Bulletin of the Iranian Mathematical Society Vol. 37 *No.* 4 (2011), *pp 109-126.*

CONNECTIONS BETWEEN $C(X)$ AND $C(Y)$, WHERE Y IS A SUBSPACE OF X

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Communicated by Mohammad Sal Moslehian

Abstract. In this paper, we introduce a method by which we can find a close connection between the set of prime z-ideals of $C(X)$ and the same of $C(Y)$, for some special subset Y of X. For instance, if $Y = Coz(f)$ for some $f \in C(X)$, then there exists a one-to-one correspondence between the set of prime z-ideals of $C(Y)$ and the set of prime z-ideals of $C(X)$ not containing f. Moreover, considering these relations, we obtain some new characterizations of classical concepts in the context of $C(X)$. For example, X is an F-space if and only if the extension $\Phi : \beta Y \to \beta X$ of the identity map $\imath : Y \to X$ is one-to-one, for each z-embedded subspace Y of X. Supposing p is a non-isolated G_δ-point in X and $Y = X \setminus \{p\}$, we prove that $M^p(X)$ contains no non-trivial maximal z-ideal if and only if $p \in \beta X$ is a quasi P-point if and only if each point of $\beta Y \setminus Y$ is a P-point with respect to Y.

1. Introduction

In this article, any topological space X is Tychonoff and any ideal of C(X) (i.e., the ring of all real valued continuous functions on X) is proper. For any $f \in C(X)$, we denote $f^{-1}\{0\}$ and $X \setminus f^{-1}\{0\}$ by $Z(f)$ and $Coz(f)$, respectively. Supposing $A \subseteq C(X)$ and $B \subseteq Z(X)$, we define $Z(A) = \{Z(f) :\ f \in A\}$ and $Z^{-1}(B) = \{f \in C(X) :\ Z(f) \in B\}$.

MSC(2000): Primary: 54C40; Secondary: 54C45, 54G05, 54G10.
Keywords: z-filter, prime z-ideal, prime z°-ideal, P-space, quasi P-space, F-space, CC-space, G_δ-point.
Received: 29 November 2009, Accepted: 3 June 2010.
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Suppose $p \in \beta X$, then by $M^p(X)$ and $O^p(X)$ we mean the sets $\{f \in C(X) : \ p \in \mathrm{cl}_{\beta X} Z(f)\}$ and $\{f \in C(X) : \ p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$, respectively. For any $p \in \beta X$, we denote the set $Z(M^p(X)) = \{Z(f) : \ f \in M^p(X)\}$ $(Z(O^p(X)) = \{Z(f) : \ f \in O^p(X)\})$ by $\mathcal{A}^p(X)$ $(\mathcal{O}^p(X))$. An ideal I in $C(X)$ is called a z-ideal (z°-ideal), if whenever $Z(f) \subseteq Z(g)$($\mathrm{int}_X Z(f) \subseteq \mathrm{int}_X Z(g)$) and $f \in I$, then $g \in I$. Clearly, a z°-ideal is a z-ideal; every minimal prime ideal is a z°-ideal and every maximal ideal, a z-ideal.

A point $p \in \beta X$ is called P-point (F-point) with respect to X, if $O^p(X)$ is a maximal ideal (prime ideal). A space X is called a P-space (F-space), if every $p \in \beta X$ is a P-point (F-point) with respect to X. Also, X is called a quasi P-space, if every prime z-ideal of $C(X)$ is maximal or minimal. The concept of quasi P-space was first defined in [1] as MZD-space and later considered as quasi P-space in [9].

A space X is called a *CC-space*, if every prime z°-ideal of $C(X)$ is minimal. The root of this concept can be traced in [3]. However, this concept was first defined in [1] as $MZ^\circ D$-space. Later, it was explained in [11] as Cozero complemented space, in [4] as m-space and in [15] as z-good space, respectively.

A subset Y of X is called *z-embedded*, if for every $Z' \in Z(Y)$ there exists a $Z \in Z(X)$ such that $Z \cap Y = Z'$. This means that $Z(Y) = \{Z \cap Y : Z \in Z(X)\}$. Every Lindelöf subspace and cozero subset of X is z-embedded, see [5] for more information. The reader is referred to [8] for undefined terms and notations.

Definition 1.1. *Let $\mathcal{F}$ be a z-filter on a topological space X. We call $\mathcal{F}$ an* invariant interior *z-filter* (*briefly I.I. z-filter*), *if whenever $\mathrm{int}_X Z_1 \subseteq \mathrm{int}_X Z_2$ and $Z_1 \in \mathcal{F}$, then $Z_2 \in \mathcal{F}$.*

(a) *If $\mathcal{F}$ is an I.I. z-filter, then $Z^{-1}(\mathcal{F})$ is a z°-ideal.*
(b) *If I is a z°-ideal, then $Z(I)$ is an I.I. z-filter.*
(c) *Let I be a z-ideal, then I is a z°-ideal if and only if $Z(I)$ is an I.I. z-filter.*

Definition 1.2. *Let X be a topological space. We denote by $\mathbf{F}(X)$ the set of all z-filters on X. Let Y be a subspace of X and $\mathcal{F} \in \mathbf{F}(X)$. We call $\mathcal{F}$ a Z_Y-filter whenever for each $Z_1, Z_2 \in Z(X)$, if $Z_1 \cap Y \subseteq Z_2 \cap Y$ and $Z_1 \in \mathcal{F}$, then $Z_2 \in \mathcal{F}$. We denote the set of all Z_Y-filter on X by $\mathbf{F}_Y(X)$.*

We can easily verify that if Y is subspace of X and $\mathbf{F}_Y(X) = \mathbf{F}(X)$, then Y is z-dense in X (i.e., every zero set of X intersects Y nontrivially).

Let $Y \subseteq X$ and $\imath$ be the identity map from Y to X. Clearly, $\imath$ has a continuous extension from βY to βX. It is obvious that $\Phi(\beta Y) = \mathrm{cl}_{\beta X} Y$ and $\Phi(\beta Y \setminus Y) = \mathrm{cl}_{\beta X} Y \setminus Y$. Define $\gamma : \mathbf{F}(Y) \to \mathbf{F}_Y(X)$ with $\gamma(\mathcal{G}) = \{Z \in Z(X) : Z \cap Y \in \mathcal{G}\}$ and $\lambda : \mathbf{F}_Y(X) \to \mathbf{F}(Y)$ such that $\lambda(\mathcal{F})$ is the z-filter on Y generated by $\{Z \cap Y : Z \in \mathcal{F}\}$. Clearly, $\psi = Z^{-1}\gamma Z$ is a map from the set of all z-ideals of $C(Y)$ to the same of $C(X)$. Throughout the paper we use Φ, γ, λ and ψ with the meanings that have been mentioned previously.

