

G-FRAMES AND HILBERT-SCHMIDT OPERATORS

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ABSTRACT. In this paper we introduce and study Besselian g -frames. We show that the kernel of associated synthesis operator for a Besselian g -frame is finite dimensional. We also introduce α -dual of a g -frame and we get some results when we use the Hilbert-Schmidt norm for the members of a g -frame in a finite dimensional Hilbert space.

1. Introduction

Frames for Hilbert spaces introduced by Duffin and schaeffer in 1952 [4]. A sequence $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist two positive constants A, B such that

$$(1.1) \quad A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{H}$. The numbers A, B are called *frame bounds*. Various generalizations of frames in Hilbert spaces have been proposed and studied recently. For example, frame of subspaces [2], Pseudo frames for subspaces [7], Bounded quasi-projectors [5], oblique frames [3] etc. Wen-chang Sun in his paper [12] introduced the concept of g -frames which include all mentioned generalizations. Members of ordinary frames are

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vectors of a Hilbert space, while members of g -frames are bounded operators. Besselian frames and near-Riesz bases in Hilbert spaces introduced by Holub [6]. Also, Besselian frame of subspaces introduced and discussed in [9]. The authors of this paper in [1], introduced the concept of near g -Riesz bases and they showed that a near g -Riesz basis is a Besselian g -frame.

In this paper by using the concept of Besselian frame and g -frame we define Besselian g -frame and investigate some of their properties. In section 2, we give the basic definitions and known results needed. In section 3, we investigate some properties of Besselian g -frames. In particular, we show that under some conditions, the kernel of associated synthesis operator for a Besselian g -frame is finite dimensional. In section 4, we introduce α -dual of a g -frame and we get some results when we use the Hilbert-Schmidt norm for the members of a g -frame in a finite dimensional Hilbert space.

2. Preliminaries

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_i\}_{i \in I}$ is a sequence of separable Hilbert spaces, where I is a subset of $\mathbb{N}$.

Definition 2.1. *The sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g -Bessel sequence if there exists $B > 0$ such that*

$$(2.1) \quad \sum_{i \in I} \|\Lambda_i f\|^2 \leq B \|f\|^2$$

for all $f \in \mathcal{H}$.

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be given. Let us define

$$\left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2} = \left\{ \{g_i\} : g_i \in \mathcal{H}_i, \sum_{i \in I} \|g_i\|^2 < \infty \right\}$$

with the inner product given by $\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle$. It is clear that $(\sum_{i \in I} \oplus \mathcal{H}_i)_{l_2}$ is a Hilbert space with respect to the pointwise operations. It is proved in [10], if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for $\mathcal{H}$, then the operator

$$T : \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2} \rightarrow H$$

defined by

$$(2.2) \quad T(\{g_i\}) = \sum_{i \in I} \Lambda_i^*(g_i)$$

is well defined, bounded and $T^*f = \{\Lambda_i f\}_{i \in I}$.

Definition 2.2. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a g -frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ if there exist two positive constants A and B such that

$$(2.3) \quad A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{H}$. We call A and B the lower and upper g -frame bounds, respectively.

We call $\{\Lambda_i\}_{i \in I}$ a tight g -frame if $A = B$ and Parseval g -frame if $A = B = 1$.

The sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for $\mathcal{H}$ if and only if the operator T defined by (2.2) is bounded and onto (see [10]). The operators T and T^* are called the synthesis and analysis operators, respectively.

Proposition 2.3. [12] Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for $\mathcal{H}$. The operator

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum_{i \in I} \Lambda_i^* \Lambda_i f$$

is a positive, bounded and invertible operator.

Proposition 2.3 implies that every $f \in \mathcal{H}$ can be represented as

$$(2.4) \quad f = SS^{-1}f = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1}f, \quad f = S^{-1}Sf = \sum_{i \in I} S^{-1} \Lambda_i^* \Lambda_i f.$$

The operator S is called the g -frame operator of $\{\Lambda_i\}_{i \in I}$.

It is easy to check that if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence, then S is well defined and $S = TT^*$. We end this section by definition of g -Riesz basis.

Definition 2.4. A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g -Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ if there exist two positive constants

A and B such that for any finite subset $F \subseteq I$ and $g_i \in \mathcal{H}_i$, $i \in F$,

$$(2.5) \quad A \sum_{i \in F} \|g_i\|^2 \leq \left\| \sum_{i \in F} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i \in F} \|g_i\|^2,$$

and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g -complete, i.e., $\{f : \Lambda_i f = 0, i \in I\} = 0$.

It is proved in [10], that $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g -complete if and only if $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$.

