum_{i:\,\min I_i>a} \frac{|f(I_i)|^p}{\lambda_i},$$

and hence

$$0 \;\le\; V^p(f;\,[a,\,a+\epsilon]) \;\le\; \sup_{a<x\le a+\epsilon} \frac{|f(a)-f(x)|^p}{\lambda_1} \;+\; V^p(f;\,(a,\,a+\epsilon]).$$

Passing to limits with $\epsilon \to 0^+$, we obtain

$$\lim_{\epsilon\to 0^+} V^p(f;\,[a,\,a+\epsilon]) \;\le\; \frac{|f(a)-f(a+)|^p}{\lambda_1}. \tag{2.2}$$

On the other hand,

$$V^p(f;\,[a,\,a+\epsilon]) \;\ge\; \frac{|f(a)-f(a+\epsilon)|^p}{\lambda_1}$$

for every $\epsilon > 0$ which implies that

$$\lim_{\epsilon\to 0^+} V^p(f;\,[a,\,a+\epsilon]) \;\ge\; \frac{|f(a)-f(a+\epsilon)|^p}{\lambda_1}. \tag{2.3}$$

It follows from (2.2) and (2.3) that (ii) holds.

The proof of (iii) is quite similar and we will leave it out. $\square$

Theorem 2.3. *Let f be of class $\Lambda BV^{(p)}$. Then, its p-Λ-variation function v is right (left) continuous at a point if and only if f is right (left) continuous at the point.*

Proof. It follows from Lemma 2.1 that if f is discontinuous at a point x, then v is similarly discontinuous at the point. On the other hand, since p-Λ-variation is subadditive with respect to intervals, the inequalities

$$v(x) \quad \leq \quad v(x+\epsilon) \quad \leq \quad v(x) \; + \; V(f;\, [x,\, x+\epsilon]),$$

which is valid for $\epsilon > 0$, lead in the case of f right-continuous at x to the conclusion

$$\lim_{\epsilon \to 0^+} v(x+\epsilon) \; = \; v(x)$$

by Lemma 2.2. That is, right-countinuity of f at x implies right-continuity of v at x.

3. **Estimates on moduli**

Our first theorem generalizes an estimate on the modulus of variation given by M. Avdispahić in [1].

Theorem 3.1. *If* $f \in \Lambda BV^{(p)}$, *then*

$$\nu(n,\, f) \quad \leq \quad n\, \frac{V(f)}{\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.$$

Proof. We get by Avdispahić's trick

$$\sum_{i=1}^{n} |f(I_i)|^p \quad \leq \quad \frac{n(V(f))^p}{\sum_{i=1}^{n} \frac{1}{\lambda_i}} \tag{3.1}$$

for any $f \in \Lambda BV^{(p)}$ and any family $\{I_i\}_{i=1}^{n}$ of nonoverlapping subintervals (cf. [7, proof of Theorem 11.11]). Now, for $p \in (1,\, +\infty)$, Hölder inequality yields

$$\sum_{i=1}^{n} |f(I_i)| \quad \leq \quad n^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} |f(I_i)|^p\right)^{\frac{1}{p}}.$$

Thus, it follows from (3.1) that

$$\sum_{i=1}^{n} |f(I_i)| \quad \leq \quad n\, \frac{V(f)}{\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}$$

for any $f \in \Lambda BV^{(p)}$ and any family $\{I_i\}_{i=1}^{n}$ of nonoverlapping subintervals of [0, 1] which implies the required inequality. $\square$

The estimate given in Theorem 3.1 enables us to give a short proof of the Helly's selection theorem for $\Lambda BV^{(p)}$, a proof based on a very general form of selection principles for functions of a real variable found by V. V. Chistyakov not long ago.