In the main theorem of his paper [13] Kohls considered some connections between $C(X)$ and $C(Y)$, where $Y = X \setminus \{p\}$ and p is a non-isolated G_δ-point of X. In this paper, we try to extend this point of view. In Section 2, we explain the primary properties of functions Φ, γ, λ and ψ, for arbitrary subspaces Y of X. In Section 3, we use those properties to show that γ (λ) is one-to-one (onto) if and only if the subspace Y of X is z-embedded. In the same section, we will observe that there exists a close connection between (prime) z-ideals of $C(Y)$ and (prime) z-ideals of $C(X)$, for any z-embedded subset Y of X. In Section 4, we will show that if $f \in C(X)$ and $Y = Coz(f)$, then there exists a one-to-one correspondence between prime z-ideals of $C(Y)$ and prime z-ideals of $C(X)$ not containing f. In this section, we obtain some useful results as the applications. For instance, we show that the main theorem of Kohls in [13, 2.3] is not only necessary but also sufficient, and consider some applications of those relations mentioned earlier to determine non-trivial maximal prime z-ideals; some spaces and points such as quasi P-space (point), CC-space (point), P-space (point) and F-space (point). In Section 5, we draw our attention to z-embedded zero subsets Y of X and considered connections between $C(X)$ and $C(Y)$.

2. **Arbitrary subsets of X**

In the following lemma, we explain primary properties of functions γ and λ.

Lemma 2.1. *Let Y be a subspace of X. Then,*

(a) $\lambda\gamma(\mathcal{G}) \subseteq \mathcal{G}$, *for each* $\mathcal{G} \in \mathbf{F}(Y)$*;*
(b) $\gamma\lambda(\mathcal{F}) = \mathcal{F}$, *for each* $\mathcal{F} \in \mathbf{F}_Y(X)$*;*
(c) *For each* $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{F}(Y)$ *If* $\mathcal{G}_1 \subseteq \mathcal{G}_2$, *then* $\gamma(\mathcal{G}_1) \subseteq \gamma(\mathcal{G}_2)$*;*
(d) $\lambda(\mathcal{F}_1) \subseteq \lambda(\mathcal{F}_2)$ *if and only if* $\mathcal{F}_1 \subseteq \mathcal{F}_2$*;*
(e) λ *is one-to-one and* γ *is onto;*

(f) *For each z-filter* $\mathcal{P} \in \mathbf{F}_Y(X)$, *if* $\lambda(\mathcal{P})$ *is prime, then* $\mathcal{P}$ *is also prime;*
(g) $\gamma(\mathcal{Q})$ *is prime, for each prime z-filter* $\mathcal{Q}$ *on* Y*;*
(h) $\gamma(\bigcap_{\alpha\in A}\mathcal{G}_\alpha) = \bigcap_{\alpha\in A}\gamma(\mathcal{G}_\alpha)$, *for each family* $\{\mathcal{G}_\alpha\}_{\alpha\in A}$ *of z-filters on* Y*;*
(i) *If* Y *is open, then* $\gamma(\mathcal{G})$ *is an I.I. z-filter, for each I.I. z-filter* $\mathcal{G}$ *on* Y.

Proof. (a). If $Z' \in \lambda\gamma(\mathcal{G})$, then there exist $Z \in \gamma(\mathcal{G})$ such that $Z \cap Y \subseteq Z'$ so $Z \cap Y \in \mathcal{G}$ and therefore $Z' \in \mathcal{G}$.

(b). If $Z \in \gamma\lambda(\mathcal{F})$, then $Z \cap Y \in \lambda(\mathcal{F})$, so there is $Z' \in \mathcal{F}$ such that $Z' \cap Y \subseteq Z \cap Y$ and hence $Z \in \mathcal{F}$. Thus, $\gamma\lambda(\mathcal{F}) \subseteq \mathcal{F}$. If $Z \in \mathcal{F}$, then $Z \cap Y \in \lambda(\mathcal{F})$ and consequently $Z \in \gamma\lambda(\mathcal{F})$. Thus, $\gamma\lambda(\mathcal{F}) \supseteq \mathcal{F}$. Therefore, $\gamma\lambda(\mathcal{F}) = \mathcal{F}$.

(c) is clear. (d) and (e) follows from (b) and (c).

(f). If $Z_1 \cup Z_2 \in \mathcal{P}$, then $(Z_1 \cap Y) \cup (Z_2 \cap Y) \in \lambda(\mathcal{P})$. Since $\lambda(\mathcal{P})$ is prime, without loss of generality, we can assume that $Z_1 \cap Y \in \lambda(\mathcal{P})$. Thus, there is $Z' \in \mathcal{P}$ such that $Z' \cap Y \subseteq Z_1 \cap Y$, so $Z_1 \in \mathcal{P}$.

(g). If $Z_1 \cup Z_2 \in \gamma(\mathcal{Q})$, then $(Z_1 \cap Y) \cup (Z_2 \cap Y) \in \mathcal{Q}$ and so $Z_1 \cap Y \in \mathcal{Q}$ or $Z_2 \cap Y \in \mathcal{Q}$. Thus, $Z_1 \in \gamma(\mathcal{Q})$ or $Z_2 \in \gamma(\mathcal{Q})$.

(h). By (c), it is clear that $\gamma(\bigcap_{\alpha\in A}\mathcal{G}_\alpha) \subseteq \bigcap_{\alpha\in A}\gamma(\mathcal{G}_\alpha)$. If $Z \in \bigcap_{\alpha\in A}\gamma(\mathcal{G}_\alpha)$, then

$$\begin{aligned} &\forall\alpha \in A, \quad Z \in \gamma(\mathcal{G}_\alpha) \quad \Rightarrow \quad \forall\alpha \in A, \quad Z \cap Y \in \mathcal{G}_\alpha \\ &\Rightarrow \quad Z \cap Y \in \bigcap_{\alpha\in A}\mathcal{G}_\alpha \quad \Rightarrow \quad Z \in \gamma(\bigcap_{\alpha\in A}\mathcal{G}_\alpha). \end{aligned}$$

(i). Suppose $Z_1^\circ \subseteq Z_2^\circ$ and $Z_1 \in \gamma(\mathcal{G})$. Then, $Z_1 \cap Y \in \mathcal{G}$ and $(Z_1 \cap Y)^\circ = Z_1^\circ \cap Y \subseteq Z_2^\circ \cap Y = (Z_2 \cap Y)^\circ$ and so $Z_2 \cap Y \in \mathcal{G}$. Hence, $Z_2 \in \gamma(\mathcal{G})$. $\square$

An immediate conclusion of the above lemma is the fact that $\lambda(\mathcal{F})$ is the smallest z-filter $\mathcal{G}$ that $\gamma(\mathcal{G}) = \mathcal{F}$ (i.e., $\lambda(\mathcal{F}) = \bigcap\{\mathcal{G} \in \mathbf{F}(Y) : \gamma(\mathcal{G}) = \mathcal{F}\}$).

Proposition 2.2. *Let* Y *be a subspace of* X *and suppose that for every z-filter* $\mathcal{Q}$ *on* Y, $\mathcal{Q}$ *is prime whenever* $\gamma(\mathcal{Q})$ *is prime. Then,* γ *is one-to-one on the set of minimal prime z-filters.*

Proof. Suppose that $\gamma(\mathcal{Q}_1) = \gamma(\mathcal{Q}_2)$, where $\mathcal{Q}_1$ and $\mathcal{Q}_1$ are minimal prime z-filters on Y. Thus,

$$\gamma(\mathcal{Q}_1 \cap \mathcal{Q}_2) = \gamma(\mathcal{Q}_1) \cap \gamma(\mathcal{Q}_2) = \gamma(\mathcal{Q}_1).$$

Since $\gamma(\mathcal{Q}_1 \cap \mathcal{Q}_2)$ is prime, $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is prime. It follows that $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ or $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ and consequently $\mathcal{Q}_1 = \mathcal{Q}_2$. □

Now, in the following theorem, we explain the relation between functions γ and Φ. Note that if $\mathcal{F}$ is a z-filter on X, then one can easily see that there exists a family $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ of prime z-filters on X such that $\mathcal{F} = \cap_{\alpha \in A} \mathcal{H}_\alpha$. In fact, this is true more generally for $\mathcal{P}$-filters, see [18] for some information about $\mathcal{P}$-filters.

