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# GENERALIZED $\sigma$-DERIVATION ON BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule. We say that a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a generalized $\sigma$-derivation whenever there exists a $\sigma$-derivation $d : \mathcal{A} \to \mathcal{M}$ such that $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in \mathcal{A}$. Giving some facts concerning generalized $\sigma$-derivations, we prove that if $\mathcal{A}$ is unital and if $\delta : \mathcal{A} \to \mathcal{A}$ is a generalized $\sigma$-derivation and there exists an element $a \in \mathcal{A}$ such that *d(a)* is invertible, then $\delta$ is continuous if and only if $d$ is continuous. We also show that if $\mathcal{M}$ is unital, has no zero divisor and $\delta : \mathcal{A} \to \mathcal{M}$ is a generalized $\sigma$-derivation such that $d(\mathbf{1}) \neq 0$, then $ker(\delta)$ is a bi-ideal of $\mathcal{A}$ and $ker(\delta) = ker(\sigma) = ker(d)$, where $\mathbf{1}$ denotes the unit element of $\mathcal{A}$.

## 1. **Introduction**

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule. Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{M}$ is a $\sigma$-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in \mathcal{A}$ (see [7], [8]). A $\sigma$-derivation $d$ is said to be inner if there exists an element $u \in \mathcal{M}$ such that $d(a) = u\sigma(a) - \sigma(a)u$, for all $a \in \mathcal{A}$. Suppose $\mathcal{M}$ is a Banach right $\mathcal{A}$-module. A linear mapping $\delta : \mathcal{M} \to \mathcal{M}$ is called a generalized derivation if there is a derivation $d : \mathcal{A} \to \mathcal{A}$ such that

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$\delta(xa) = \delta(x)a + xd(a)$ $(x \in \mathcal{M},\ a \in \mathcal{A})$ (for more details see [1, 5]). Generalized inner derivation is defined in [5, 6] as follows:
A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is a generalized inner derivation if $\delta(x) = bx - xa$, for some $a, b \in \mathcal{A}$. Getting idea from generalized derivation, we define a generalized $\sigma$-derivation. Now, suppose $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule, then a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a generalized $\sigma$-derivation if there exists a $\sigma$-derivation $d : \mathcal{A} \to \mathcal{M}$ such that $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in \mathcal{A}$. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a homomorphism(algebra morphism). A linear mapping $T : \mathcal{M} \to \mathcal{M}$ is called a $\varphi$-morphism if $T(xa) = T(x)\varphi(a)$, for all $a \in \mathcal{A}$, $x \in \mathcal{M}$. Using the extension of the definition of $\varphi$-morphism, we define $\sigma$-algebraic map $T : \mathcal{A} \to \mathcal{M}$ as follows: A linear mapping $T : \mathcal{A} \to \mathcal{M}$ is a $\sigma$-algebraic map if there exists a linear mapping $\sigma : \mathcal{A} \to \mathcal{A}$ such that $T(ab) = T(a)\sigma(b)$, for all $a, b \in \mathcal{A}$. It is clear that if $\sigma$ is an endomorphism, then $T$ will be a $\sigma$-morphism in the aforementioned sense. Obviously, generalized $\sigma$-derivation covers the notion of generalized derivation (in case $\sigma = id$, the identity operator on $\mathcal{A}$), notion of a $\sigma$-derivation (in case $\delta = d$), notion of a derivation (in case $\delta = d$, $\sigma = id$), notion of a $\sigma$-algebraic map (in case $d = 0$) and the notion of a modular left centralizer (in case $d = 0$, $\sigma = id$). Thus, it is interesting to investigate properties of this general notion. We shall prove a theorem about the relation between separating space of $\sigma$-derivation $d$ and $\sigma$-algebraic map $T$ and generalized $\sigma$-derivation $\delta$ by Niknam's paper (you can refer to [9]).

## 2. $\sigma$-algebraic maps

Throughout the paper $\mathcal{A}$ and $\mathcal{M}$ denote a Banach algebra and a Banach $\mathcal{A}$-bimodule, respectively. If $\mathcal{A}$ is unital, then $\mathbf{1}$ will show the unit element of $\mathcal{A}$. Recall that if $E$ is a subset of an algebra $B$, the *right annihilator* $ran(E)$ of $E$ (respectively the *left annihilator* $lan(E)$ of $E$) is defined to be $\{b \in B\colon Eb = \{0\}\}$ (respectively $\{b \in B\colon bE = \{0\}\}$). The set $ann(E) := ran(E) \bigcap lan(E)$ is called the *annihilator* of $E$. Suppose $S \subseteq \mathcal{M}$. The right annihilator $ran(S)$ of $S$ is defined to be $\{a \in \mathcal{A} : Sa = 0\}$. Similarly, we define the left annihilator of $S$. We also recall that if $Y$ and $Z$ are normed spaces and $T : Y \to Z$ is a linear mapping, then the set of all $z \in Z$ such that there is a sequence $\{y_n\} \subseteq Y$ with $y_n \to 0$ and $Ty_n \to z$ is called the separating space $S(T)$ of $T$. Clearly, $S(T) = \bigcap_{n=1}^{\infty} \overline{\{T(y) : \|y\| < \frac{1}{n}\}}$ is a closed linear space. If $Y$ and $Z$ are Banach spaces, by the closed graph theorem, $T$ is continuous if and only if $S(T) = \{0\}$.