3. Besselian g -frames

As usual, we denote by $l^2(I)$ the Hilbert space of all square-summable sequences of scalars $\{c_i\}_{i \in I}$. If $\{f_i\}_{i \in I}$ is a frame for $\mathcal{H}$, then $\sum_{i \in I} c_i f_i$ converges if $\{c_i\}_{i \in I} \in l^2(I)$. But the converse is not true in general (see [6]). We say that a frame $\{f_i\}_{i \in I}$ for $\mathcal{H}$ is

- Besselian if, whenever $\sum_{i \in I} c_i f_i$ converges, then $\{c_i\}_{i \in I} \in l^2(I)$;
- a near-Riesz basis, if there is a finite set σ for which $\{f_i\}_{i \in I \setminus \sigma}$ is a Riesz basis for $\mathcal{H}$.

We recall the following characterization of frames which are near-Riesz bases.

Theorem 3.1. [6] *If $\{f_i\}_{i \in I}$ is a frame in $\mathcal{H}$, the following are equivalent:*

- (i) $\{f_i\}_{i \in I}$ is a near-Riesz basis for $\mathcal{H}$;
- (ii) $\{f_i\}_{i \in I}$ is Besselian;
- (iii) $\sum_{i \in I} c_i f_i$ converges if and only if $\{c_i\}_{i \in I} \in l^2(I)$.

Besselian frame of subspaces introduced and discussed in [9]. Here we introduce the concept of Besselian g -frames.

Definition 3.2. *Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$. We call Λ a Besselian g -frame if, whenever $\sum_{i \in I} \Lambda_i^* g_i$ converges, then*

$$\{g_i\}_{i \in I} \in \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2}.$$

Let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be given. Then $\{\Lambda_i^* e_{ij}\}_{i \in I, j \in J_i}$ is a frame (res. Riesz

basis) for $\mathcal{H}$ if and only if $\{\Lambda_i\}_{i \in I}$ is a g -frame (res. g -Riesz basis) for $\mathcal{H}$ (see [12]).

Theorem 3.3. *Suppose that $\dim \mathcal{H}_i < \infty$ for each $i \in I$. Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a Besselian g -frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ and T be the associated synthesis operator for Λ . Then $\text{Ker} T$ is finite dimensional.*

Proof. Let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. Then $\{\Lambda_i^* e_{ij}\}_{i \in I, j \in J_i}$ is a frame for $\mathcal{H}$. Suppose that $\sum_{i \in I} \sum_{j \in J_i} c_{ij} \Lambda_i^* e_{ij}$ converges. Since Λ is a Besselian g -frame, we get $\left\{ \sum_{j \in J_i} c_{ij} e_{ij} \right\}_{i \in I} \in \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2}$. So

$$\sum_{i \in I} \sum_{j \in J_i} |c_{ij}|^2 = \sum_{i \in I} \left\| \sum_{j \in J_i} c_{ij} e_{ij} \right\|^2 < \infty.$$

Hence $\{\Lambda_i^* e_{ij}\}_{i \in I, j \in J_i}$ is Besselian. Let Q be the associated synthesis operator for $\{\Lambda_i^* e_{ij}\}_{i \in I, j \in J_i}$, then $\dim \text{Ker} Q < \infty$ [6, Theorem 2.3]. Let us define $E_{ij} \in \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2}$ by

$$(3.1) \quad (E_{ij})_k = \begin{cases} e_{ij}, & i = k \\ 0, & i \neq k \end{cases}$$

for all $i, j, k \in I$. It is easy to check that $\{E_{ij}\}_{i \in I, j \in J_i}$ is an orthonormal basis for $\left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{l_2}$ (see[10]). By the definition of Q and T , it is clear that

$$Q(\{c_{ij}\}_{i \in I, j \in J_i}) = \sum_{i \in I} \sum_{j \in J_i} c_{ij} \Lambda_i^* e_{ij} = T \left(\sum_{i \in I} \sum_{j \in J_i} c_{ij} E_{ij} \right).$$

Now we consider the mapping

$$\varphi : \text{Ker} Q \rightarrow \text{Ker} T, \quad \varphi(\{c_{ij}\}_{i \in I, j \in J_i}) = \sum_{i \in I} \sum_{j \in J_i} c_{ij} E_{ij}.$$

It is obvious that φ is linear and injective. We claim that φ is surjective. Let $\{g_i\}_{i \in I} \in \text{Ker} T$. Then $g_i \in \mathcal{H}_i$ and $g_i = \sum_{j \in J_i} \lambda_{ij} e_{ij}$ for each $i \in I$. Since $\|g_i\|^2 = \sum_{j \in J_i} |\lambda_{ij}|^2$, we have $\sum_{i \in I} \sum_{j \in J_i} |\lambda_{ij}|^2 = \sum_{i \in I} \|g_i\|^2 < \infty$.

∞ . Therefore $\{\lambda_{ij}\}_{i \in I, j \in J_i} \in l^2$ and

$$\begin{aligned} Q(\{\lambda_{ij}\}_{i \in I, j \in J_i}) &= T\left(\sum_{i \in I} \sum_{j \in J_i} \lambda_{ij} E_{ij}\right) = T(\{g_i\}_{i \in I}) = 0, \\ \varphi(\{\lambda_{ij}\}_{i \in I, j \in J_i}) &= \sum_{i \in I} \sum_{j \in J_i} \lambda_{ij} E_{ij} = \{g_i\}_{i \in I}. \end{aligned}$$

Hence $\dim \text{Ker} T = \dim \text{Ker} Q < \infty$ and the proof is completed. $\square$

In the next theorem we get characterizations of generalized frames, Riesz bases, and frames.