Theorem 2.3. *Let Y be a subspace of X. For each z-filter $\mathcal{G}$ in $\mathbf{F}(Y)$, if $\mathcal{G} \to b$, then $\gamma(\mathcal{G}) \to \Phi(b)$. So, for each prime z-filter $\mathcal{Q}$ in $\mathbf{F}(Y)$, $\gamma(\mathcal{Q}) \to a$ if and only if $\mathcal{Q}$ converges to one point of $\Phi^{-1}(a)$.*

Proof. First, assume that $\mathcal{G}$ is a prime z-filter on Y converging to b. Then, by Lemma 2.1, $\gamma(\mathcal{G})$ is prime and so there exists $a \in \beta X$ such that $\gamma(\mathcal{G}) \to a$. We must show $\Phi(b) = a$. Otherwise, there exist two open sets U and V in βX such that

$$U \cap V = \emptyset \quad , \quad a \in U \quad , \quad \Phi(b) \in V \qquad (1).$$

Since $\gamma(\mathcal{G}) \to a$, $\mathcal{G} \to b$ and Φ is continuous there exist $Z \in \gamma(\mathcal{G})$ such that $Z \subseteq U$ and $Z' \in \mathcal{G}$ such that $Z' \subseteq \Phi^{-1}(V)$. Thus,

$$Z \cap Y \subseteq U \cap Y \quad , \quad Z \cap Y \in \mathcal{G} \qquad (2)$$

$$\text{and} \qquad Z' \subseteq \Phi^{-1}(V) \cap Y = V \cap Y. \quad (3)$$

Then, by (1), (2) and (3), it follows that $Z' \cap (Z \cap Y) \in \mathcal{G}$ and

$$Z' \cap (Z \cap Y) \subseteq (V \cap Y) \cap (U \cap Y) \subseteq V \cap U = \emptyset$$

which is a contradiction. Now, Suppose that $\mathcal{G} \in \mathbf{F}(Y)$ and $\mathcal{G} \to b$. There is a family $\{\mathcal{Q}_\alpha\}_{\alpha \in A}$ of prime z-filters on Y such that $\mathcal{G} = \bigcap_{\alpha \in A} \mathcal{Q}_\alpha$. Clearly, $\mathcal{Q}_\alpha \to b$, for every $\alpha \in A$. So, by the previous $\gamma(\mathcal{Q}_\alpha) \to \Phi(b)$, for each $\alpha \in A$. By Lemma 2.1, it follows that $\gamma(\mathcal{G}) = \bigcap_{\alpha \in A} \gamma(\mathcal{Q}_\alpha) \to \Phi(b)$. □

The converse of Theorem 2.3 is not true, for example, let $b_1, b_2 \in \beta Y$ be distinct and $\Phi(b_1) = \Phi(b_2) = a$. Then, $\mathcal{G} = \mathcal{A}^{b_1}(Y) \cap \mathcal{A}^{b_2}(Y)$ does not converge whereas $\gamma(\mathcal{G}) = \gamma(\mathcal{A}^{b_1}(Y)) \cap \gamma(\mathcal{A}^{b_2}(Y)) \to a$.

Theorem 2.4. *Let Y be a subspace of X.*

(a) *If for each $Z \in \mathcal{A}^p(X)$ we have $Z \cap Y \neq \emptyset$, then there is a z-filter $\mathcal{G}$ on Y such that $\gamma(\mathcal{G}) = \mathcal{A}^p(X)$.*

(b) *Suppose that $\mathcal{P} \subseteq \mathcal{A}^p(X)$ is a prime z-filter on X, $Z \cap Y \neq \emptyset$, for each $Z \in \mathcal{P}$ and there exists $Z_\circ \notin \mathcal{P}$ such that $Z_\circ \supseteq X \setminus Y$. Then, there is a z-filter $\mathcal{G}$ on Y such that $\gamma(\mathcal{G}) = \mathcal{P}$.*

Proof. (a). By part (e) of Lemma 2.1, it is enough to show that $\mathcal{A}^p(X) \in \mathbf{F}_Y(X)$. Suppose $Z_1 \cap Y \subseteq Z_2 \cap Y$ and $Z_1 \in \mathcal{A}^p(X)$. If $Z_2 \notin \mathcal{A}^p(X)$, then for some $Z_\circ \in \mathcal{A}^p(X)$ we have $Z_\circ \cap Z_2 = \emptyset$. Thus,

$$Z_\circ \cap Z_1 \cap Y \subseteq Z_\circ \cap Z_2 \cap Y = \emptyset$$

which is a contradiction. Hence, $\mathcal{A}^p(X) \in \mathbf{F}_Y(X)$.

(b). Suppose $Z_1 \cap Y \subseteq Z_2 \cap Y$ and $Z_1 \in \mathcal{P}$, then

$$Z_1 \subseteq Z_2 \cup (X \setminus Y) \subseteq Z_2 \cup Z_\circ.$$

Thus, $Z_2 \cup Z_\circ \in \mathcal{P}$ and therefore $Z_2 \in \mathcal{P}$. Hence, $\mathcal{P} \in \mathbf{F}_Y(X)$ and by part (e) of Lemma 2.1, we are done. $\square$

By the above discussion and using the map ψ, the following is immediate.

Theorem 2.5. *Suppose Y is subspace of X. Then,*

(a) *If $I \subseteq J$, then $\psi(I) \subseteq \psi(J)$, for every pair of z-ideals I, J of $C(Y)$;*

(b) *$\psi(Q)$ is a prime z-ideal in $C(X)$, for each prime z-ideal Q of $C(Y)$;*

(c) *$\psi(\bigcap_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} \psi(I_\alpha)$, for each family $\{I_\alpha\}_{\alpha \in A}$ of z-ideals of $C(Y)$;*

(d) *If Y is open, then $\psi(I)$ is a z°-ideal in $C(X)$, for each z°-ideal of $C(Y)$;*

(e) *Let I be a z-ideal of $C(Y)$. If $I \supseteq O^q(Y)$, then $\psi(I) \supseteq O^{\Phi(q)}(X)$;*

(f) *If Q is a prime z-ideal of $C(Y)$, then $\psi(Q) \supseteq O^p(X)$ if and only if $Q \supseteq O^q(Y)$, for some $q \in \Phi^{-1}(\{p\})$;*

(g) *If for each $f \in M^p(X)$ we have $f|_Y$ is not unit, then there is a z-ideal I of $C(Y)$ such that $\psi(I) = M^p(X)$;*

(h) *Suppose that $P \subseteq M^p(X)$ is a prime z-ideal of $C(X)$, $f|_Y$ is not unit for each $f \in P$ and there exists $f_\circ \notin P$ such that $f_\circ(X \setminus Y) = \{0\}$. Then, there exists a z-ideal I of $C(Y)$ such that $\psi(I) = P$.*