**Definition 2.1.** *A linear operator $T : \mathcal{A} \to \mathcal{M}$ is called a $\sigma$-algebraic map if there is a linear mapping $\sigma : \mathcal{A} \to \mathcal{A}$ such that $T(ab) = T(a)\sigma(b)$, for all $a, b \in \mathcal{A}$. If $\sigma$ is an endomorphism on $\mathcal{A}$, then $T$ is called a $\sigma$-morphism. It is clear that if $T$ is a $\sigma$-algebraic map on a unital algebra, then $ker(\sigma) \subseteq ker(T)$.*

**Example 2.2.** *Suppose $\sigma : \mathcal{A} \to \mathcal{A}$ is an endomorphism and $T$: $\mathcal{A} \to \mathcal{M}$ is a modular map, i.e., $T(ab) = T(a)b$ $(a, b \in \mathcal{A})$. Then, $T_1 = T\sigma$ is a $\sigma$-algebraic map.*

**Example 2.3.** *Let $ß = \mathcal{A} \times \mathcal{A}$, then ß is a Banach algebra by the following action and norm: $(a, b) \bullet (c, d) = (ac, bd)$ and $\|(a, b)\| = \|a\| + \|b\|$. Suppose $I$ is an ideal of ß, then we know that $\frac{ß}{I}$ is a ß-bimodule by the following actions: $((a, b) + I).(c, d) = (ac, bd) + I, (c, d).((a, b) + I) = (ca, db) + I$. We define $\sigma : ß \to ß$ by $\sigma(a, b) = (a, \frac{a+b}{2})$ and $T : ß \to \frac{ß}{I}$ by $T(a, b) = (a, 0) + I$. Then,* T *is a $\sigma$-algebraic map.*

**Example 2.4.** *Suppose $T$, $\sigma : C([0, 1]) \to C([0, 1])$ are defined by*

$$T(f)(t) = \begin{cases} f(2t)h_0(t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(1)h_0(\frac{1}{2}) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\sigma(f)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

*where, $h_0$ is a fixed element of $C([0, 1])$. It is clear that $T$ is a $\sigma$-algebraic map.*

**Proposition 2.5.** *Suppose $\mathcal{A}$ is a unital algebra and $T : \mathcal{A} \to \mathcal{A}$ is a $\sigma$-algebraic map such that $T(\mathbf{1}) = \mathbf{1}$. Then, $T = \sigma$ and $\sigma$ is an endomorphism.*

*Proof.* $T(a) = T(\mathbf{1})\sigma(a) = \sigma(a)$, for all $a \in \mathcal{A}$, it leads to $\sigma = T$ and we have $\sigma(ab) = T(ab) = T(a)\sigma(b) = \sigma(a)\sigma(b)$. □

**Theorem 2.6.** *Suppose $T : \mathcal{A} \to \mathcal{M}$ is a $\sigma$-algebraic map. Then,*

(i) *$T(\mathcal{A})S(\sigma) \subseteq S(T)$.*
(ii) *$S(T)\sigma(\mathcal{A}) \subseteq S(T)$.*
(iii) *If $\mathcal{M} = \mathcal{A}$ and $T(\mathbf{1})$ is invertible, then $S(T) = T(\mathbf{1})S(\sigma)$ and $T$ is surjective if and only if $\sigma$ is surjective, furthermore $S(T) = \mathcal{A}$ if and only if $S(\sigma) = \mathcal{A}$.*

(iv) *If $\mathcal{M} = \mathcal{A}$ and $\sigma$ is surjective, then $T(\mathcal{A})$ is a right ideal of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is unital, then $T(\mathcal{A})$ is a right ideal generated by $T(\mathbf{1})$.*

*Proof.* (i) Assume that $a \in S(\sigma)$. Then, there is a sequence $\{a_n\} \subseteq \mathcal{A}$ such that $a_n \to 0$ and $\sigma(a_n) \to a$. We have $T(ba_n) = T(b)\sigma(a_n) \to T(b)a$, for all $b \in \mathcal{A}$, it implies that $T(\mathcal{A})S(\sigma) \subseteq S(T)$.
(ii) The proof is similar to the proof of (i).
(iii) Assume $a \in S(T)$. Then, there is a sequence $\{a_n\} \subseteq \mathcal{A}$ such that $a_n \to 0$ and $T(a_n) \to a$. We have $T(\mathbf{1})\sigma(a_n) = T(a_n) \to a$. Since $T(\mathbf{1})$ is invertible, we obtain $\sigma(a_n) \to (T(\mathbf{1}))^{-1}a$, it means that $S(T) \subseteq T(\mathbf{1})S(\sigma)$. Now, Assume that $a \in S(\sigma)$, then, there is a sequence $\{a_n\} \subseteq \mathcal{A}$ such that $a_n \to 0$ and $\sigma(a_n) \to a$. We have $T(\mathbf{1})\sigma(a_n) \to T(\mathbf{1})a$, it means that $T(a_n) \to T(\mathbf{1})a$. We obtain $T(\mathbf{1})S(\sigma) \subseteq S(T)$. Therefore, $S(T) = T(\mathbf{1})S(\sigma)$. Suppose $T$ is surjective and $b \in \mathcal{A}$. Then, $T(\mathbf{1})b \in \mathcal{A}$. Since $T$ is surjective, there exists an element $a \in \mathcal{A}$ such that $T(a) = T(\mathbf{1})b$. Therefore, $b = T(\mathbf{1})^{-1}T(a) = T(\mathbf{1})^{-1}T(\mathbf{1})\sigma(a) = \sigma(a)$. Hence, $\sigma$ is a surjective map. Conversely, suppose $\sigma$ is a surjective mapping. We know that $\mathcal{A} = T(\mathbf{1})\mathcal{A}$; therefore, $T(\mathcal{A}) = T(\mathbf{1})\sigma(\mathcal{A}) = T(\mathbf{1})\mathcal{A} = \mathcal{A}$. In conclusion, $T$ is surjective. By a similar procedure, we are able to prove $S(T) = \mathcal{A}$ if and only if $S(\sigma) = \mathcal{A}$.
(iv) The proof of this part is like the former one. □