Theorem 3.4. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be given and let $\{g_{ij}\}_{j \in K_i}$ be a frame (res. Riesz basis) for $\mathcal{H}_i$ with bounds A_i, B_i for each $i \in I$ such that $0 < A = \inf_{i \in I} A_i$ and $B = \sup_{i \in I} B_i < \infty$. Then $\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$ is a frame (res. Riesz basis) for $\mathcal{H}$ if and only if $\{\Lambda_i\}_{i \in I}$ is a g -frame (res. g -Riesz basis) for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$.*

Proof. (1) Let $f \in \mathcal{H}$ and $\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$ be a frame for $\mathcal{H}$ with bounds $0 < C \leq D$. Then

$$\begin{aligned} A \sum_{i \in I} \|\Lambda_i f\|^2 &\leq \sum_{i \in I} A_i \|\Lambda_i f\|^2 \leq \sum_{i \in I} \sum_{j \in K_i} |\langle g_{ij}, \Lambda_i f \rangle|^2 \\ &= \sum_{i \in I} \sum_{j \in K_i} |\langle \Lambda_i^* g_{ij}, f \rangle|^2 \leq D \|f\|^2 \end{aligned}$$

and

$$\begin{aligned} C \|f\|^2 &\leq \sum_{i \in I} \sum_{j \in K_i} |\langle \Lambda_i^* g_{ij}, f \rangle|^2 = \sum_{i \in I} \sum_{j \in K_i} |\langle g_{ij}, \Lambda_i f \rangle|^2 \\ &\leq \sum_{i \in I} B_i \|\Lambda_i f\|^2 \leq B \sum_{i \in I} \|\Lambda_i f\|^2. \end{aligned}$$

Hence

$$\frac{C}{B} \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq \frac{D}{A} \|f\|^2.$$

Next we assume that $\{\Lambda_i\}_{i \in I}$ is a g -frame for $\mathcal{H}$ with bounds $0 < C_0 \leq D_0$. By the same argument we have

$$AC_0 \|f\|^2 \leq \sum_{i \in I} \sum_{j \in K_i} |\langle \Lambda_i^* g_{ij}, f \rangle|^2 \leq BD_0 \|f\|^2$$

for all $f \in \mathcal{H}$.

(2) Suppose that $\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$ is a Riesz basis for $\mathcal{H}$ with Riesz basis bounds $0 < C \leq D$. Let $F \subseteq I$ be a finite subset of I and $g_i \in \mathcal{H}_i$ for each $i \in F$. Then we have $g_i = \sum_{j \in J_i} \lambda_{ij} g_{ij}$ where $\{\lambda_{ij}\} \in l^2(K_i)$. Since $\{g_{ij}\}_{j \in K_i}$ is a Riesz basis for $\mathcal{H}_i$, we have

$$(3.2) \quad \begin{aligned} A \sum_{j \in K_i} |\lambda_{ij}|^2 &\leq A_i \sum_{j \in K_i} |\lambda_{ij}|^2 \leq \left\| \sum_{j \in K_i} \lambda_{ij} g_{ij} \right\|^2 = \|g_i\|^2, \\ \left\| \sum_{j \in K_i} \lambda_{ij} g_{ij} \right\|^2 &\leq B_i \sum_{j \in K_i} |\lambda_{ij}|^2 \leq B \sum_{j \in K_i} |\lambda_{ij}|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{C}{B} \sum_{i \in F} \|g_i\|^2 &\leq C \sum_{i \in F} \sum_{j \in K_i} |\lambda_{ij}|^2 \leq \left\| \sum_{i \in F} \sum_{j \in K_i} \lambda_{ij} \Lambda_i^* g_{ij} \right\|^2 = \left\| \sum_{i \in F} \Lambda_i^* g_i \right\|^2, \\ \left\| \sum_{i \in F} \sum_{j \in K_i} \lambda_{ij} \Lambda_i^* g_{ij} \right\|^2 &\leq D \sum_{i \in F} \sum_{j \in K_i} |\lambda_{ij}|^2 \leq \frac{D}{A} \sum_{i \in F} \|g_i\|^2. \end{aligned}$$

Since $\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$ is a Riesz basis for $\mathcal{H}$, we have $\overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i} = \mathcal{H}$ and so $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$. Hence $\{\Lambda_i\}_{i \in I}$ is a g -Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$.