3. z-embedded subsets of X

Theorem 3.1. *Let Y be a subspace of X. The following statements are equivalent.*

(a) *γ is one-to-one.*
(b) *λ is onto.*
(c) *Y is z-embedded.*
(d) *For each z-filters $\mathcal{G}_1$ and $\mathcal{G}_2$ on Y, if $\gamma(\mathcal{G}_1) \subseteq \gamma(\mathcal{G}_2)$, then $\mathcal{G}_1 \subseteq \mathcal{G}_2$.*
(e) *For each prime z-filters $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on Y, if $\gamma(\mathcal{Q}_1) \subseteq \gamma(\mathcal{Q}_2)$, then $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$.*
(f) *For each prime z-filters $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on Y, if $\gamma(\mathcal{Q}_1) = \gamma(\mathcal{Q}_2)$, then $\mathcal{Q}_1 = \mathcal{Q}_2$.*
(g) *$\lambda\gamma(\mathcal{Q}) = \mathcal{Q}$, for each prime z-filter $\mathcal{Q}$ on Y.*
(h) *$\lambda\gamma(\mathcal{G}) = \mathcal{G}$, for each z-filter $\mathcal{G}$ on Y (i.e., $\gamma^{-1} = \lambda$).*

Proof. (a) $\Rightarrow$ (b). It follows from part (b) of Lemma 2.1.

(b) $\Rightarrow$ (c). Suppose that $Z' \in Z(Y)$. Consider the z-filter $\mathcal{G}$ on Y generated by Z'. Since λ is onto, there is a z-filter $\mathcal{F}$ on X such that $\lambda(\mathcal{F}) = \mathcal{G}$, hence $Z \cap Y \subseteq Z'$, for some $Z \in \mathcal{F}$. Since $\mathcal{G}$ is the z-filter generated by Z' and $Z \cap Y \in \mathcal{G}$, it follows that $Z' \subseteq Z \cap Y$, so $Z \cap Y = Z'$. Therefore, Y is z-embedded.

(c) $\Rightarrow$ (d). Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be z-filters on Y and $\gamma(\mathcal{G}_1) \subseteq \gamma(\mathcal{G}_2)$. Now, suppose $Z' \in \mathcal{G}_1$. By hypothesis, there exists $Z \in Z(X)$ such that $Z' = Z \cap Y$. Obviously $Z \in \gamma(\mathcal{G}_1)$ and consequently $Z \in \gamma(\mathcal{G}_2)$. Hence, $Z' \in \mathcal{G}_2$.

(d) $\Rightarrow$ (e) $\Rightarrow$ (f). These implications are clear.

(f) $\Rightarrow$ (g). By part (b) of Lemma 2.1, for every prime z-filter $\mathcal{Q}$ on Y, we have $\gamma\lambda\gamma(\mathcal{Q}) = \gamma(\mathcal{Q})$ and so by (f), $\lambda\gamma(\mathcal{Q}) = \mathcal{Q}$.

(g) $\Rightarrow$ (h). Since every z-filter is an intersection of prime z-filters, it is also easy to prove.

(h) $\Rightarrow$ (a). This implications is evident. $\square$

Corollary 3.2. *Let Y be a z-embedded subspace of X. For each z-filter $\mathcal{Q}$, $\gamma(\mathcal{Q})$ is prime if and only if $\mathcal{Q}$ is prime.*

Proof. By part (g) of Lemma 2.1 it is sufficient to show that if $\gamma(\mathcal{Q})$ is prime, then $\mathcal{Q}$ is prime. The z-filter $\mathcal{Q}$ is an intersection of prime z-filters $\{\mathcal{Q}_\alpha\}_{\alpha\in A}$. Since $\gamma(\mathcal{Q}) = \cap_{\alpha\in A}\gamma(\mathcal{Q}_\alpha)$, and $\gamma(\mathcal{Q})$ and $\{\gamma(\mathcal{Q}_\alpha)\}_{\alpha\in A}$

are prime, $\{\gamma(\mathcal{Q}_\alpha)\}_{\alpha\in A}$ forms a chain. Hence, , by part (d) of Theorem 3.1, $\{\mathcal{Q}_\alpha\}_{\alpha\in A}$ forms a chain and $\mathcal{Q} = \cap_{\alpha\in A}\mathcal{Q}_\alpha$. Thus, $\mathcal{Q}$ is prime. □

As an immediate consequence of Corollary 3.2, it follows that every z-embedded subset of an F-space is also an F-space. Although, this fact can be obtained from [6, Proposition 4.5] in which Blair and Hager showed that X is an F-space if and only if every z-embedded subset is C^*-embedded. Now, we give a different proof for this and an extra equivalent condition.

Theorem 3.3. *The following statements are equivalent.*
(*a*) *X is an F-space.*
(*b*) *Φ is one-to-one for every z-embedded subspace Y of X.*
(*c*) *Every z-embedded subspace Y of X is C^*-embedded.*

Proof. (a) $\Rightarrow$ (b). If $\Phi(q_1) = p = \Phi(q_2)$, then by Theorem 2.3, $\gamma(\mathcal{A}^{q_1}(X)\cap \mathcal{A}^{q_2}(X)) \to p$, so $\mathcal{O}^p \subseteq \gamma(\mathcal{A}^{q_1}(X) \cap \mathcal{A}^{q_2}(X))$. Since X is an F-space, it follows that $\gamma(\mathcal{A}^{q_1}(X) \cap \mathcal{A}^{q_2}(X))$ is prime, since Y is z-embedded, we conclude that $\mathcal{A}^{q_1}(X) \cap \mathcal{A}^{q_2}(X)$ is prime and hence $q_1 = q_2$.

(b) $\Rightarrow$ (c). If Φ is one-to-one, then βY is homeomorphic with $\mathrm{cl}_{\beta X}Y$ and so βY is C^*-embedded in βX. Thus, Y is C^*-embedded in X.

(c) $\Rightarrow$ (a). If U is cozero subset of X, then U is z-embedded and hence it is C^*-embedded in X. Therefore, by [8, 14.25], X is an F-space. □

Theorem 3.4. *Let Y be a z-embedded subspace of X.*

(a) *If for each $Z \in \mathcal{A}^p(X)$ we have $Z \cap Y \neq \emptyset$, then $\gamma(\mathcal{A}^q(Y)) = \mathcal{A}^p(X)$ in which $\Phi(q) = p$.*

(b) *Suppose $\mathcal{P} \subseteq \mathcal{A}^p(X)$ is a prime z-filter on X, $Z\cap Y \neq \emptyset$ for each $Z \in \mathcal{P}$ and there exists $Z_\circ \notin \mathcal{P}$ such that $Z_\circ \supseteq X \setminus Y$, then there is a prime z-filter $\mathcal{Q}$ on Y such that $\mathcal{Q} \subseteq \mathcal{A}^q(Y)$ and $\gamma(\mathcal{Q}) = \mathcal{P}$, for some unique $q \in \Phi^{-1}(\{p\})$.*

Proof. (a). By Theorem 2.4, there is a z-filter $\mathcal{G}$ on Y such that $\gamma(\mathcal{G}) = \mathcal{A}^p(X)$. By Theorem 3.1, γ is one-to-one and thus $\mathcal{G} = \mathcal{A}^p(Y)$, for some $q \in \beta Y$. By Theorem 2.3, $\Phi(q) = p$.

