Bulletin of the Iranian Mathematical Society Vol. 35 No. 2 (2009), pp 191-208.

QUENCHING TIME FOR A NONLOCAL DIFFUSION PROBLEM WITH LARGE INITIAL DATA

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Communicated by Mohammad Asadzadeh

ABSTRACT. We are concerned with the study of the following nonlocal diffusion problem,

$$\begin{cases} u_t = J * u - u + f(u) \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{in} \quad (\mathbb{R}^N - \Omega) \times (0,T), \\ u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $J * u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$, $J : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a kernel which is nonnegative, symmetric ($J(z) = J(-z)$), bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$ and $f : (-\infty, b) \rightarrow (0,\infty)$ is a C^1 convex, increasing function, $\lim_{s\to b} f(s) = \infty$, $\int_0^b \frac{ds}{f(s)} < \infty$ with b a positive constant. The initial datum $u_0 \in C^1(\overline{\Omega})$ is nonnegative in Ω, with $\|u_0\|_\infty = \sup_{x\in\Omega} |u_0(x)| < b$. Under some assumptions, we show that if $\|u_0\|_\infty$ is large enough, then the solution of the above problem quenches in a finite time, and its quenching time goes to that of the solution of the differential equation,

$$\alpha^{'}(t) = f(\alpha(t)), \quad t > 0, \quad \alpha(0) = \|u_0\|_\infty,$$

as $\|u_0\|_\infty$ tends to b. Finally, we give some numerical results to illustrate our analysis.

MSC(2000): Primary: 35B40, 45A07; Secondary: 35G10.
Keywords: Nonlocal diffusions, asymptotic behavior, quenching, numerical quenching time.
Received: 05 June 2008, Accepted: 10 December 2008.
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1. **Introduction**

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem,

$$u_t = J * u - u + f(u) \quad \text{in} \quad \Omega \times (0, T), \tag{1.1}$$

$$u = 0 \quad \text{in} \quad (\mathbb{R}^N - \Omega) \times (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

where $J * u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$, $J : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a kernel which is nonnegative, symmetric $(J(z) = J(-z))$, bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$ and $f : (-\infty, b) \rightarrow (0, \infty)$ is a C^1 convex, increasing function, $\lim_{s\to b} f(s) = \infty$, $\int_0^b \frac{ds}{f(s)} < \infty$ with b a positive constant. The initial datum $u_0 \in C^1(\overline{\Omega})$ is nonnegative in Ω and $\|u_0\|_\infty = \sup_{x\in\Omega} |u_0(x)| < b$. Here, $(0, T)$ is the maximal time interval on which the solution u of (1.1)–(1.3) exists. The time T may be finite or infinite. If T is infinite, then we say that the solution u exists globally. If T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} \|u(\cdot, t)\|_\infty = b,$$

where $\|u(\cdot,t)\|_\infty = \max_{x\in\Omega} |u(x,t)|$. In the latter case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Recently, nonlocal diffusion problems have been the subject of investigations for many authors (see [3]–[8], [11]–[16], [19]–[21], [23], [24], [35], [38], and the references cited therein). Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy,$$

and variations of it, have been used by several authors to model diffusion processes (see [5], [11], [19], [20]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point x, at the time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then, the convolution $J * u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate at which they are leaving location x to travel to any other site (see [19]). Solutions of nonlinear parabolic equations (local diffusion) which quench in a finite time have been widely studied by many authors (see [9], [17], [18], [31], and the references cited therein). Here, we are interested in the asymptotic behavior of the quenching time when the L^∞ norm of the initial datum is large enough for the nonlocal diffusion problem described by (1.1)–(1.3). Note that the determination of the quenching time is an interesting question, and theoretically, it is not possible to obtain this time. In the case of one dimensional space, if one constructs suitable schemes, it is possible to obtain a good approximation of the quenching time (see for instance, [34]). In [34], Nabongo and Boni showed that the quenching time approaches that of the solution of a semidiscrete scheme when the mesh size goes to zero. It is worth noting that the quenching time of the semidiscrete scheme is the same as that of the solution of a differential system and can be determined easily using standard methods. In the case of several dimensional spaces, the problem is generally more complicated because of the geometry of the domain. However, in certain cases, it is possible to approach the quenching time by that of the solution of a certain differential equation. Our work was motivated by the paper of Friedman and Lacey [22], where they considered the following initial-boundary value problem,