Conversely, let $\{\Lambda_i\}_{i \in I}$ be a g -Riesz basis for $\mathcal{H}$ with bounds $0 < C_0 \leq D_0$ and $\{c_{ij}\}$ be a finite scalar sequence. Then

$$C_0 \sum_i \left\| \sum_j c_{ij} g_{ij} \right\|^2 \leq \left\| \sum_i \Lambda_i^* \left(\sum_j c_{ij} g_{ij} \right) \right\|^2 \leq D_0 \sum_i \left\| \sum_j c_{ij} g_{ij} \right\|^2$$

and

$$A \sum_j |c_{ij}|^2 \leq A_i \sum_j |c_{ij}|^2 \leq \left\| \sum_j c_{ij} g_{ij} \right\|^2 \leq B_i \sum_j |c_{ij}|^2 \leq B \sum_j |c_{ij}|^2.$$

Hence

$$AC_0 \sum_{i,j} |c_{ij}|^2 \leq \left\| \sum_{i,j} c_{ij} \Lambda_i^* g_{ij} \right\|^2 \leq BD_0 \sum_{i,j} |c_{ij}|^2.$$

Moreover, we have $\mathcal{H} = \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$. So $\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$ is a Riesz basis for $\mathcal{H}$.

For any sequence $\{\mathcal{H}_i\}_{i \in I}$ of Hilbert spaces, we can find a Hilbert space $\mathcal{K}$ to contain all the $\mathcal{H}_i$ by setting $\mathcal{K} = (\sum_{i \in I} \oplus \mathcal{H}_i)_{l_2}$. $\square$

Proposition 3.5. *Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ and $E \subseteq I$ such that*

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j, \quad i, j \in E.$$

Then $f = \sum_{i \in E} \Lambda_i^ \Lambda_i f$ for all $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in E}$.*

Proof. First of all, the series $\sum_{i \in E} \Lambda_i^* \Lambda_i f$ are convergent for all $f \in \mathcal{H}$. To see this, let J be a finite subset of E . Then

$$\left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\|^2 = \sum_{i \in J} \|\Lambda_i f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2$$

for all $f \in \mathcal{H}$. Since $\{\Lambda_i\}_{i \in I}$ is a g -frame for $\mathcal{H}$, we get $\sum_{i \in E} \Lambda_i^* \Lambda_i f$ converges. Let $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in E}$, then $f = \sum_{i \in E} \Lambda_i^* g_i$ where $g_i \in \mathcal{H}_i$ and the set $\{i \in E : \Lambda_i^* g_i \neq 0\}$ is finite. We show that $g_i = \Lambda_i f$ for $i \in E$. Let $h \in \mathcal{H}_i$, then

$$\begin{aligned} \langle \Lambda_i f, h \rangle &= \left\langle \sum_{k \in E} \Lambda_i \Lambda_k^* g_k, h \right\rangle = \sum_{k \in E} \langle \Lambda_k^* g_k, \Lambda_i^* h \rangle \\ &= \langle \Lambda_i^* g_i, \Lambda_i^* h \rangle = \langle g_i, h \rangle. \end{aligned}$$

So $g_i = \Lambda_i f$ for $i \in E$ and $f = \sum_{i \in E} \Lambda_i^* \Lambda_i f$.

For the case $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in E}$, there exists a sequence $\{f_n\}$ in $\text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in E}$ such that $f_n \rightarrow f$ as $n \rightarrow \infty$. Let B be the upper g -frame bound for Λ . We have

$$\begin{aligned} \left\| \sum_{i \in E} \Lambda_i^* \Lambda_i f_n - \sum_{i \in E} \Lambda_i^* \Lambda_i f \right\|^2 &= \left\| \sum_{i \in E} \Lambda_i^* \Lambda_i (f_n - f) \right\|^2 \\ &= \sum_{i \in E} \|\Lambda_i (f_n - f)\|^2 \\ &\leq B \|f_n - f\|^2 \rightarrow 0. \end{aligned}$$

Hence $f = \sum_{i \in E} \Lambda_i^* \Lambda_i f$. □

Definition 3.6. *A g -frame $\{\Lambda_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ is called a g -Riesz frame if for every $J \subseteq I$, $\{\Lambda_i\}_{i \in J}$ is a g -frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in J}$ with uniform g -frame bounds A, B .*

Proposition 3.7. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for $\mathcal{H}$ with bounds $0 < C \leq D$ such that*

$$(3.3) \quad \langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j, \quad i, j \in I.$$

Then Λ is a g -Riesz frame with bounds 1 and D . Moreover, if $\{g_{ij}\}_{j \in K_i}$ is a Riesz frame for $\mathcal{H}_i$ with bounds A_i, B_i for each $i \in I$ and $0 < A = \inf_{i \in I} A_i$, $B = \sup_{i \in I} B_i < \infty$, then $\{\Lambda_i^* g_{ij}\}_{i \in I, j \in K_i}$ is a Riesz frame for $\mathcal{H}$.

