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# BEST PROXIMITY PAIR AND COINCIDENCE POINT THEOREMS FOR NONEXPANSIVE SET-VALUED MAPS IN HILBERT SPACES

A. AMINI-HARANDI

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Abstract. This paper is concerned with the best proximity pair problem in Hilbert spaces. Given two subsets $A$ and $B$ of a Hilbert space $H$ and the set-valued maps $F : A \to 2^B$ and $G : A_0 \to 2^{A_0}$, where $A_0 = \{x \in A : \|x - y\| = d(A, B)$ for some $y \in B\}$, best proximity pair theorems provide sufficient conditions that ensure the existence of an $x_0 \in A$ such that

$$d(G(x_0), F(x_0)) = d(A, B).$$

## 1. Introduction

Let $(M, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $M$. Let $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Let

$$B_0 := \{b \in B \ : \ d(a, b) = d(A, B) \text{ for some } a \in A\},$$

and

$$A_0 := \{a \in A \ : \ d(a, b) = d(A, B) \text{ for some } b \in B\}.$$

Let $G : A_0 \to 2^{A_0}$ and $F : A \to 2^B$ be set valued maps. $(G(x_0), F(x_0))$ is called a *best proximity pair*, if $d(G(x_0), F(x_0)) = d(A, B)$. Best proximity pair theorems analyse the conditions on $F$, $G$, $A$ and $B$ under which the problem of minimizing the real valued function $x \to d(G(x), F(x))$

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has a solution. In the setting of normed spaces and hyperconvex metric spaces, the best proximity pair problem has been studied by many authors, see [1, 2, 3, 5, 7, 8].

Let $H$ be a Hilbert space and $A, B \subseteq H$. It is well-know that if $A$ and $B$ are compact subsets of $M$, then there exist $a_0 \in A$ and $b_0 \in B$ such that $d(A, B) = d(a_0, b_0)$. Therefore, in this case

$$d(A, B) = 0 \Leftrightarrow A \cap B \neq \emptyset.$$

Let $M$ be a metric space and let $\mathcal{M}$ denotes the family of nonempty, closed bounded subsets of $M$. Let $A, B \in \mathcal{M}$. The Hausdorff metric $d_H$ on $\mathcal{M}$ defined by

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\},$$

where $N_\epsilon(A)$ denotes the closed $\epsilon$-neighborhood of $A$, that is, $N_\epsilon(A) = \{x \in M : d(x, A) \leq \epsilon\}$. Let $X$ and $Y$ be topological spaces with $C \subseteq Y$. Let $G : X \to 2^Y$ be a set-valued map with nonempty values. The inverse image of $C$ under $G$ is

$$G^-(C) = \{x \in X : G(x) \cap B \neq \emptyset\}.$$

A set-valued map $F : A \to 2^B$ is said to be *nonexpansive*, if for each $x, y \in A$

$$d_H(F(x), F(y)) \leq \|x - y\|.$$

Given a nonempty closed convex subset $A$ of a Hilbert space $H$, $P_A$ will always denote the nearest point projection of $H$ onto $A$. We will use the well-known fact that $P_A$ is nonexpansive and so is continuous.

**Lemma 1.1.** ([*5, Lemma 3.1*]) *Let $A$ be a nonempty closed convex subset of a Hilbert space $H$. If $C$ and $D$ are nonempty closed and bounded subsets of $H$, then*

$$d_H(P_A(C), P_A(D)) \leq d_H(C, D).$$

Let $(X, \|.\|)$ be a reflexive Banach space and $A \subseteq X$ be nonempty, closed, convex and bounded. It is well-known that for each $x \in X$, $P_A(x) \neq \emptyset$. Here we give the proof for the completeness. For each $n \in \mathbb{N}$, let $A_n(x) = \{y \in A : d(x, y) \leq d(x, A) + \frac{1}{n}\}$. Notice that $(A_n(x))$ is a decreasing sequence of nonempty closed, convex bounded subsets of the reflexive Banach space $X$, so by Šmulina theorem we have [4, page 433]

$$P_A(x) = \bigcap_{n=1}^{\infty} A_n(x) \neq \emptyset.$$

**Lemma 1.2.** ([5, *Lemma 3.2*]) *Let $X$ be a reflexive Banach space. Let $A$ be a nonempty bounded closed convex subset of $X$, and let $B$ be a nonempty closed convex subset of $X$. Then, $A_0$ and $B_0$ are nonempty and satisfy*

$$P_B(A_0) \subseteq B_0 \text{ and } P_A(B_0) \subseteq A_0.$$

Recall that a Banach space $X$ is *uniformly convex*, if given $\epsilon > 0$ there is a $\delta > 0$ such that whenever $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

**Theorem 1.3.** ([*6*]) *Let $X$ be a uniformly convex Banach space, Let $K$ be a bounded, closed and convex subset of $X$, and suppose $F : K \to 2^K$ is a compact-valued, nonexpansive set-valued map. Then, $F$ has a fixed point.*

## 2. Main results

We first present a coincidence point theorem for nonexpansive set-valued self maps.