**Corollary 2.7.** *Suppose $T : \mathcal{A} \to \mathcal{A}$ is a $\sigma$-algebraic map.*

(i) *If $\mathcal{A}$ is unital and $T(\mathbf{1})$ is invertible, then $T$ is continuous if and only if $\sigma$ is continuous.*
(ii) *Suppose $\mathcal{A}$ is unital and $T(\mathbf{1}) = \mathbf{1}$. If $\sigma$ is a surjective map, then $S(\sigma)$ is a bi-ideal of $\mathcal{A}$.*
(iii) *If $T$ is continuous and $ran(T(\mathcal{A})) = \{0\}$, then $\sigma$ is continuous.*
(iv) *If $\sigma$ is surjective, then $S(T)$ is a right ideal of $\mathcal{A}$.*

*Proof.* (i) We can prove this part by (iii) of previous theorem.
(ii) This part is derived from Proposition 2.5 and (i), (ii) of previous theorem.
(iii) This part can be proved by (i) of the former theorem.
(iv) We obtain this part by (ii) of the former theorem. □

**Definition 2.8.** *A Banach algebra $\mathcal{A}$ has the Cohen's factorization property if for all sequences $\{a_n\} \subseteq \mathcal{A}$ such that $a_n \to 0$ there exist an element $c \in \mathcal{A}$ and a sequence $\{b_n\} \subseteq \mathcal{A}$ such that $a_n = cb_n$, for all positive integer $n$ and $b_n \to 0$. If $\mathcal{A}$ has a bounded left approximate identity,* *then Corollary 11.12 of* [2] *results that it has the Cohen's factorization property.*

**Theorem 2.9.** *Suppose $\mathcal{A}$ has the Cohen's factorization property and $T : \mathcal{A} \to \mathcal{M}$ is a non-zero $\sigma$-algebraic map. If $ran(T(\mathcal{A})) = \{0\}$, then $T$ is continuous if and only if $\sigma$ is continuous.*

*Proof.* Suppose $\sigma$ is continuous and let $\{a_n\}$ be a sequence in $\mathcal{A}$ converging to zero in the norm topology. By Cohen's factorization property, there exist a sequence $\{b_n\}$ and an element $c \in \mathcal{A}$ such that $b_n \to 0$ and $a_n = cb_n$. We have $T(a_n) = T(c)\sigma(b_n) \to 0$; thus, by the closed graph theorem, $T$ is continuous. Conversely, suppose $T$ is continuous. By part (iii) of Corollary 2.7 , $\sigma$ is continuous. □

## 3. Generalized $\sigma$-derivations

**Definition 3.1.** *A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a generalized $\sigma$-derivation if there exists a $\sigma$-derivation $d : \mathcal{A} \to \mathcal{M}$ such that $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)d(b)$, for all $a, b \in \mathcal{A}$.*

For convenience, we say that such a generalized $\sigma$-derivation $\delta$ is a $(\sigma, d)$-derivation. In general, the $\sigma$-derivation $d : \mathcal{A} \to \mathcal{M}$ is not unique and it may happen that $\delta$(respectively$d$) is continuous but $d$(respectively$\delta$) is discontinuous. For instance, assume that the actions of $\mathcal{A}$ on $\mathcal{M}$ and of $\mathcal{A}$ on $\mathcal{A}$ are trivial, i.e., $\mathcal{MA} = \mathcal{AM} = \{0\}$ and $\mathcal{AA} = \{0\}$. Then, every linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation, for all $\sigma$-derivation $d : \mathcal{A} \to \mathcal{M}$.

**Example 3.2.** *Suppose $\sigma : \mathcal{A} \to \mathcal{A}$ is an endomorphism and $\delta : \mathcal{A} \to \mathcal{M}$ is a generalized derivation, i.e., $\delta(ab) = \delta(a)b + ad(b)$, for some derivation $d : \mathcal{A} \to \mathcal{M}$. We know that $d_1 = d\sigma$ is a $\sigma$-derivation. If $\delta_1 = \delta\sigma$, then $\delta_1$ is a $(\sigma, d_1)$-derivation.*

**Example 3.3.** *Suppose $T : \mathcal{A} \to \mathcal{M}$ is a $\sigma$-algebraic map and $d : \mathcal{A} \to \mathcal{M}$ is a $\sigma$-derivation. Then, $\delta = d + T$ is a $(\sigma, d)$-derivation.*

**Example 3.4.** *Suppose ß and $\frac{ß}{I}$ are the symbols which are introduced in Example 2.3. We define $d : ß \to \frac{ß}{I}$ by $d(a, b) = (0, a - b) + I$, $\sigma : ß \to ß$ by $\sigma(a, b) = (a, \frac{a+b}{2})$ and $\delta : ß \to \frac{ß}{I}$ by $\delta(a, b) = (a, a - b) + I$. Then, a straightforward verification shows that $\delta$ is a $(\sigma, d)$-derivation.*

**Theorem 3.5.** *A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation if and only if there exist a $\sigma$-derivation $d : \mathcal{A} \to \mathcal{M}$ and a $\sigma$-algebraic map $T : \mathcal{A} \to \mathcal{M}$ such that $\delta = d + T$.*