Proof. Let $E \subseteq I$ and $W = \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in E}$. By Proposition 3.5 we have

$$\|f\|^2 = \left\| \sum_{i \in E} \Lambda_i^* \Lambda_i f \right\|^2 = \sum_{i \in E} \|\Lambda_i f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq D \|f\|^2$$

for all $f \in W$. Now we assume that $\{g_{ij}\}_{j \in K_i}$ is a Riesz frame for $\mathcal{H}_i$ and $I_0 \subseteq I$. We show that $\{\Lambda_i^* g_{ij}\}_{i \in I_0, j \in K_i^1}$ is a frame for $\overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i \in I_0, j \in K_i^1}$ with uniform frame bounds A and BD , where $K_i^1 \subseteq K_i$ for each $i \in I_0$. Let $f \in \text{span}\{\Lambda_i^* g_{ij}\}_{i \in I_0, j \in K_i^1}$ and $k \in I_0$. Then there is a finite scalar sequence $\{c_{ij}\}$ such that $\Lambda_k f = \sum_{i,j} c_{ij} \Lambda_k \Lambda_i^* g_{ij}$. It follows from (3.3) that

$$\Lambda_l f = \sum_j c_{lj} \Lambda_l \Lambda_l^* g_{lj} = \sum_j c_{lj} g_{lj}.$$

Therefore $\Lambda_l f \in \overline{\text{span}}\{g_{lj}\}_{j \in K_l^1}$ for all $f \in \overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i \in I_0, j \in K_i^1}$ and all $l \in I_0$. Since $\{\Lambda_i\}_{i \in I}$ is a g -Riesz frame we have

$$(3.4) \quad \|f\|^2 \leq \sum_{i \in I_0} \|\Lambda_i f\|^2 \leq D \|f\|^2$$

for all $f \in \overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i \in I_0, j \in K_i^1}$. Also $\{g_{ij}\}_{j \in K_i}$ is a Riesz frame for $\mathcal{H}_i$ so

$$(3.5) \quad A_i \|\Lambda_i f\|^2 \leq \sum_{j \in K_i^1} |\langle \Lambda_i f, g_{ij} \rangle|^2 \leq B_i \|\Lambda_i f\|^2$$

for all $f \in \mathcal{H}$. Therefore (3.4) and (3.5) imply that

$$\begin{aligned} A \|f\|^2 &\leq \sum_{i \in I_0} A_i \|\Lambda_i f\|^2 \\ &\leq \sum_{i \in I_0} \sum_{j \in K_i^1} |\langle f, \Lambda_i^* g_{ij} \rangle|^2 \leq \sum_{i \in I_0} B_i \|\Lambda_i f\|^2 \leq BD \|f\|^2 \end{aligned}$$

for all $f \in \overline{\text{span}}\{\Lambda_i^* g_{ij}\}_{i \in I_0, j \in K_i^1}$

□

4. α -dual of g -frames and Hilbert-Schmidt operators

In this section $\mathcal{H}$ denotes a finite dimensional Hilbert space. We also denote the norm of a Hilbert-Schmidt operator T by $\|T\|_2$.

Definition 4.1. Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be g -frames for $\mathcal{H}$. We say that $\{\Theta_i\}_{i \in I}$ is a dual g -frame (or simply dual) of $\{\Lambda_i\}_{i \in I}$ if

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i f$$

holds for all $f \in \mathcal{H}$.

It is easy to show that if $\{\Theta_i\}_{i \in I}$ is a dual g -frame of $\{\Lambda_i\}_{i \in I}$, then $\{\Lambda_i\}_{i \in I}$ will be a dual g -frame of $\{\Theta_i\}_{i \in I}$.

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for $\mathcal{H}$ with g -frame operator S . Then (2.4) shows that $\{\Lambda_i S^{-1}\}_{i \in I}$ is a dual g -frame of $\{\Lambda_i\}_{i \in I}$. $\{\Lambda_i S^{-1}\}_{i \in I}$ is called *canonical dual* g -frame of $\{\Lambda_i\}_{i \in I}$.

Proposition 4.2. Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a dual of g -frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$. Then

$$\sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i \in I} \|\Lambda_i\|_2^2 + \sum_{i \in I} \|\Theta_i\|_2^2 - 2 \dim \mathcal{H}.$$

Proof. Suppose that $\{e_n\}_{n=1}^M$ is an orthonormal basis for $\mathcal{H}$. We have

$$\begin{aligned} \sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 &= \sum_{i \in I} \sum_n \|(\Lambda_i - \Theta_i)e_n\|^2 \\ &= \sum_n \sum_{i \in I} \langle (\Lambda_i - \Theta_i)e_n, (\Lambda_i - \Theta_i)e_n \rangle \\ &= \sum_{i \in I} \sum_n \|\Lambda_i e_n\|^2 + \sum_{i \in I} \sum_n \|\Theta_i e_n\|^2 \\ &\quad - \sum_n \sum_{i \in I} \langle \Lambda_i^* \Theta_i e_n, e_n \rangle - \sum_n \sum_{i \in I} \langle e_n, \Lambda_i^* \Theta_i e_n \rangle \\ &= \sum_{i \in I} \|\Lambda_i\|_2^2 + \sum_{i \in I} \|\Theta_i\|_2^2 - 2 \dim \mathcal{H}. \end{aligned}$$