(b). By Theorem 2.4, there is a z-filter $\mathcal{Q}$ on Y such that $\gamma(\mathcal{Q}) = \mathcal{P}$. By Corollary 3.2, $\mathcal{Q}$ is a prime z-filter on Y and by Theorem 2.3, there exists $q \in \Phi^{-1}\{p\}$ such that $\mathcal{Q} \subseteq \mathcal{A}^q(Y)$. Obviously, by Theorem 3.1, $\mathcal{Q}$ is unique and so q is too. □

4. **Cozero subsets of X**

In this section, supposing $f \in C(X)$ and $Y = X \setminus Z(f)$, we obtain a one to one correspondence between the set of all (prime) z-ideals of $C(X)$ not containing f and the set of all (prime) z-ideals of $C(Y)$.

Theorem 4.1. *Let $Z_\circ \in Z(X)$, $Y = X \setminus Z_\circ$. Then,*

(a) $\mathbf{F}_Y(X)$ *is the set of all z-filters on X not containing $Z_\circ$;*
(b) γ *is invertible; more precisely, $\gamma^{-1} = \lambda$;*
(c) *Suppose that $p \in cl_{\beta X} Y$, γ is a one-to-one map from the set of all prime z-filters on Y converging to one point of $\Phi^{-1}(\{p\})$ onto the set of all prime z-filters of $\mathbf{F}_Y(X)$ converging to p;*
(d) *If $\mathcal{Q}$ is a prime z-filter on Y and there exists a prime z-filter $\mathcal{P}$ on X such that $\mathcal{P} \subseteq \gamma(\mathcal{Q})$, then $\mathcal{Q}$ belongs to the range of γ;*
(e) *A z-filter $\mathcal{Q}$ on Y is a z-ultrafilter, if and only if $\gamma(\mathcal{Q})$ is maximal in $\mathbf{F}_Y(X)$. Also, $\mathcal{Q}$ is a minimal prime z-filter if and only if $\gamma(\mathcal{Q})$ is too;*
(f) *Assuming $p \in cl_{\beta X} Y$, γ is a one-to-one map from the set of all prime I. I. z-filters on Y converging to one point of $\Phi^{-1}(\{p\})$ onto the set of all prime I. I. z-filters of $\mathbf{F}_Y(X)$ converging to p.*

Proof. (a). Suppose $\mathcal{F} \in \mathbf{F}_Y(X)$, then clearly $Z_\circ \notin \mathcal{F}$. Now, let $\mathcal{F}$ be a z-filter on X such that $Z_\circ \notin \mathcal{F}$ and $Z_1 \cap Y \subseteq Z_2 \cap Y$ in which $Z_1, Z_2 \in Z(X)$ and $Z_1 \in \mathcal{F}$, we have to show $Z_2 \in \mathcal{F}$. Since every z-filter is an intersection of prime z-filters, without loss of generality, we can suppose that $\mathcal{F}$ is a prime z-filter. Clearly, by hypothesis, $Z_1 \cup Z_\circ \subseteq Z_2 \cup Z_\circ$ and $Z_1 \cup Z_\circ \in \mathcal{F}$. Hence, $Z_2 \cup Z_\circ \in \mathcal{F}$ and consequently $Z_2 \in \mathcal{F}$.

(b) is obtained from Theorem 3.1, (c) is obtained from Theorem 2.3, and (d) and (e) are evident. To prove (f), by part (i) of Lemma 2.1, it is enough to show that if $\gamma(\mathcal{G})$ is an I. I. z-filter, then $\mathcal{G}$ is an I. I. z-filter. Suppose $\mathrm{int}_Y(Z_1 \cap Y) \subseteq \mathrm{int}_Y(Z_2 \cap Y)$ and $Z_1 \cap Y \in \mathcal{G}$. Since Y is open in X and $X \setminus Y = Z_\circ$, then

$$\begin{gathered}\mathrm{int}_X Z_1 \cap Y = \mathrm{int}_Y(Z_1 \cap Y) \subseteq \mathrm{int}_Y(Z_2 \cap Y) = \mathrm{int}_X Z_2 \cap Y \\ \Rightarrow \quad \mathrm{int}_X Z_1 \subseteq \mathrm{int}_X(Z_2 \cup Z_\circ) \quad \Rightarrow \quad Z_2 \cup Z_\circ \in \gamma(\mathcal{G}) \\ \Rightarrow \quad Z_2 \in \gamma(\mathcal{G}) \quad \Rightarrow \quad Z_2 \cap Y \in \mathcal{G}.\end{gathered}$$

□

The following is an immediate consequence of Theorems 2.5 and 4.1.

Theorem 4.2. *Suppose $f \in C(X)$, $Y = X \setminus Z(f)$. Then,*

(a) *The map ψ induces a one-to-one correspondence, invariant inclusion, between the set of prime z-ideals of $C(Y)$ and the set of prime z-ideals of $C(X)$ not containing f;*

(b) *If P is a prime z-ideal of $C(X)$ such that $f \notin P$ and $O^p(X) \subseteq P$, where $p \in cl_{\beta X} Y$, then $\psi^{-1}(P)$ is a prime z-ideal of $C(Y)$ containing $O^q(Y)$, where $\Phi(q) = p$;*

(c) *P is a prime z°-ideal in $C(Y)$ if and only if $\psi(P)$ is so in $C(X)$;*

(d) *The set of all prime z-ideals of $C(X)$ containing $O^p(X)$ and not containing f is equal to*

$$\bigcup_{\Phi(q)=p} \{\psi(Q) : \ Q \text{ is a prime ideal of } C(Y) \text{ and } O^q(Y) \subseteq Q\}$$

and $\{\psi(Q) : \ O^{q_1}(Y) \subseteq Q\} \cap \{\psi(Q) : O^{q_2}(Y) \subseteq Q\} = \emptyset$ if and only if $\Phi(q_1) \neq \Phi(q_2)$.

An Immediate conclusion of the above theorem is the following.

Corollary 4.3. *Suppose that $Y \subseteq X$ and $p \in int_X Y$. Then, there is a one-to-one correspondence between (prime) z-ideals of $C(X)$ containing $O^p(X)$ and (prime) z-ideals of $C(Y)$ containing $O^p(Y)$.*

Proposition 4.4. *Suppose $f \in C(X)$, $Y = X \setminus Z(f)$ and Q_1, Q_2 are two prime z-ideals in $C(Y)$. Then,*

(a) $Q_1 + Q_2 = C(Y)$ if and only if $f \in \psi(Q_1) + \psi(Q_2)$;

(b) If $Q_1 + Q_2 \neq C(Y)$, then $\psi(Q_1 + Q_2) = \psi(Q_1) + \psi(Q_2)$.