*Proof.* Suppose $\delta$ is a $(\sigma, d)$-derivation on $\mathcal{A}$. Then, there exists a $\sigma$-derivation $d$ on $\mathcal{A}$ such that $\delta$ is a $(\sigma, d)$-derivation. Putting $T = \delta - d$ we have

$$\begin{aligned} T(ab) &= (\delta - d)(ab) \\ &= \delta(a)\sigma(b) + \sigma(a)d(b) - d(a)\sigma(b) - \sigma(a)d(b) \\ &= (\delta(a) - d(a))\sigma(b) \\ &= T(a)\sigma(b) \end{aligned}$$

for all $a, b \in \mathcal{A}$. Thus, $T$ is a $\sigma$-algebraic map and $\delta = d+T$. Conversely, let $d$ be a $\sigma$-derivation and $T$ be a $\sigma$-algebraic map on $\mathcal{A}$ and put $\delta = d + T$. Then, clearly, $\delta$ is a linear mapping and

$$\begin{aligned} \delta(ab) &= d(ab) + T(ab) \\ &= d(a)\sigma(b) + \sigma(a)d(b) + T(a)\sigma(b) \\ &= (d(a) + T(a))\sigma(b) + \sigma(a)d(b) \\ &= \delta(a)\sigma(b) + \sigma(a)d(b) \end{aligned}$$

for all $a, b \in \mathcal{A}$. Therefore, $\delta$ is a $(\sigma, d)$-derivation. □

**Theorem 3.6.** *Let $\mathcal{A}$ have the Cohen's factorization property and let $\delta$ be a $(\sigma, d)$-derivation on $\mathcal{A}$ such that $\sigma$ is continuous. Then, $\delta$ is continuous if and only if $d$ is continuous.*

*Proof.* By Theorem 3.5, $T = \delta - d$ is a $\sigma$-algebraic map on $\mathcal{A}$. Since $\sigma$ is continuous by Theorem 2.9, $T$ is continuous. Therefore, $\delta$ is continuous if and only if $d$ is continuous. □

**Theorem 3.7.** *Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a homomorphism and $\delta : \mathcal{A} \to \mathcal{M}$ be a $(\sigma, d)$-derivation. Then,*

(i) *$\mathcal{M}$ equipped with the module multiplications $a.x = \sigma(a)x$ and $x.a = x\sigma(a)$ is a $\mathcal{A}$-bimodule denoted by $\widetilde{M}$.*
(ii) *$\delta : \mathcal{A} \to \widetilde{M}$ is a generalized derivation and $d : \mathcal{A} \to \widetilde{M}$ is an ordinary derivation.*
(iii) *$E = \mathcal{A} \bigoplus \widetilde{M}$ equipped with the multiplication $(a, x)(b, y) = (ab, a.y + x.b)$ is an algebra and $\varphi_d : \mathcal{A} \to E$ defined by $\varphi_d(a) = (a, d(a))$ is an injective homomorphism and $\varphi_\delta : \mathcal{A} \to E$ defined by*

$\varphi_\delta(a) = (a, \delta(a))$ *is an injective* $\varphi_d$*-morphism, i.e.,* $\varphi_\delta(ab) = \varphi_\delta(a)\varphi_d(b)$ $(a, b \in \mathcal{A})$.

(iv) *If* $\mathcal{M}$ *has a norm,* $\sigma$ *is continuous and* $E$ *is equipped by the norm* $\|(a, x)\| = \|a\| + sup\{\|x\|, \|b.x\|, \|x.c\|, \|b.x.c\| : b, c \in \mathcal{A}, \|b\| \leq 1, \|c\| \leq 1\}$*, then* $\varphi_\delta$ *is continuous if and only if* $\delta$ *is continuous. Thus, if every injective* $\varphi$*-morphism of* $\mathcal{A}$ *into a Banach algebra is continuous, then every* $(\sigma, d)$*-derivation of* $\mathcal{A}$ *into a Banach* $\mathcal{A}$*-bimodule is continuous.*

*Proof.* Straightforward (see [7]). □

**Theorem 3.8.** *In this theorem the notations are the same as in Theorem 3.7. Then,*

(i) $\varphi_\delta(\mathcal{A})S(\varphi_d) \subseteq S(\varphi_\delta)$.

(ii) $S(\varphi_\delta)\varphi_d(\mathcal{A}) \subseteq S(\varphi_\delta)$.

(iii) *If* $\mathcal{M}$ *is unital and* $\sigma(\mathbf{1}) = \mathbf{1}$*, then* $S(\varphi_\delta) = \varphi_\delta(\mathbf{1})S(\varphi_d)$ *and* $\varphi_\delta$ *is surjective if and only if* $\varphi_d$ *is surjective. Furthermore,* $S(\varphi_d) = E$ *if and only if* $S(\varphi_\delta) = E$.

*Proof.* The proof is like that of Theorem 2.6. But, note that if $\sigma(\mathbf{1}) = \mathbf{1}$, then $(\mathbf{1},0)$ is the unit element of $E$ and $(\varphi_\delta(\mathbf{1}))^{-1} = (\mathbf{1}, -\delta(\mathbf{1}))$. □

**Definition 3.9.** *Let* $\sigma : \mathcal{A} \to \mathcal{A}$ *be an arbitrary linear mapping and suppose that x,y are two elements of* $\mathcal{M}$ *satisfying* $x(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))y = y(\sigma(ab) - \sigma(a)\sigma(b))$*, for all* $a, b \in \mathcal{A}$*. Then, the* $(\sigma, d)$*-derivation* $\delta : \mathcal{A} \to \mathcal{M}$ *defined by* $\delta(a) = x\sigma(a) - \sigma(a)y$ *is called a generalized inner* $\sigma$*-derivation. In fact,* $\delta$ *is a* $(\sigma, d_y)$*-derivation, where* $d_y(a) = y\sigma(a) - \sigma(a)y$*, for all* $a \in \mathcal{A}$.
*It is clear that, if* $\sigma$ *is an endomorphism, then x,y can be arbitrary elements of* $\mathcal{M}$.