□

Corollary 4.3. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two Parseval g -frames for $\mathcal{H}$. If $\{\Theta_i\}_{i \in I}$ is a dual of $\{\Lambda_i\}_{i \in I}$, then $\Lambda_i = \Theta_i$ for all $i \in I$.*

Proof. Since $\mathcal{H}$ is a finite dimensional and $\{\Lambda_i\}_{i \in I}$ is a Parseval g -frame for $\mathcal{H}$, we have $\sum_{i \in I} \|\Lambda_i\|_2^2 = \dim \mathcal{H}$ (see [11]). Hence the result follows by Proposition 4.2. $\square$

Corollary 4.4. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a dual of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$, where $\{\Theta_i\}_{i \in I}$ and $\{\Lambda_i\}_{i \in I}$ are two tight g -frames for $\mathcal{H}$ with bounds B_Θ and B_Λ , respectively. Then $B_\Theta + B_\Lambda = 2$ if and only if $\Lambda_i = \Theta_i$ for all $i \in I$.*

Proof. Since $\sum_{i \in I} \|\Lambda_i\|_2^2 = B_\Lambda \dim \mathcal{H}$ and $\sum_{i \in I} \|\Theta_i\|_2^2 = B_\Theta \dim \mathcal{H}$, the result follows from Proposition 4.2. $\square$

Remark 4.5. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g -frames for $\mathcal{H}$ with the associated synthesis operators T_Λ and T_Θ , respectively. Using the proof of Proposition 4.2, we get*

$$\sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i \in I} \|\Lambda_i\|_2^2 + \sum_{i \in I} \|\Theta_i\|_2^2 - \text{tr}(T_\Theta T_\Lambda^*) - \text{tr}(T_\Lambda T_\Theta^*).$$

Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for $\mathcal{H}$ with the frame operator S . It is clear that $\{\Lambda_i S^{\alpha-1}\}_{i \in I}$ is a g -frame for $\mathcal{H}$ with the property $\sum_{i \in I} \Lambda_i^* \Lambda_i S^{\alpha-1} = S^\alpha f$ for all $f \in \mathcal{H}$. For $\alpha = 0$ we get the canonical dual g -frame of $\{\Lambda_i\}_{i \in I}$.

Definition 4.6. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for $\mathcal{H}$ with g -frame operator S_Λ . A g -frame $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a α -dual of $\{\Lambda_i\}_{i \in I}$ if $\sum_{i \in I} \Lambda_i^* \Theta_i = S_\Lambda^\alpha f$ for all $f \in \mathcal{H}$.*

It is clear that $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i \in I}$ is a α -dual of $\{\Lambda_i\}_{i \in I}$. The canonical dual g -frame of $\{\Lambda_i\}_{i \in I}$ has some interesting properties between other dual g -frames of $\{\Lambda_i\}_{i \in I}$ (see [12]). We will show that the α -dual frame $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i \in I}$ has some minimal properties between other α -dual frames of $\{\Lambda_i\}_{i \in I}$.

Proposition 4.7. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a α -dual of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ with g -frame operator S_Θ . Then*

$$(4.1) \quad \sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \|S_\Lambda^{\frac{2\alpha-1}{2}}\|_2^2 \leq \|S_\Theta^{\frac{1}{2}}\|_2^2 = \sum_{i \in I} \|\Theta_i\|_2^2.$$

The equality in (4.1) holds if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ for all $i \in I$.

Proof. Let $\{e_n\}_{n=1}^M$ be an orthonormal basis for $\mathcal{H}$. We have

$$\begin{aligned}
 \sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 &= \sum_{i \in I} \sum_n \langle \Lambda_i S_\Lambda^{\alpha-1} e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle \\
 &= \sum_{i \in I} \sum_n \langle \Lambda_i^* \Lambda_i S_\Lambda^{\alpha-1} e_n, S_\Lambda^{\alpha-1} e_n \rangle \\
 &= \sum_n \langle e_n, S_\Lambda^{2\alpha-1} e_n \rangle \\
 &= \sum_n \langle S_\Lambda^{\frac{2\alpha-1}{2}} e_n, S_\Lambda^{\frac{2\alpha-1}{2}} e_n \rangle = \|S_\Lambda^{\frac{2\alpha-1}{2}}\|_2^2, \\
 \sum_{i \in I} \|\Theta_i\|_2^2 &= \sum_{i \in I} \sum_n \langle \Theta_i^* \Theta_i e_n, e_n \rangle = \sum_n \langle S_\Theta e_n, e_n \rangle = \|S_\Theta^{\frac{1}{2}}\|_2^2.
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 (4.2) \quad \sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 &= \sum_{i \in I} \sum_n \langle \Lambda_i S_\Lambda^{\alpha-1} e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle \\
 &= \sum_n \langle S_\Lambda^\alpha e_n, S_\Lambda^{\alpha-1} e_n \rangle \\
 &= \sum_n \sum_{i \in I} \langle \Lambda_i^* \Theta_i e_n, S_\Lambda^{\alpha-1} e_n \rangle \\
 &= \sum_n \sum_{i \in I} \langle \Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle \\
 &\leq \left(\sum_n \sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1} e_n\|^2 \right)^{\frac{1}{2}} \left(\sum_n \sum_{i \in I} \|\Theta_i e_n\|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

So $\sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 \leq \sum_{i \in I} \|\Theta_i\|_2^2$ and we obtain (4.1).