Proof. (a). Assume that $Q_1 + Q_2 = C(Y)$ and $f \notin \psi(Q_1) + \psi(Q_2)$. So,

$$Q_1 + Q_2 = \psi^{-1}(\psi(Q_1)) + \psi^{-1}(\psi(Q_2)) \subseteq \psi^{-1}(\psi(Q_1) + \psi(Q_2)) \neq C(Y)$$
$$\Rightarrow \quad Q_1 + Q_2 \neq C(Y)$$

which is a contradiction. Now, suppose $f \in \psi(Q_1) + \psi(Q_2)$. So, there exist $h_1 \in \psi(Q_1)$ and $h_2 \in \psi(Q_2)$ such that $f = h_1 + h_2$. Then, clearly $h_1|_Y \in Q_1$, $h_2|_Y \in Q_2$ and $Z(h_1) \cap Z(h_2) \subseteq Z(f)$. Hence,

$$Z(h_1|_Y) \cap Z(h_2|_Y) = Z(h_1) \cap Z(h_2) \cap Y \subseteq Z(f) \cap Y = \emptyset$$
$$\Rightarrow \ Z(h_1|_Y) \cap Z(h_2|_Y) = \emptyset \ \Rightarrow \ P + Q = C(Y).$$

(b). Suppose that $Q_1 + Q_2 \neq C(Y)$. It is enough to prove that $\psi(Q_1 + Q_2) \subseteq \psi(Q_1) + \psi(Q_2)$. If $g \in \psi(Q_1 + Q_2)$, then $Z(g) \cap Y \in$

$Z(Q_1 + Q_2)$ and consequently there exist $h_1 \in Q_1$ and $h_2 \in Q_2$ so that $Z(h_1) \cap Z(h_2) \subseteq Z(h_1 + h_2) = Z(g) \cap Y$. Without loss of generality, there exist $g_1 \in \psi(Q_1)$ and $g_2 \in \psi(Q_2)$ such that g_1, g_2 are nonnegative, $Z(g_1) \cap Y = Z(h_1)$ and $Z(g_2) \cap Y = Z(h_2)$. Therefore,

$$Z(g) \cap Y \supseteq Z(h_1) \cap Z(h_2) = Z(g_1) \cap Z(g_2) \cap Y = Z(g_1 + g_2) \cap Y$$

$$\Rightarrow \quad Z(g_1 + g_2) \subseteq Z(g) \cup Z(f) = Z(fg). \qquad (1)$$

Since $Q_1 + Q_2 \neq C(Y)$, by (a), $f \notin \psi(Q_1) + \psi(Q_2)$ and consequently $\psi(Q_1) + \psi(Q_2)$ is a prime z-ideal and so by (1),

$$fg \in \psi(Q_1) + \psi(Q_2) \quad \Rightarrow \quad g \in \psi(Q_1) + \psi(Q_2).$$

Hence, $\psi(Q_1 + Q_2) \subseteq \psi(Q_1) + \psi(Q_2)$. □

We conclude this section with some applications.

Corollary 4.5. *Suppose that $Z = Z(f) \in Z(X)$, $Y = X \setminus Y$ and $p \in cl_{\beta X}Y$. Every point of $\Phi^{-1}\{p\}$ is an F-point with respect to Y if and only if for every two distinct minimal prime ideals P_1, P_2 containing $O^p(X)$ and not containing f, we have $f \in P_1 + P_2$*

Proof. ($\Rightarrow$). Assume that every point of $\Phi^{-1}\{p\}$ is an F-point and P_1, P_2 are two distinct minimal prime ideal containing $O^p(X)$, then by Theorem 4.1,(c) $\psi^{-1}(P_1) \supseteq O^{q_1}(Y)$ and $\psi^{-1}(P_2) \supseteq O^{q_2}(Y)$, in which $\Phi(q_1) = \Phi(q_2) = p$, are two distinct minimal prime ideal of $C(Y)$. Since q_1 and q_2 are F-point, it follows that $q_1 \neq q_2$ and hence $\psi^{-1}(P_1) + \psi^{-1}(P_2) = C(Y)$. Therefore, by Proposition 4.4, $f \in P_1 + P_2$.

($\Leftarrow$). If $\Phi(q) = p$ and Q_1, Q_2 are minimal prime ideal containing $O^q(Y)$, then by Theorem 4.1,(c) $\psi(Q_1)$, $\psi(Q_2)$ are minimal prime ideals containing $O^p(X)$. Since $Q_1 + Q_2 \subseteq M^q(Y) \neq C(Y)$, by part (a) of Proposition 4.4, $f \notin \psi(Q_1) + \psi(Q_2)$. Hence, by hypothesis, we must have $\psi(Q_1) = \psi(Q_2)$ and so $Q_1 = Q_2$. Therefore, q is an F-point. □

An immediate conclusion of the above corollary is the following.

Corollary 4.6. *Let $p \in X$ be a non-isolated G_δ-point and $Y = X \setminus \{p\}$. Every point of $\Phi^{-1}\{p\}$ is an F-point with respect to Y if and only if every two distinct chains of prime ideals over $O^p(X)$ have one element $M^p(X)$ in common.*

Proposition 2.3 in [13] states that "Let p be a non-isolated G_δ-point of X, and let I and J be prime z-ideals contained in M_p such that $\gamma^{-1}(Z(I))$, and $\gamma^{-1}(Z(J))$ are contained in distinct z-ultrafilters on $X \setminus \{p\}$, then the chain of prime ideals of $C(X)$ containing I and the chain of prime ideals of $C(X)$ containing J have only the element M_p in common." By Proposition 4.4, the proof of this is obvious. In the following we prove that the condition is not only necessary but also sufficient.

Corollary 4.7. *Let p be a non-isolated G_δ-point of X, and let I and J be prime z-ideals contained in M_p. Then, $\gamma^{-1}(Z(I))$ and $\gamma^{-1}(Z(J))$ are contained in distinct z-ultrafilters on $X \setminus \{p\}$ if and only if the chain of prime ideals of $C(X)$ containing I and the chain of prime ideals of $C(X)$ containing J have only the element M_p in common.*

Proof. To see this, let f be such that $\{p\} = Z(f)$ and $Y = X \setminus Z(f)$. Note that the fact "two chain of prime ideals containing I and J, separately, have only the element M_p in common" is equivalent to the fact $I + J = M_p$. So, by Proposition 4.4, we can continue

$$I + J = M_p \quad \Leftrightarrow \quad \psi\psi^{-1}(I) + \psi\psi^{-1}(J) = M_p \Leftrightarrow$$

$$f \in \psi\psi^{-1}(I) + \psi\psi^{-1}(J) \quad \Leftrightarrow \quad \psi^{-1}(I) + \psi^{-1}(J) = C(Y)$$

and the latest equation is equivalent to the fact that $\gamma^{-1}Z(I)$ and $\gamma^{-1}Z(J)$ are in distinct z-ultrafilters . $\square$

Remark 4.8. *We call $p \in \beta X$ is a CC-point with respect to X, if every prime z°-ideal containing $O^p(X)$ is a minimal prime ideal. Obviously X is a CC-space if and only if every $p \in \beta X$ is a CC-point with respect to X. Therefore, $\beta\mathbb{N}$ is a CC-space whereas we know that the Stone-Čech compactification of every infinite discrete space is not quasi P-space* (*see* [9]) *and consequently $\beta\mathbb{N}$ is not quasi P-space.*

Corollary 4.9. *Suppose that $Y = X \setminus Z$ for some zero-set Z of X and $p \in cl_{\beta X} Y$. Then,*

(a) *If p is an F-point with respect to X, then $\Phi^{-1}\{p\}$ has only one point;*

(b) *If p is a quasi P-point with respect to X, then q is a quasi P-point with respect to Y, for every $q \in \Phi^{-1}\{p\}$;*

(c) *If p is a quasi P-point with respect to X and $p \in \mathrm{cl}_{\beta X} Z$, then q is a P-point, for each $q \in \Phi^{-1}\{p\}$;*
(d) *If p is CC-point with respect to X, then q is CC-point with respect to Y, for every $q \in \Phi^{-1}\{p\}$.*

Proof. (a). If $\Phi(q_1) = p = \Phi(q_2)$, then $\gamma(\mathcal{A}^{q_1}(X) \cap \mathcal{A}^{q_2}(X)) \to p$. So, similar to the proof of part "(a)⇒(b)" of Theorem 3.3, it follows that $q_1 = q_2$.