**Theorem 3.10.** *Suppose* $\delta : \mathcal{A} \to \mathcal{A}$ *is a generalized inner* $\sigma$*-derivation. If* $0 < \|x\| < 1$ *and* $0 < \|y\| < 1$*, then*

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} x^k \delta(a) y^{n-1-k}$$

*is a generalized inner* $\sigma$*-derivation.*

*Proof.* We may assume that $\mathcal{A}$ is unital. In fact, if $\mathcal{A}$ has no identity, we shall consider the unitization $\mathcal{A}_1 = \mathcal{A} \bigoplus \mathbb{C}$ of $\mathcal{A}$. First of all, by induction on $n$, we prove that

$$x^n\sigma(a) - \sigma(a)y^n = \sum_{k=0}^{n-1} x^k\delta(a)y^{n-1-k} \tag{3.1}$$

If $n = 1$, then (3.1) is clear. Now, suppose that (3.1) is true, for $n$. We have

$$\begin{aligned} x^{n+1}\sigma(a) - \sigma(a)y^{n+1} &= x(x^n\sigma(a) - \sigma(a)y^n) + (x\sigma(a) - \sigma(a)y)y^n \\ &= x\sum_{k=0}^{n-1} x^k\delta(a)y^{n-1-k} + \delta(a)y^n \\ &= \sum_{k=0}^{n-1} x^{k+1}\delta(a)y^{n-1-k} + \delta(a)y^n \\ &= \sum_{k=1}^{n} x^k\delta(a)y^{n-k} + \delta(a)y^n \\ &= \sum_{k=0}^{n} x^k\delta(a)y^{n-k} \ . \end{aligned}$$

We know that if $\|x\| < 1$, then $(\mathbf{1} - x)^{-1} = \mathbf{1} + \sum_{n=1}^{\infty} x^n$. Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} x^k\delta(a)y^{n-1-k} &= \sum_{n=1}^{\infty} x^n\sigma(a) - \sigma(a)y^n \\ &= (\mathbf{1} - x)^{-1}\sigma(a) - \sigma(a)(\mathbf{1} - y)^{-1}. \end{aligned}$$

□

We can prove theorems like Theorem 3.10, for $\sigma$-derivations and generalized derivations.

The proofs of Lemma 3.11 and Theorem 3.12 are similar to the proofs of Lemma 2.2 and Lemma 2.3 in [7], respectively.

**Lemma 3.11.** *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a $(\sigma, d)$-derivation. Then, $\delta(a)(\sigma(bc) - \sigma(b)\sigma(c)) = (\sigma(ab) - \sigma(a)\sigma(b))d(c)$, for all $a, b, c \in \mathcal{A}$.*

**Theorem 3.12.** *Suppose $\delta$ is a $(\sigma, d)$-derivation such that $\sigma : \mathcal{A} \to \mathcal{A}$ is a continuous mapping. Then, for each $a \in S(\delta), a_1 \in S(d)$ and $b, c \in \mathcal{A}$ we have*

(i) $a(\sigma(bc) - \sigma(b)\sigma(c)) = 0$

(ii) $(\sigma(bc) - \sigma(b)\sigma(c))a_1 = a_1(\sigma(bc) - \sigma(b)\sigma(c)) = 0$

**Corollary 3.13.** *Suppose $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation.*

(i) *If $ran(S(\delta)) \bigcap ann(S(d)) = \{0\}$, then $\sigma$ is an endomorphism.*
(ii) *If $lan(\{\sigma(bc) - \sigma(b)\sigma(c) : b, c \in \mathcal{A}\}) = \{0\}$, then $d$ and $\delta$ are continuous.*

*Proof.* Straightforward. □

**Theorem 3.14.** *Suppose $\mathcal{A}$ is a simple algebra and has the Cohen's factorization property. If $\delta : \mathcal{A} \to \mathcal{A}$ is a $(\sigma, d)$-derivation such that $\sigma$ is a surjective continuous linear mapping, then $\delta$ is continuous or $\sigma$ is an endomorphism.*

*Proof.* First, note that $S(d)$ is a bi-ideal of $\mathcal{A}$ (it is proved in Proposition 2.5 of [7]). Therefore, $S(d)$ is $\{0\}$ or $\mathcal{A}$. If $S(d) = \{0\}$, then $d$ is continuous. We show that $\delta$ is continuous. Suppose $\{a_n\}$ is an arbitrary sequence in $\mathcal{A}$ such that $a_n \to 0$. By Cohen's factorization property, there exist a sequence $\{b_n\}$ and an element $c$ in $\mathcal{A}$ such that $b_n \to 0$ and $a_n = cb_n \ \ (n \in \mathbb{N})$. Then, $\delta(a_n) = \delta(cb_n) = \delta(c)\sigma(b_n) + \sigma(c)d(b_n) \to 0$. Thus, by the closed graph theorem, $\delta$ is continuous. Now, suppose $S(d) = \mathcal{A}$. By Theorem 3.12, we know that $\{\sigma(bc) - \sigma(b)\sigma(c) : b, c \in \mathcal{A}\} \subseteq ann(S(d)) = ann(\mathcal{A})$. Since $\mathcal{A}$ is a bi-ideal, $ann(\mathcal{A})$ is a bi-ideal of $\mathcal{A}$; therefore, $ann(\mathcal{A})$ is $\{0\}$ or $\mathcal{A}$. If $ann(\mathcal{A}) = \mathcal{A}$, then $\mathcal{A}\mathcal{A} = \{0\}$ which is a contradiction and if $ann(\mathcal{A}) = \{0\}$, then $\sigma$ is an endomorphism. □

**Definition 3.15.** *An $\mathcal{A}$-bimodule $\mathcal{M}$ has no zero divisor if $ax = 0$ or $xa = 0$, then a= 0 or $x = 0$ $(a \in \mathcal{A}), x \in \mathcal{M})$.*