If $\sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \sum_{i \in I} \|\Theta_i\|_2^2$, then it follows from (4.2) that

$$\begin{aligned} \sum_n \sum_{i \in I} \langle \Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle &= \sum_n \sum_{i \in I} |\langle \Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle| \\ &= \left(\sum_n \sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1} e_n\|^2 \right)^{\frac{1}{2}} \\ &\quad \left(\sum_n \sum_{i \in I} \|\Theta_i e_n\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

So $\langle \Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle \geq 0$ and $\langle \Theta_i e_n, \Lambda_i S_\Lambda^{\alpha-1} e_n \rangle = \|\Theta_i e_n\| \|\Lambda_i S_\Lambda^{\alpha-1} e_n\|$ for all i, n . Therefore there exist $\lambda, \lambda_{i,n} \geq 0$ such that

$$\Lambda_i S_\Lambda^{\alpha-1} e_n = \lambda_{i,n} \Theta_i e_n, \quad \|\Lambda_i S_\Lambda^{\alpha-1} e_n\| = \lambda \|\Theta_i e_n\|$$

for all i, n . Hence $\lambda_{i,n} = \lambda$ and we conclude that $\Lambda_i S_\Lambda^{\alpha-1} e_n = \lambda \Theta_i e_n$ for all i, n . Since $\sum_{i \in I} \|\Lambda_i S_\Lambda^{\alpha-1}\|_2^2 = \sum_{i \in I} \|\Theta_i\|_2^2$, we get $\lambda = 1$ and so $\Lambda_i S_\Lambda^{\alpha-1} e_n = \Theta_i e_n$ for all i, n . $\square$

Corollary 4.8. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for $\mathcal{H}$ with g -frame operator S_Λ . Then*

$$\begin{aligned} \sum_{i \in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2 &= \\ \min \left\{ \sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 : \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\} \text{ is a } \alpha\text{-dual of } \{\Lambda_i\}_{i \in I} \right\}. \end{aligned}$$

Moreover, if $\{\Theta_i\}_{i \in I}$ is a α -dual of $\{\Lambda_i\}_{i \in I}$, then $\sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i \in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ for all $i \in I$.

Proof. Since

$$\sum_{i \in I} \sum_n \langle \Theta_i e_n, \Lambda_i e_n \rangle = \sum_n \langle S_\Lambda^\alpha e_n, e_n \rangle = \sum_{i \in I} \sum_n \langle \Lambda_i S_\Lambda^{\alpha-1} e_n, \Lambda_i e_n \rangle,$$

by Proposition 4.7 we have

$$\begin{aligned}
\sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 &= \sum_{i \in I} \sum_n (\|\Lambda_i e_n\|^2 + \|\Theta_i e_n\|^2 - 2\Re\langle \Theta_i e_n, \Lambda_i e_n \rangle) \\
&\geq \sum_{i \in I} \sum_n (\|\Lambda_i e_n\|^2 + \|\Lambda_i S_\Lambda^{\alpha-1} e_n\|^2 - 2\Re\langle \Lambda_i S_\Lambda^{\alpha-1} e_n, \Lambda_i e_n \rangle) \\
&= \sum_{i \in I} \sum_n \|(\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}) e_n\|^2 \\
&= \sum_{i \in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2.
\end{aligned}$$

Therefore the above inequality implies that $\sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i \in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\sum_{i \in I} \sum_n \|\Lambda_i S_\Lambda^{\alpha-1} e_n\|^2 = \sum_{i \in I} \sum_n \|\Theta_i e_n\|^2$. Hence by Proposition 4.7, $\sum_{i \in I} \|\Lambda_i - \Theta_i\|_2^2 = \sum_{i \in I} \|\Lambda_i - \Lambda_i S_\Lambda^{\alpha-1}\|_2^2$ if and only if $\Theta_i = \Lambda_i S_\Lambda^{\alpha-1}$ for all $i \in I$. $\square$

Corollary 4.9. *Let $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a dual of g -frame $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$. Then*

$$(4.3) \quad \sum_{i \in I} \|\Theta_i\|_2^2 = \|S_\Theta^{\frac{1}{2}}\|_2^2 \geq \|S_\Lambda^{-\frac{1}{2}}\|_2^2 = \sum_{i \in I} \|\Lambda_i S_\Lambda^{-1}\|_2^2$$

where S_Λ and S_Θ are the g -frame operators of $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$, respectively. Moreover, the following are equivalent

- (i) $\sum_{i \in I} \|\Theta_i\|_2^2 = \sum_{i \in I} \|\Lambda_i S_\Lambda^{-1}\|_2^2$;
- (ii) $\Theta_i = \Lambda_i S_\Lambda^{-1}$ for all $i \in I$;
- (iii) $S_\Theta = S_\Lambda^{-1}$.