(b). Applying Theorem 4.2, it is obvious.

(c). Suppose that $Z = Z(f)$ and $p \in \mathrm{cl}_{\beta X} Z$. By part (b), $\psi(M^q(Y))$ is maximal or minimal. Since $f \in M^p(X)$, it follows that $\psi(M^q(Y)) \neq M^p(X)$ and so $\psi(M^q(Y))$ is minimal. Hence, q is a P-point.

(d). Applying Theorem 4.2, it is obvious. □

Corollary 4.10. *Let $Z \in Z(X)$ and $Y = X \setminus Z$ be a P-space.*
(a) *If P is a non minimal prime z-ideal in $C(X)$, then $Z \in Z(P)$.*
(b) *If Z is finite, then X is a quasi P-space.*

Proof. (a). If, on the contrary, $Z \notin Z(P)$, then $\psi^{-1}(P)$ is a non minimal prime ideal of $C(Y)$ which is a contradiction.

(b). Applying (a), clearly it follows. □

Corollary 4.11. *Suppose that $Z \in Z(X)$, $Y = X \setminus Z$ and $int_X Z = \emptyset$. Then, Y is CC-space if and only if X is too.*

Proof. By part (d) of Corollary 4.9, It is obvious. □

Suppose that I is an ideal of a ring $C(X)$. We call I a maximal nontrivial z-ideal, if it is maximal in the set of all nonmaximal nonminimal prime z-ideals of $C(X)$. Clearly, every maximal nontrivial z-ideal is prime. In the following we verify the existence of the maximal nontrivial z-ideals in $C(X)$.

Theorem 4.12. *Suppose p is a non-isolated G_δ-point in X and $Y = X \setminus \{p\}$. Then, the following statements are equivalent.*
(a) *p is a quasi P-point with respect to X.*
(b) *There is no maximal nontrivial prime z-ideal contained in $M^p(X)$.*
(c) *Every point of $\beta Y \setminus Y$ is a P-point with respect to Y.*

Proof. Note that since p is a G_δ-point, there exists $f \in C(X)$ such that $Z(f) = \{p\}$ and clearly the set of all prime z-ideals containing $O^p(X)$ and not containing f is equal to the set of all prime z-ideals properly contained in $M^p(X)$.

(a) $\Rightarrow$ (b). This implication is clear.

(b) $\Rightarrow$ (c). If $\Phi(q) = p$ and q is not a P-point, then by Theorem 4.2, $\psi(M^p(Y))$ is maximal in the set of nontrivial prime z-ideals contained in $M^p(X)$.

(c) $\Rightarrow$ (a). Since $\Phi^{-1}(\{p\}) = \beta Y \setminus Y$, by Theorem 4.2, it is clear. □

Assume that $\alpha\mathbb{N} = \mathbb{N} \cup \{\omega\}$ is the one-point compactification of $\mathbb{N}$, then ω is non-isolated G_δ-point and $\mathbb{N}$ is a P-space, thus ω is a quasi P-point and the set of nontrivial prime z-ideals has no maximal element. Since $\mathbb{N}$ has 2^c free maximal ideals, by Theorem 4.2, there are 2^c minimal prime z-ideals containing $O^\omega(\alpha\mathbb{N})$ (See [8, 14G]).

Corollary 4.13. *Suppose $p \in X$ is non-isolated G_δ-point, Y is a countable subset of X contained in a zero set $Z \in Z(X)$ and $p \notin Y$. If $p \in cl_X Y$ and $int_X Z = \emptyset$, then there exists a maximal nontrivial prime z-ideal contained in $M^p(X)$.*

Proof. Since Y is countable, Y is Lindelöf and therefore Y is z-embedded in X. Because $p \in \mathrm{cl}_X Y$, there is $q \in \beta Y$ such that $\Phi(q) = p$ and hence $\psi(M^q(Y))$ is a prime z-ideal containing $O^p(X)$. Since $p \notin Y$, p is a G_δ-point and $\mathrm{int}_X Z = \emptyset$, it follows that $\psi(M^q(Y))$ is neither maximal nor minimal and so p is not a quasi P-point with respect to X. Therefore, by Theorem 4.12, we are done. □

Example 4.14. *If $Y = \{\frac{1}{n} : n \in \mathbb{N}\}$, then clearly $Y \cup \{0\} \in Z(\mathbb{R})$ and $int_{\mathbb{R}}(Y \cup \{0\}) = \emptyset$. Thus, by the above corollary, there exists a maximal nontrivial prime z-ideal contained in $M^0(\mathbb{R})$, see* [8, 14I].

Recall that $x \in X$ is said to be an almost P-point, if every zero set containing x has non empty interior. We denote by $AP(X)$ the set of all almost P-points of X. One can see that $X \setminus AP(X)$ is a z-open subset of X (i.e., $\forall x \in X \setminus AP(X), \quad \exists Z \in Z(X) \quad x \in Z \subseteq X \setminus AP(X)$), the subspace $AP(X)$ is not necessarily an almost P-space but, if $AP(X)$ is dense in X, then it is an almost P-space. We call X weakly almost

P-space, if $AP(X)$ is dense in X. A point $p \in \beta X$ is called weakly CC-point with respect to X, if any prime z°-ideal contained in $M^p(X)$ is maximal or minimal prime ideal. Also X is said to be weakly CC-space, if every point of βX is a weakly CC-point with respect to X.

Proposition 4.15. *Let X be a weakly almost P-space and $X \setminus AP(X)$ be finite. If we put $Y = AP(X)$ and $A = X \setminus Y$, then*

(a) *A is a zero-set in X;*
(b) *X is a CC-space if and only if Y is too;*
(c) *Every nonmaximal prime ideal in $C(X)$ is a z-ideal if and only if it is a z°-ideal;*
(d) *X is a weakly CC-space if and only if it is a quasi P-space.*

Proof. (a). Since A is z-open, it is a finite union zero-sets and necessarily A is a zero-set.

(b). By part (a) and Corollary 4.11, it is clear.

(c). Suppose P is a nonmaximal prime z-ideal in a $C(X)$. Since A is finite, $A \notin Z(P)$ (Note that if $A \in Z(P)$, then P is maximal ideal). So, $Z(P) \in \mathbf{F}_Y(X)$ and by Theorem 4.2, $\psi^{-1}(P)$ is a z-ideal and consequently is a z°-ideal in $C(Y)$. Therefore, $\psi\psi^{-1}(P) = P$ is a z°-ideal in $C(X)$. The converse is true everywhere.

(d). It follows from the part (c). □

5. Zero subsets of X

In this section, we consider zero-subsets of X by other method. First, note that, by Theorems 1.17 and 1.18 of [8], every z-embedded zero subset of X is C-imbedded in X.