**Theorem 3.16.** *Suppose $\mathcal{M}$ has no zero divisor and $\mathcal{A}$ has the Cohen's factorization property and suppose that $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation. If $d$ is non-zero, then $d$ is continuous if and only if $\delta$ is continuous.*

*Proof.* Suppose $\delta$ is continuous and $a \in \mathcal{A}$ such that $d(a) \neq 0$. Let $\{a_n\}$ be an arbitrary sequence in $\mathcal{A}$ such that $a_n \to 0$ and $\sigma(a_n) \to c$. We must prove $c = 0$. Since $a_n a \to 0$ and $\delta$ is continuous, we have $\delta(a_n)\sigma(a) + \sigma(a_n)d(a) = \delta(a_n a) \to 0$. It implies that $cd(a) = 0$. It concludes $d(a) = 0$ which is a contradiction or $c = 0$. Therefore, by the closed graph theorem, $\sigma$ is continuous. Theorems 2.9 and 3.5 imply that the $\sigma$-algebraic map $T = \delta - d$ is continuous. Hence, $d$ is continuous. Conversely, suppose $d$ is continuous and let $\{a_n\}$ be an arbitrary sequence in $\mathcal{A}$ such that $a_n \to 0$ and $\sigma(a_n) \to c$. Since $d$ is continuous, $d(a_n)\sigma(a) + \sigma(a_n)d(a) = d(a_n a) \to 0$. It implies that

$cd(a) = 0$ and it follows that $c = 0$. Hence, $\sigma$ is continuous. The proof is complete by continuity of the $\sigma$-algebraic map $T = \delta - d$. $\square$

**Theorem 3.17.** *Suppose $\mathcal{A}$ is unital and $\delta : \mathcal{A} \to \mathcal{A}$ is a $(\sigma, d)$-derivation. If there exists an element $a \in \mathcal{A}$ such that $d(a)$ is invertible, then $\delta$ is continuous if and only if $d$ is continuous.*

*Proof.* Suppose $d$ is continuous and $a$ is an element in $\mathcal{A}$ such that $d(a)$ is invertible. We show that $\sigma$ is continuous. Let $\{a_n\}$ be an arbitrary sequence such that $a_n \to 0$ and $\sigma(a_n) \to c$. We have $d(a)\sigma(a_n) + \sigma(a)d(a_n) = d(aa_n) \to 0$. Thus, $d(a)c = 0$. Since $d(a)$ is invertible, $c = 0$. By the closed graph theorem, $\sigma$ is continuous. Theorem 2.9 implies that the $\sigma$-algebraic map $T = \delta - d$ is continuous. Hence, $\delta$ is continuous. Conversely, suppose $\delta$ is continuous and let $\{a_n\}$ be an arbitrary sequence in $\mathcal{A}$ such that $a_n \to 0$ and $\sigma(a_n) \to c$. We have $\delta(a_n)\sigma(a) + \sigma(a_n)d(a) = \delta(a_n a) \to 0$ thus, $cd(a) = 0$. Since $d(a)$ is invertible, $c = 0$. Hence, $\sigma$ is continuous; therefore, the $\sigma$-algebraic map $T = \delta - d$ is continuous and so $d$ is continuous. $\square$

**Proposition 3.18.** *Suppose $\mathcal{A}$ is unital and $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation. If $\sigma(\mathbf{1}) = 0$, then $\delta$ and $d$ are equal to zero.*

*Proof.* It is clear that $d(\mathbf{1}) = 0$. We have $d(a) = d(a)\sigma(\mathbf{1}) + \sigma(a)d(\mathbf{1}) = 0$, for all $a \in \mathcal{A}$, it means that $d = 0$. Now, we can see $\delta(\mathbf{1}) = 0$ and it follows that $\delta = 0$. $\square$

**Theorem 3.19.** *Suppose $\mathcal{M}$ is unital and has no zero divisor and suppose that $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation. If $d(\mathbf{1}) \neq 0$, then $ker(\delta)$ is a bi-ideal of $\mathcal{A}$ and $ker(d) = ker(\sigma) = ker(\delta)$.*

*Proof.* First of all, we show that if $d : \mathcal{A} \to \mathcal{M}$ is a non-zero $\sigma$-derivation, then $d(\mathbf{1}) = 0$ if and only if $\sigma(\mathbf{1}) = \mathbf{1}$. Suppose $\sigma(\mathbf{1}) = \mathbf{1}$, it is clear that $d(\mathbf{1}) = 0$. Now, suppose that $d(\mathbf{1}) = 0$ and $a$ is an element in $\mathcal{A}$ such that $d(a) \neq 0$. We have $d(a) = d(a)\sigma(\mathbf{1}) + \sigma(a)d(\mathbf{1}) = d(a)\sigma(\mathbf{1})$, it means that $d(a)(\mathbf{1} - \sigma(\mathbf{1})) = 0$. This equality implies that $\sigma(\mathbf{1}) = \mathbf{1}$. Therefore, we have $d(\mathbf{1}) \neq 0$ if and only if $\sigma(\mathbf{1}) \neq \mathbf{1}$. Let $a \in ker(\sigma)$, we have $d(a) = d(\mathbf{1})\sigma(a) + \sigma(\mathbf{1})d(a) = \sigma(\mathbf{1})d(a)$, it means that $(\mathbf{1} - \sigma(\mathbf{1}))d(a) = 0$. It follows that $d(a) = 0$, i.e., $a \in ker(d)$. Thus, $ker(\sigma) \subseteq ker(d)$. Now, assume that $a \in ker(d)$. By a similar procedure, we obtain $ker(d) \subseteq ker(\sigma)$. Hence, $ker(d) = ker(\sigma)$. We prove that $ker\ (d)$ is a bi-ideal of $\mathcal{A}$. Suppose that $a \in ker(d)$ and $b \in \mathcal{A}$, we have $d(ab) = d(a)\sigma(b) + \sigma(a)d(b) = 0$; hence, $ab \in ker(d)$. Similarly, $ba \in ker(d)$; therefore, $ker\ (d)$ is a bi-ideal of $\mathcal{A}$. Now, we show that $ker(\sigma) = ker(\delta)$.