Theorem 3.2. *If (f_j) is a sequence in $\Lambda BV^{(p)}$ with $\|f_j\|_{\Lambda,\,p} \le M$, then there exists a subsequence (f_{j_k}) converging pointwise to a function f in $\Lambda BV^{(p)}$ with $\|f\|_{\Lambda,\,p} \le M$.*

Proof. Given any sequence $(f_j) \subset \Lambda BV^{(p)}$ uniformly bounded in p-Λ-variation, we get by Theorem 3.1

$$\limsup_{j\to\infty} \nu(n,\, f_j) \;\le\; \limsup_{j\to\infty} n \, \frac{V(f_j)}{\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} \;=\; o(n).$$

On the other hand, it is elementary that if the sequence (f_j) is uniformly bounded in p-Λ-variation norm, then it is uniformly bounded (cf. [14, proof of Lemma 1.6]), and hence pointwise precompact. Thus, by [6, Thm.1], there is a subsequence (f_{j_k}) of (f_j) that converges pointwise to a function f. Since the p-Λ-variation is sequentially lower semicontinuous with respect to pointwise convergence, we have

$$\|f\|_{\Lambda,\,p} \;\le\; \liminf_{k\to\infty} \|f_{j_k}\|_{\Lambda,\,p} \;\le\; M,$$

and therefore the pointwise limit belongs to $\Lambda BV^{(p)}$ which completes the proof.

Another useful consequence of Theorem 3.1 is the following estimate on the order of magnitude of Fourier coefficients $\hat{f}(n)$ of $\Lambda BV^{(p)}$ functions (cf. [11, Corollary]). Before we state it, it is crucial (and easy) to note that the estimate given in Theorem 3.1 is independent of the fixed domain interval, so it can be $[0,\, 2\pi]$ instead of $[0,\, 1]$.

Theorem 3.3. *If $f \in \Lambda BV^{(p)}$, then*

$$|\hat{f}(n)| \;\le\; \frac{V(f)}{2\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.$$

Proof. It follows directly from Theorem 3.1 and from the following result of Z. A. Chanturia [5]: if $f \in V[h]$, then $|\hat{f}(n)| \;\le\; \frac{1}{2}\frac{\nu(n,\, f)}{n}$ (see also [7, Theorem 11.19]). $\square$

It is worth noticing that the inequality of Theorem 3.1 combined with [4, Theorem 5] provides a quick proof of the fact that all functions of bounded p-Λ-variation are regulated.

We conclude the section with an estimate on the integral modulus of continuity for functions in the Waterman-Shiba class. An estimate of equal strength was established by S. L. Wang [17, Theorem 4] in the case $p = 1$ first. Unfortunately, there are two misprints in the statement of his theorem, including a replacement of big "O" by little "o", but they can be found easily by reading the correct proof there. Later M. Schramm and D. Waterman extended the estimate to $p \geq 1$ [11, Thm.1]. Our proof shows that it is a consequence of Theorem 3.1.

Theorem 3.4. *Let f in $\Lambda BV^{(p)}$ be 1-periodic. Then,*

$$\omega_1(\delta;\ f) \leq \frac{2V(f)}{\left(\sum_{i=1}^{\lfloor \frac{1}{\delta} \rfloor} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.$$

Proof. Given a positive integer n, we get

$$\nu(n,\ f,\ [0,\ 1+\frac{1}{n}]) \ \leq\ \nu(n,\ f,\ [0,2]) \ \leq\ 2\nu(n,,f,\ [0,\ 1]), \tag{3.2}$$

because f is 1-periodic and because the modulus of variation is subadditive with respect to intervals. Therefore,

$$\begin{aligned}
\omega_1(\tfrac{1}{n},\ f) &\leq \sup_{0<h\leq\frac{1}{n}} \int_0^1 |f(x+h) - f(x)|dx \\
&= \sup_{0<h\leq\frac{1}{n}} \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(x+h) - f(x)|dx \\
&= \sup_{0<h\leq\frac{1}{n}} \sum_{k=1}^{n} \int_0^{\frac{1}{n}} |f(x+\tfrac{k-1}{n}+h) - f(x+\tfrac{k-1}{n})|dx \\
&= \sup_{0<h\leq\frac{1}{n}} \int_0^{\frac{1}{n}} \sum_{k=1}^{n} |f(x+\tfrac{k-1}{n}+h) - f(x+\tfrac{k-1}{n})|dx \\
&\leq \sup_{0<h\leq\frac{1}{n}} \int_0^{\frac{1}{n}} \nu(n,\ f,\ [0,\ 1+\tfrac{1}{n}])dx \\
&\overset{(3.2)}{\leq} \frac{2}{n}\,\nu(n,\ f)
\end{aligned}$$

$$\overset{\text{(Theorem 3.1)}}{\leq} \quad \frac{2V(f)}{\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.$$