$$u_t = \varepsilon \Delta u + g(u) \quad \text{in} \quad \Omega \times (0,T),$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega,$$

where Δ is the Laplacian, $g : [0,\infty) \to (0,\infty)$ is a C^1 convex, increasing function, with $\int_0^\infty \frac{ds}{g(s)} < \infty$. The initial datum u_0 is a positive and continuous function in Ω. Under some additional conditions on the initial datum, they proved that if ε was small enough, then the solution u of the above problem blew up in a finite time, and its blow-up time went to that of the solution of the following differential equation,

$$\alpha'(t) = g(\alpha(t)), \quad t > 0, \quad \alpha(0) = \|u_0\|_\infty, \tag{1.4}$$

as ε tended to zero (we say that a solution blows up in a finite time if it attains the value infinity in a finite time). Comparable studies can also be found in [26] and [27]. In [26], $\|u_0\|_\infty$ is taken as a parameter. In this case, it is shown that the blow-up time tends to that of the solution of a certain differential equation when $\|u_0\|_\infty$ tends to infinity. In the same way, in [33], Nabongo and Boni considered the initial-boundary value problem below,

$$u_t = \varepsilon Lu + f(u) \quad \text{in} \quad \Omega \times (0, T),$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega,$$

where the initial datum $u_0 \in C^1(\overline{\Omega})$ is nonnegative in Ω, L is an elliptic operator and f is the same function given in the introduction of the paper. They showed that when ε was small enough, then the solution of the above problem quenched in a finite time, and its quenching time went to that of the solution of the following differential equation,

$$\alpha'(t) = f(\alpha(t)), \quad t > 0, \quad \alpha(0) = \|u_0\|_\infty, \tag{1.5}$$

as ε approached zero. To prove this result in the case where the initial datum is not null, the idea consists of showing that the solution of the above problem quenches in a small neighborhood where the initial datum attains its maximum. To reveal that quenching occurs, the authors used the method of Kaplan (see, [29]) where the eigenvalue introduced takes large values. Here, we establish a similar result taking as a parameter the L^∞ norm of the initial datum. More precisely, in the case with $f(s) = (b-s)^{-p}$ and p a positive constant, we show that, if $b - \|u_0\|_\infty < \min\{1, (2b^p)^{-1/p}\}$, then the solution u of (1.1)-(1.3) quenches in a finite time T obeying the following estimates,

$$0 \leq T - T_{u_0} \leq (\tfrac{b2^p}{p+1} + 1)(b - \|u_0\|_\infty)^{2p+1} + o((b - \|u_0\|_\infty)^{2p+1}), \tag{1.6}$$

where $T_{u_0} = \frac{(b-\|u_0\|_\infty)^{p+1}}{p+1}$ is the quenching time of the solution $\alpha(t)$ of the differential equation defined as follows:

$$\alpha'(t) = (b - \alpha(t))^{-p}, \quad t > 0, \quad \alpha(0) = \|u_0\|_\infty. \tag{1.7}$$

To prove this result, we take the advantage that it is possible to modify the method of Kaplan and to adapt this modified method to our problem showing that the solution quenches in a small neighborhood. Contrary to the method used by Nabongo and Boni in [33], here, we use a bounded eigenvalue which allows us to obtain the estimates given in (1.6) for any positive constant p. It is worth mentioning that it is not possible to obtain the estimates as those given by (1.6) in the case of local problems, that is, if one considers for instance the following initial-boundary problem,

$$u_t = Lu + (b - u)^{-p} \quad \text{in} \quad \Omega \times (0, T), \tag{1.8}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.9}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \tag{1.10}$$