**Theorem 2.1.** *Let $H$ be a Hilbert space and $K$ be a closed, bounded convex subset of $H$. Let $F : K \to 2^K$ be a nonexpansive set-valued map with nonempty compact values. Let $G : K \to 2^K$ be an onto, set-valued map for which $G^-(C)$ is compact for each compact set $C \subseteq K$. Assume that for each compact subsets $C$ and $D$ of $K$*

$$d_H(G^-(C), G^-(D)) \leq d_H(C, D).$$

*Then, there exists a $x_0 \in K$ with*

$$F(x_0) \cap G(x_0) \neq \emptyset.$$

*Proof.* Since

$$F(x_0) \cap G(x_0) \neq \emptyset \Leftrightarrow x_0 \in G^-(F(x_0)) = \{x \in H : G(x) \cap F(x_0) \neq \emptyset\}),$$

then, the conclusion follows, if we show that the set-valued map $J(x) = G^-(F(x)) : K \to 2^K$ has a fixed point. Since $G$ is onto, then $J(x) \neq \emptyset$. For each $x \in K$, since $F(x)$ is compact, then $J(x) = G^-(F(x))$ is compact. Now, we show that $J$ is nonexpansive. For each $x, y \in K$ we have

$$d_H(J(x), J(y)) = d_H(G^-(F(x)), G^-(F(y))) \leq$$
$$d_H(F(x), F(y)) \leq \|x - y\|.$$

Therefore, $J$ satisfies all conditions of Theorem 1.3 and so has a fixed point. □

Now, we obtain a best proximity pair theorem for nonexpansive set-valued maps in Hilbert spaces.

**Theorem 2.2.** *Let $H$ be a Hilbert space. Let $A$ be a nonempty bounded closed convex subset of $H$, and let $B$ be a nonempty closed convex subset of $H$. Let $F : A \to 2^B$ be a nonexpansive set-valued map with nonempty compact values. Let $G : A_0 \to 2^{A_0}$ be an onto set-valued map for which $G^-(C)$ is compact for each compact set $C \subseteq A_0$. Assume that for each compact subsets $C$ and $D$ of $A_0$*

$$d_H(G^-(C), G^-(D)) \leq d_H(C, D).$$

*Assume that $F(A_0) \subseteq B_0$. Then, there exists a $x_0 \in A_0$ such that*

$$d(G(x_0), F(x_0)) = d(A, B).$$

*Proof.* By Lemma 1.2, $A_0$ and $B_0$ are nonempty. Let us show that $A_0$ is closed. To this end, let $x_n \in A_0$ be a convergent sequence, say, $x_n \to x_0 \in A$. Then, for each $n \in \mathbb{N}$, there exists $y_n \in B$ such that $d(x_n, y_n) = d(A, B)$. Thus, $\{y_n\}$ is a bounded sequence in $B$ (note that $\{x_n\}$ is bounded). Since bounded subsets of a reflexive Banach space are weakly sequentially compact [4, Theorem 28, page 68], then passing to a subsequence, if necessary, we may assume that $(y_n)$ converges weakly, say to $y_0 \in B$. Since $\|.\|$ is weakly lower semicontinuous, then we get

$$\|x_0 - y_0\| \leq \lim_{n\to\infty} \|x_n - y_n\| = d(A, B).$$

Therefore, $\|x_0 - y_0\| = d(A, B)$, and so $x_0 \in A_0$. From Lemma 1.2, $P_A(B_0) \subseteq A_0$ and by Lemma 1.1,

$$d_H(P_A(F(x)), P_A(F(y))) \leq d_H(F(x), F(y)) \leq \|x - y\|.$$

Then, the map $P_A(F(.)) : A_0 \to A_0$ is a nonxpansive set-valued map. Moreover, $A_0$ is a nonempty closed bounded convex subsets of $H$, and for each $x \in A_0$, $P_A(F(x))$ is a compact subset of $A_0$ (note $F(x)$ is compact and $P_A$ is continuous). Hence, by Theorem 2.1 there exists a $x_0 \in A_0$ such that

$$P_A(F(x_0)) \cap G(x_0) \neq \emptyset.$$

Let $z_0 \in P_A(F(x_0)) \cap G(x_0)$, then there exists $y_0 \in F(x_0)$ so that $z_0 = P_{A_0}(y_0)$. Since $x_0 \in A_0$ and $y_0 \in F(x_0) \subseteq B_0$, there exists $a_0 \in A_0$ such that $d(a_0, y_0) = d(A, B)$. Therefore,

$$d(A,B) \leq d(G(x_0), F(x_0)) \leq d(z_0, F(x_0)) \leq$$

$$d(P_{A_0}(y_0), y_0)) \leq d(a_0, y_0) = d(A,B)$$

Thus,

$$d(G(x_0), F(x_0)) = d(A,B).$$

□

**Remark 2.3.** *Let $A$ be a nonempty bounded, closed convex subset of $H$. Let $G : A_0 \to A_0$ be an onto isometry. We show that $G$ satisfies all the conditions of Theorem 2.2. Let $C$ be a compact subset of $A_0$. Since $G$ is an isometry, then $G^-(C) = G^{-1}(C)$ is compact (note $G^{-1}$ is isometry and so is continuous). Since $G : A_0 \to A_0$ is isometry, then $d_H(G^-(C), G^-(D)) = d_H(C, D)$, for compact subsets $C$ and $D$ of $A_0$.*

If we take $G = I$, Theorem 2.2 reduces to Theorem 3.3 of Kirk, Reich and Veeramani [5].

**Theorem 2.4.** *Let $H$ be a Hilbert space. Let $A$ be a nonempty bounded closed convex subset of $H$, and let $B$ be a nonempty closed convex subset of $H$. Let $F : A \to 2^B$ be a nonexpansive set-valued map with nonempty compact values. Assume that $F(A_0) \subseteq B_0$. Then, there exists a $x_0 \in A_0$ such that*

$$d(x_0, F(x_0)) = d(A,B).$$

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**A. Amini-Harandi**

Department of Mathematics, University of Shahrekord, P.O. Box 115, Shahrekord, Iran

and

School of Mathematics, Institute for Research in Fundamental Sciences (IPM),

P.O. Box 19395-5746, Tehran, Iran

Email: aminih_a@yahoo.com