Finally, given a $\delta \in (0,\,1]$, we set $n := \lfloor \frac{1}{\delta} \rfloor$ (that is, n is the integer part of the number $\frac{1}{\delta}$) and obtain

$$\omega_1(\delta,\, f) \;\leq\; \omega_1(\frac{1}{n},\, f) \;\leq\; \frac{2V(f)}{\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} \;=\; \frac{2V(f)}{\left(\sum_{i=1}^{\lfloor \frac{1}{\delta} \rfloor} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.$$

□

4. The embedding $\Lambda BV^{(p)} \subset H_1^{\omega}$

Our main result provides a characterization of the embedding of Waterman-Shiba classes into classes of functions with given integral modulus of continuity. Our proof of the following theorem follows the lines of the proof invented by U. Goginava for characterizing the inclusion $\Lambda BV \subset H_1^{\omega}$ [8, Theorem 1] very closely.

Theorem 4.1. *The inclusion* $\Lambda BV^{(p)} \subset H_1^{\omega}$ *holds if and only if*

$$\liminf_{n\to\infty} \omega^p(\tfrac{1}{n}) \sum_{i=1}^{n} \frac{1}{\lambda_i} \quad > \; 0. \tag{4.1}$$

Proof. If (4.1) holds, then $\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{-\frac{1}{p}} = \mathrm{O}(\omega(\frac{1}{n}))$, and hence
$\omega_1(\frac{1}{n},\, f)$
$= \mathrm{O}(\omega(\frac{1}{n}))$ for any $f \in \Lambda BV^{(p)}$ by Theorem 3.4 which proves that $\Lambda BV^{(p)} \subset H_1^{\omega}$.

We are now going to show that if the condition (4.1) is not satisfied, then there is a function $f \in \Lambda BV^{(p)}$ that does not belong to H_1^{ω}. In fact, if (4.1) fails, then there is an increasing sequence of positive integers (m_k) such that

$$\lim_{k\to\infty} \omega^p(\tfrac{1}{m_k}) \sum_{i=1}^{m_k} \frac{1}{\lambda_i} \quad = \quad 0.$$

Then, setting $n_0 := 1$, we can choose a subsequence (n_k) of (m_k) such that

$$14 n_{k-1} \ < n_k; \tag{4.2}$$

$$\omega^p(\tfrac{1}{n_k}) \sum_{i=1}^{n_k} \frac{1}{\lambda_i} \quad < \quad \left(\frac{1}{n_{k-1}}\right)^{4p} \tag{4.3}$$

for all k. Define even numbers

$$r(n_k) \ := \ \max\left\{ j \in 2\mathbb{N} : \ \frac{2j+1}{n_k} \le \frac{1}{n_{k-1}} \right\}$$

inductively. It follows from the definition that

$$\frac{2(r(n_k)+2)+1}{n_k} \quad > \ \frac{1}{n_{k-1}}$$

which is equivalent to

$$r(n_k) - 1 \quad > \quad \frac{n_k}{2n_{k-1}} \ - \ \frac{7}{2}. \tag{4.4}$$

Next, define the functions

$$f_k(x) \ := \ \begin{cases} (-1)^j c_k (n_k x - 2j + 1) & \text{for } \ x \in [\frac{2j-1}{n_k}, \frac{2j}{n_k}],\ 1 \le j \le r(n_k), \\ (-1)^{j+1} c_k (n_k x - 2j - 1) & \text{for } \ x \in [\frac{2j}{n_k}, \frac{2j+1}{n_k}],\ 1 \le j \le r(n_k), \\ 0 & \text{otherwise}, \end{cases}$$

where

$$c_k \quad := \quad \left(\frac{\omega(\frac{1}{n_k})}{\left(\sum_{i=1}^{n_k} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} \right)^{\frac{1}{2}}. \tag{4.5}$$