Theorem 5.1. *Let $f_\circ \in C(X)$, $Y = Z(f_\circ)$ be a z-embedded subset of X and $\varphi : C(X) \to C(Y)$ such that $\varphi(f) = f|_Y$. Then,*

(a) *φ is a one-to-one correspondence preserving inclusion between the set of (prime) z-ideals of $C(Y)$ and the set of (prime) z-ideals of $C(X)$ containing $f_\circ$;*
(b) *$\varphi^{-1}(O^p(Y)) = Z^{-1}Z(O^p(X), f_\circ)$, for each $p \in \beta Y$.*

Proof. (a). First, we note that since Y is a z-embedded zero set, it is C-embedded and hence the restriction homomorphism $\varphi : C(X) \to C(Y)$ is onto. Now, Let I' be a z-ideal of $C(Y)$, then $\varphi^{-1}(I')$ is a z-ideal of $C(X)$. Moreover, it is clear that $\varphi(f_\circ) = 0 \in I'$ and so $f_\circ \in \varphi^{-1}(I')$. Conversely, let I be a z-ideal in $C(X)$ containing $f_\circ$. Suppose $Z(\varphi(f)) \subseteq Z(\varphi(g))$ and $f \in I$. Therefore,

$$\begin{aligned} Z(f|_Y) \subseteq Z(g|_Y) \quad &\Rightarrow \quad Z(f) \cap Z(f_\circ) \subseteq Z(g) \cap Z(f_\circ) \\ &\Rightarrow \quad Z(f^2 + f_\circ^2) \subseteq Z(g^2 + f_\circ^2). \end{aligned}$$

Since $f^2 + f_\circ^2 \in I$, it follows that $g^2 + f_\circ^2 \in I$. Hence, $g \in I$ and so $\varphi(g) \in \varphi(I)$. Therefore, $\varphi(I)$ is a z-ideal. Now, let P be a prime z-ideal in $C(X)$ containing $f_\circ$. Suppose $\varphi(f)\varphi(g) \in \varphi(P)$, then

$$\varphi(f)\varphi(g) = \varphi(fg) \in \varphi(P) \quad \Rightarrow \quad \exists h \in P \quad \ni \quad \varphi(fg - h) = 0$$
$$\Rightarrow \quad (fg - h)|_Y = 0 \quad \Rightarrow \quad Z(fg) \cap Z(f_\circ) = Z(h) \cap Z(f_\circ)$$
$$\Rightarrow \quad Z(f^2g^2 + f_\circ^2) = Z(h^2 + f_\circ^2) \quad \Rightarrow \quad f^2g^2 + f_\circ^2 \in P \quad \Rightarrow \quad f^2g^2 \in P$$
$$\Rightarrow \quad f \in P \text{ or } \quad g \in P \quad \Rightarrow \quad \varphi(f) \in \varphi(P) \text{ or } \quad \varphi(g) \in \varphi(P).$$

Therefore, $\varphi(P)$ is a prime z-ideal. To prove that φ is one-to-one and preserves inclusion, it is enough to show that if I and J are two z-ideals of $C(X)$ containing $f_\circ$ and $\varphi(I) \subseteq \varphi(J)$, then $I \subseteq J$. To see this, suppose that $f \in I$, then $\varphi(f) \in \varphi(I) \subseteq \varphi(J)$ and hence

$$\exists g \in J \quad \ni \quad \varphi(f - g) = 0$$
$$\Rightarrow \quad Z(f) \cap Z(f_\circ) = Z(g) \cap Z(f_\circ) \quad \Rightarrow \quad Z(f^2 + f_\circ^2) = Z(g^2 + f_\circ^2)$$
$$\Rightarrow \quad f^2 + f_\circ^2 \in J \quad \Rightarrow \quad f^2 \in J \quad \Rightarrow \quad f \in J.$$

Therefore, $I \subseteq J$.

(b). Suppose $f \in O^p(X)$, then $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$ and hence there exists an open subset U of βX such that $p \in U \subseteq \mathrm{cl}_{\beta X}Z(f)$. Thus,

$$\begin{aligned} p \in U \cap \beta Y &\subseteq \mathrm{cl}_{\beta X}Z(f) \cap \beta Y = \mathrm{cl}_{\beta X}Z(f) \cap \mathrm{cl}_{\beta X}Z(f_\circ) \\ &= \mathrm{cl}_{\beta X}(Z(f) \cap Z(f_\circ)) = \mathrm{cl}_{\beta X}Z(f|_Y) = \mathrm{cl}_{\beta Y}Z(f|_Y) \end{aligned}$$
$$\Rightarrow \; p \in \mathrm{int}_{\beta Y}\mathrm{cl}_{\beta Y}Z(f|_Y) \; \Rightarrow \; \varphi(f) \in O^p(Y) \; \Rightarrow \; f \in \varphi^{-1}(O^p(Y)).$$

Therefore, $O^p(X) \subseteq \varphi^{-1}(O^p(Y))$ and since $\varphi^{-1}(O^p(Y))$ is a z-ideal containing $f_\circ$, it follows that

$$Z^{-1}Z(O^p(X), f_\circ) \subseteq \varphi^{-1}(O^p(Y)). \qquad (1)$$

Now, assume that $f \in \varphi^{-1}(O^p(Y)$, then $f|_Y \in O^p(Y)$ and hence $p \in \mathrm{int}_{\beta Y}\mathrm{cl}_{\beta Y}Z(f|_Y)$. So, there exists an open subset U of βX such that $p \in U \cap \beta Y \subseteq \mathrm{cl}_{\beta Y}Z(f|_Y) = \mathrm{cl}_{\beta X}(Z(f) \cap Z(f_\circ))$. Therefore, there exists $g^\beta \in O^p(\beta X)$ such that $p \in Z(g^\beta) \cap \beta Y \subseteq \mathrm{cl}_{\beta Y}Z(f|_Y)$. Thus,

$$Z(g) \cap Y \subseteq Z(f|_Y) \quad \Rightarrow \quad Z(g^2 + f_\circ^2) \subseteq Z(f|_Y) = Z(f^2 + f_\circ^2)$$
$$\Rightarrow \quad f^2 + f_\circ^2 \in Z^{-1}Z(O^p(X), f_\circ) \quad \Rightarrow \quad f \in Z^{-1}Z(O^p(X), f_\circ)$$

$$\therefore \qquad \varphi^{-1}(O^p(Y)) \subseteq Z^{-1}Z(O^p(X), f_\circ). \qquad (2)$$

The equality follows from (1) and (2). □

Corollary 5.2. *Let $Y = Z(f_\circ)$ be a z-embedded subset of X and $p \in \beta Y$, then supposing $I = Z^{-1}Z(O^p(X), f_\circ)$*

(a) *p is a P-point with respect to Y if and only if $I = M^p(X)$;*
(b) *p is an F-point with respect to Y if and only if I is a prime ideal;*
(c) *p is a quasi P-point with respect to Y if and only if for every prime z-ideal P containing I either $P \in Min(I)$ or $P = M^p(X)$.*

Proof. By Theorem 5.1, it is clear. □

Acknowledgments

We are very grateful to the referee whose valuable suggestions and comments improved an earlier version of this article.

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