Suppose that $a \in ker(\sigma)$, we have $\delta(a) = \delta(\mathbf{1})\sigma(a) + \sigma(\mathbf{1})d(a) = 0$; it means that $ker(\sigma) \subseteq ker(\delta)$. Now, suppose that $a \in ker(\delta)$. We have $\delta(a)\sigma(\mathbf{1}) + \sigma(a)d(\mathbf{1}) = \delta(a) = 0$, it means that $\sigma(a)d(\mathbf{1}) = 0$ hence, $a \in ker(\sigma)$. Therefore, $ker(\delta) \subseteq ker(\sigma)$. It follows that $ker(\delta) = ker(\sigma) = ker(d)$. □

**Corollary 3.20.** *Suppose that $\mathcal{M}$ is unital, has no zero divisor and $\mathcal{A}$ is a simple algebra.*

(i) *If $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation such that $d(\mathbf{1}) \neq 0$, then $d$, $\sigma$ and $\delta$ are injective.*

(ii) *If $\mathcal{M} = \mathcal{A}$ and $\delta : \mathcal{A} \to \mathcal{A}$ is a $(\sigma, d)$-derivation such that $d(\mathbf{1}) \neq 0$, then there is no positive integer $n$ such that $\delta^n$ or $\sigma^n$ or $d^n$ are equal to zero.*

*Proof.* Straightforward. □

**Theorem 3.21.** *Suppose that $\mathcal{A}$ is unital.*

(i) *If $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation such that $\delta(\mathbf{1}) = d(\mathbf{1})$, then $\delta = d$.*

(ii) *If $\delta : \mathcal{A} \to \mathcal{A}$ is a $(\sigma, d)$-derivation such that $d(\mathbf{1}) = \mathbf{1}$, then $\delta = d$ and $d$ is an endomorphism.*

*Proof.* (i) The proof of this part is straightforward.
(ii) Since $d(\mathbf{1}) = \mathbf{1}$, it follows that $\sigma(\mathbf{1}) = \frac{\mathbf{1}}{2}$. By Theorem 3.5, $T = \delta - d$ is a $\sigma$-algebraic map; therefore, $T(a) = T(a)\sigma(\mathbf{1}) = \frac{T(a)}{2}$, for all $a \in \mathcal{A}$. It follows that $T = 0$ and in conclusion $\delta = d$. We have $d(a) = d(a)\sigma(\mathbf{1}) + \sigma(a)d(\mathbf{1}) = \frac{d(a)}{2} + \sigma(a)$. Thus, $\frac{d(a)}{2} = \sigma(a)$, for all $a \in \mathcal{A}$. By this fact we have,

$$\begin{aligned} d(ab) &= d(a)\sigma(b) + \sigma(a)d(b) \\ &= d(a)\frac{d(b)}{2} + \frac{d(a)}{2}d(b) \\ &= d(a)d(b). \end{aligned}$$

It means that $d$ is an endomorphism. □

**Theorem 3.22.** *Suppose that $\mathcal{A}$ is unital and $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation such that $\sigma$ is continuous. If $\sigma(\mathbf{1}) = \mathbf{1}$, then $\delta$ is continuous if and only if $d$ is continuous.*

*Proof.* It is clear that $\delta - d = \delta(\mathbf{1})\sigma$. Since $\sigma$ is continuous, $\delta - d$ is continuous. Therefore, by Proposition 5.2.3 of [4], we have $S(\delta) = S(d)$. Hence, $\delta$ is continuous if and only if $d$ is continuous. □

**Proposition 3.23.** *Suppose that $\mathcal{A}$ is unital and $\delta : \mathcal{A} \to \mathcal{A}$ is a $(\sigma, d)$-derivation such that $\delta(\mathbf{1})$ is invertible and $\sigma(\mathbf{1}) = \mathbf{1}$. Then,*

(i) *$\delta(a)$ is not equal to $d(a)$, for all $a \in (ker(\sigma))^C$, where $(ker(\sigma))^C$ is the complement of $ker(\sigma)$.*
(ii) *If $\delta$ and $d$ are continuous, then $ker(\sigma)$ is not dense in $\mathcal{A}$.*
(iii) *$\sigma$ is an endomorphism.*

*Proof.* (i) Arguing by contradiction, suppose that there is an element $b \in (ker(\sigma))^C$ such that $\delta(b) = d(b)$. Since $\sigma(\mathbf{1}) = \mathbf{1}$, we have $\delta(a) - d(a) = \delta(\mathbf{1})\sigma(a)$, for all $a \in \mathcal{A}$; hence, $\delta(\mathbf{1})\sigma(b) = 0$. It follows that $\sigma(b) = 0$ which is a contradiction.
(ii) Arguing by contradiction, suppose that $ker(\sigma)$ is dense in $\mathcal{A}$. If $a \in ker(\sigma)$, we have $\delta(a) = \sigma(\mathbf{1})d(a)$ and $d(a) = \sigma(\mathbf{1})d(a)$. It means that $\delta = d$ on $ker(\sigma)$; hence, $\delta = d$ on $\mathcal{A}$. Assume $b \in (ker(\sigma))^C$. Since $\sigma(\mathbf{1}) = \mathbf{1}$, we have $\delta(a) - d(a) = \delta(\mathbf{1})\sigma(a)$, i.e., $\delta(\mathbf{1})\sigma(a) = 0$, for all $a \in \mathcal{A}$. It follows that $\delta(\mathbf{1})\sigma(b) = 0$. We conclude $\sigma(b) = 0$ which is a contradiction.
(iii) By $\sigma(\mathbf{1}) = \mathbf{1}$, we have $\delta - d = \delta(\mathbf{1})\sigma$. According to Theorem 3.5, $T = \delta - d$ is a $\sigma$-algebraic map. Therefore, we have $\delta(\mathbf{1})\sigma(ab) = \delta(\mathbf{1})\sigma(a)\sigma(b)$. Since $\delta(\mathbf{1})$ is invertible, $\sigma$ is an endomorphism. □