It is easy to see that

$$r(n_k) \quad < \quad n_k \tag{4.6}$$

for all k and that the sequence (c_k) decreases to 0. Thus, $f(x) := \sum_{k=1}^{\infty} f_k(x)$ is a well-defined continuous function on $[0, 1]$. We extend it to a 1-periodic function on $\mathbb{R}$. Since

$$\begin{aligned} V^p(f,\,[0,1]) \;&\le\; V^p(f,\,[0,\,\frac{2r(n_1)}{n_1}]) \;+\; V^p(f,\,[\frac{2r(n_1)}{n_1},\,1]) \\ &<\; \sum_{i=1}^{\infty}\sum_{j=1}^{r(n_i)} \frac{(2c_i)^p}{\lambda_j} \;+\; \frac{c_1}{\lambda_1} \\ &\overset{(4.6),(4.5)}{<}\; 2^p \sum_{i=1}^{\infty} \sqrt{\omega^p(\tfrac{1}{n_i})\sum_{j=1}^{n_i}\frac{1}{\lambda_j}} \;+\; \frac{c_1}{\lambda_1} \\ &\overset{(4.3)}{<}\; 2^p \sum_{i=1}^{\infty}\frac{1}{n_{i-1}^{2p}} \;+\; \frac{c_1}{\lambda_1} \;<\; +\infty, \end{aligned}$$

the function f belongs to $\Lambda BV^{(p)}$.

We are now going to show that $f \notin H_1^{\omega}$. Since $f_k(x+\frac{2}{n_k}) = -f_k(x)$ for $x \in [\frac{1}{n_k},\, \frac{2r(n_k)-1}{n_k}]$, we get

$$\begin{aligned} \int_0^1 |f(x+\tfrac{2}{n_k}) - f(x)|\,dx \;&\ge\; \int_{\frac{1}{n_k}}^{\frac{2r(n_k)-1}{n_k}} |f(x+\tfrac{2}{n_k}) - f(x)|\,dx \\ &=\; 2\int_{\frac{1}{n_k}}^{\frac{2r(n_k)-1}{n_k}} |f(x)|\,dx \;=\; 2\cdot\frac{1}{2}\cdot c_k\cdot\frac{2}{n_k}\cdot(r(n_k)-1) \\ &\overset{(4.4)}{\ge}\; \frac{2c_k}{n_k}\left(\frac{n_k}{2n_{k-1}} - \frac{7}{2}\right) \;\overset{(4.2)}{\ge}\; \frac{2c_k}{n_k}\frac{n_k}{4n_{k-1}} \;=\; \frac{c_k}{2n_{k-1}}, \end{aligned}$$

and finally

$$\begin{aligned} \frac{\omega_1(\frac{2}{n_k},\,f)}{\omega(\frac{2}{n_k})} \;&\ge\; \frac{\omega_1(\frac{2}{n_k},\,f)}{2\omega(\frac{1}{n_k})} \;\ge\; \frac{c_k}{2n_{k-1}}\frac{1}{\omega(\frac{1}{n_k})} \\ &=\; \frac{1}{4}\left(\left(\sum_{i=1}^{n_k}\frac{1}{\lambda_i}\right)^{\frac{1}{p}}\omega(\tfrac{1}{n_k})\right)^{-\frac{1}{p}}\cdot\frac{1}{n_{k-1}} \;\overset{(4.3)}{\ge}\; \frac{n_{k-1}}{4} \xrightarrow[k\to\infty]{} +\infty \end{aligned}$$

which completes the proof of the theorem. $\square$

It will be interesting to find conditions necessary and sufficient for more general inclusions $\Lambda BV^{(p)} \subset H_q^{\omega}$ for $q \in [1, +\infty)$ as U. Goginava did for the Waterman class ΛBV [9].

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