Proposition 4.10. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame and $\{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a Parseval g -frame for $\mathcal{H}$. Then*

- (i) $\text{tr}(T_\Theta T_\Lambda^*) + \text{tr}(T_\Lambda T_\Theta^*) \leq 2\|S_\Lambda^{\frac{1}{4}}\|_2^2$;
- (ii) $\text{tr}(T_\Theta T_\Lambda^*) + \text{tr}(T_\Lambda T_\Theta^*) = 2\|S_\Lambda^{\frac{1}{4}}\|_2^2$ if and only if $\Theta_i = \Lambda_i S_\Lambda^{-\frac{1}{2}}$ for all $i \in I$.

Proof. By Remark 4.5, $\text{tr}(T_\Theta T_\Lambda^*) + \text{tr}(T_\Lambda T_\Theta^*)$ is real. Since $\{\Theta_i\}_{i \in I}$ is a Parseval g -frame for $\mathcal{H}$, we have $\|T_\Theta\| = \|T_\Theta^*\| = 1$. Let us denote the trace-class norm by $\|\cdot\|_1$. By a simple computation we get $\|T_\Lambda^*\|_1 = \|S_\Lambda^{\frac{1}{4}}\|_2^2$. By [8; Theorems 2.4.14 and 2.4.16], we get

$$|\text{tr}(T_\Theta T_\Lambda^*)|, |\text{tr}(T_\Lambda T_\Theta^*)| \leq \|S_\Lambda^{\frac{1}{4}}\|_2^2.$$

Therefore (i) is proved. To prove (ii), let $\text{tr}(\mathbf{T}_\Theta \mathbf{T}_\Lambda^*) + \text{tr}(\mathbf{T}_\Lambda \mathbf{T}_\Theta^*) = 2\|\mathbf{S}_\Lambda^{\frac{1}{4}}\|_2^2$. It follows from (i) that $\text{tr}(\mathbf{T}_\Theta \mathbf{T}_\Lambda^*) = \|\mathbf{S}_\Lambda^{\frac{1}{4}}\|_2^2$. Hence we get the result by Corollary 2.6 of [11]. $\square$

Corollary 4.11. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for $\mathcal{H}$ with g -frame operator S_Λ . Then*

$$\max \{ \Re \text{tr}(\mathbf{T}_\Theta \mathbf{T}_\Lambda^*) : \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\} \text{ is a Parseval } g\text{-frame for } \mathcal{H} \} \\ = \|S_\Lambda^{\frac{1}{4}}\|_2^2.$$

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REFERENCES

- [1] M.R. Abdollahpour and A. Najati, Besselian g -frames and near g -Riesz bases, *Appl. Anal. Discrete Math* **5** (2011) 259-270.
- [2] P.G. Casazza and G. Kutyniok, Frames of subspaces, *Contemp. Math.* **345** (2004) 87-113.
- [3] O. Christensen and Y.C. Eldar, Oblique dual frames and shift invariant-spaces, *Appl. Comput. Harmon. Anal.* **17** (2004) 48-68.
- [4] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952) 341-366.
- [5] M. Fornasier, Decompositions of Hilbert space: local construction of global frames, *Proc. Int. Conf. On constructive function theory, varna (2002)*, B. Bojanov Ed., DARBA, Sofia, (2003) 275-281.
- [6] J. Holub, Pre-frame operators, Besselian frames and near-Riesz bases, *Proc. Amer. Math. Soc.* **122** (1994) 779-785.
- [7] S. Li and H. Ogawa, Pseudoframes for subspaces with applications, *J. Fourier Anal. Appl.* **10** (2004) 409-431.
- [8] Gerard J. Murphy, C^* -algebras and operator Theory, Academic Press, Inc, 1990.
- [9] A. Najati, A. Rahimi and M.H. Faroughi, Continuous and discrete frames of subspaces in Hilbert spaces, *Southeast Asian Bull. Math.* **32** (2008) 305-324.
- [10] A. Najati, M.H. Faroughi, and A. Rahimi, G -frames and stability of g -frames in Hilbert spaces, *Methods Funct. Anal. Topology* **14** (2008) 271-286.
- [11] A. Najati, Approximation of g -frames in Hilbert spaces, *Afr. Diaspora J. Math.* **6** (2008) 137-145.
- [12] W. Sun, G -frames and g -Riesz bases, *J. Math. Anal. Appl.* **322** (2006) 437-452.

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