The proof of the following theorem is straightforward.

**Theorem 3.24.** *Suppose that $\delta : \mathcal{A} \to \mathcal{M}$ is a $(\sigma, d)$-derivation. Then,*

(i) *$S(\delta)\sigma(ker(d)) \subseteq S(\delta)$.*
(ii) *$\sigma(ker(\delta))S(d) \subseteq S(\delta)$.*
(iii) *$\delta(ker(\sigma))S(\sigma) \subseteq S(\delta)$.*
(iv) *$S(\sigma)d(ker(\sigma)) \subseteq S(\delta)$.*
(v) *If $\sigma$ is continuous, then $\sigma(\mathcal{A})S(d) \subseteq S(\delta)$.*
(vi) *If $d$ is continuous, then $\delta(\mathcal{A})S(\sigma) \subseteq S(\delta)$.*

**Corollary 3.25.** *(i) Suppose that $\mathcal{M}$ has no zero divisor and $\delta$ is a non-zero continuous $(\sigma, d)$-derivation on $\mathcal{A}$. If $\sigma$ is non-zero, then $\sigma$ is continuous if and only if $d$ is continuous.*
*(ii) Suppose that $\delta$ is a $(\sigma, d)$-derivation such that $d$ is continuous. If $\delta$ is continuous, then $S(\sigma) \subseteq ran(\delta(\mathcal{A}))$.*
*(iii) Suppose that $\mathcal{A}$ is unital and $\delta : \mathcal{A} \to \mathcal{A}$ is a $(\sigma, d)$-derivation such that $d$ is continuous and $\delta(\mathbf{1})$ is invertible. Then, $S(\delta) = \delta(\mathbf{1})S(\sigma)$. If $T = \delta - d$, then $S(\delta) = S(T)$.*

*Proof.* (i) Suppose that $\sigma$ is continuous. By continuity of $\delta$ and part $(v)$ of Theorem 3.24, we obtain $\sigma(\mathcal{A})S(d) = \{0\}$, i.e., $\sigma(a)b = 0$, for all

$a \in \mathcal{A}$, $b \in S(d)$. Let $a \in \mathcal{A}$ such that $\sigma(a) \neq 0$. We have $\sigma(a)b = 0$, for all $b \in S(d)$; it implies that $\sigma(a) = 0$, where it is a contradiction or $b = 0$. Since $b$ is an arbitrary element in S($d$), S($d$) is equal to $\{0\}$. Hence, $d$ is continuous. Conversely, suppose that $d$ is continuous. By the continuity of $\delta$ and part ($vi$) of Theorem 3.24, we can prove that $\sigma$ is continuous.
(ii) This part can be proved using ($vi$) of Theorem 3.24.
(iii) By part ($vi$) of Theorem 3.24, we obtain $\delta(\mathbf{1})S(\sigma) \subseteq S(\delta)$. Now, suppose that $a \in S(\delta)$, then there is a sequence $\{a_n\}$ in $\mathcal{A}$ such that $a_n \to 0$ and $\delta(a_n) \to a$. We have $\delta(\mathbf{1})\sigma(a_n) + \sigma(\mathbf{1})d(a_n) = \delta(a_n) \to a$, it implies that $\delta(\mathbf{1})\sigma(a_n) \to a$; therefore, $\sigma(a_n) \to (\delta(\mathbf{1}))^{-1}a$ and in conclusion $S(\delta) \subseteq \delta(\mathbf{1})S(\sigma)$. Therefore, $S(\delta) = \delta(\mathbf{1})S(\sigma)$. Since $d$ is continuous, Proposition 5.2.3 of [4] gives that $S(T) = S(\delta)$. □

## References


[1] Gh. Abbaspour, M. S. Moslehian and A. Niknam, Generalized derivations on modules, *Bull. Iranian Math. Soc.* **32**(1) (2006) 21-31.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, 1973.

[3] M. Brešar, On the distance of the compositions of two derivations to the generalized derivations, *Glasg. Math. J.* **33** (1991) 89-93.

[4] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, New Series, 24, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.

[5] B. Hvala, Generalized derivation in rings, *Comm. Algebra.* **26** (4) (1998) 1147-1166.

[6] M. Mathieu, *Elementary Operators and Applications*, Proc. Internat. Workshop, World Scientific Publishing Co., Inc., River Edge, NJ, 1992.

[7] M. Mirzavaziri and M. S. Moslehian, Automatic continuity of $\sigma$-derivations in $C^*$-algebras, *Proc. Amer. Math. Soc.* **134**(11) (2006) 3319-3327.

[8] M. Mirzavaziri and M. S. Moslehian, $\sigma$-derivations in Banach algebras, *Bull. Iranian Math. Soc.* **32**(1) (2006) 65-78.

[9] A. Niknam, A remark on closability of linear mappings in normed algebras, *Arch. Math.* **47** (1986) 131-134.



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