In fact, when p takes small positive values, it is not possible to control the effect of the values of the first eigenvalue of the operator L which takes large values in a small neighborhood. Some numerical experiments in the last section have confirmed this assertion. The remainder of the paper is organized as follows. In Section 2, we prove the local existence and uniqueness of the solution of (1.1)–(1.3). We also give some results about the maximum principle for nonlocal problems. In Section 3, under some conditions, we show that the solution u of (1.1)-(1.3) quenches in a finite time, and its quenching time goes to that of the solution of the differential equation defined by (1.7) as $\|u_0\|_\infty$ tends to b. Finally, in Section 4, we give some numerical results to illustrate our analysis.

2. **Local existence**

Here, we establish the existence and uniqueness of the solution of (1.1)–(1.3) in $\Omega \times (0, T)$ for small T. We also state some results about the maximum principle for nonlocal problems to be used subsequently. Let $t_0 > 0$ be fixed, and define the function space $Y_{t_0} = \{u\ ; u \in C([0, t_0], C(\overline{\Omega}))\}$ equipped with the norm defined by $\|u\|_{Y_{t_0}} = \max_{0 \le t \le t_0} \|u(\cdot, t)\|_\infty$, for $u \in Y_{t_0}$. It is easy to see that Y_{t_0} is a Banach space. Introduce the set $X_{t_0} = \{u\ ; u \in Y_{t_0}, \|u\|_{Y_{t_0}} \le b_0\}$, where $b_0 = \frac{\|u_0\|_\infty + b}{2}$. We observe that X_{t_0} is a nonempty bounded closed convex subset of Y_{t_0}. Define the map R as follows:

$$R : X_{t_0} \longrightarrow X_{t_0},$$

$$R(v(x,t)) = u_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)(v(y,s) - v(x,s))dyds + \int_0^t f(v(x,s))ds,$$

where,

$$v(x, t) = 0, \quad \text{for} \quad x \in \mathbb{R}^N - \Omega.$$

We have the following result.

Theorem 2.1. *Assume $u_0 \in C(\overline{\Omega})$. Then, R maps X_{t_0} into X_{t_0}, and R is strictly contractive if t_0 is appropriately small relative to $\|u_0\|_\infty$.*

Proof. We get

$$|R(v(x,t)) - u_0(x)| \leq 2\|v\|_{Y_{t_0}} t + f(\|v\|_{Y_{t_0}})t,$$

which implies $\|R(v)\|_{Y_{t_0}} \leq \|u_0\|_\infty + 2b_0 t_0 + f(b_0)t_0$. If

$$t_0 \leq \frac{b_0 - \|u_0\|_\infty}{2b_0 + f(b_0)}, \tag{2.1}$$

then

$$\|R(v)\|_{Y_{t_0}} \leq b_0.$$

Therefore, if (2.1) holds, then R maps X_{t_0} into X_{t_0}. Now, we prove that the map R is strictly contractive. Let $v, z \in X_{t_0}$. Setting $\alpha = v - z$, we discover,

$$\begin{aligned} |(R(v) - R(z))(x,t)| \leq{}& |\int_0^t \int_{\mathbb{R}^N} J(x-y)(\alpha(y,s) - \alpha(x,s))dyds| \\ &+ |\int_0^t (f(v(x,s)) - f(z(x,s)))ds|. \end{aligned}$$

Use Taylor's expansion to obtain:

$$|(R(v) - R(z))(x,t)| \leq 2\|\alpha\|_{Y_{t_0}} t + t\|v - z\|_{Y_{t_0}} f^{'}(\|\beta\|_{Y_{t_0}}),$$

where β is a value between v and z. We deduce,

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq 2\|\alpha\|_{Y_{t_0}} t_0 + t_0\|v - z\|_{X_{t_0}} f^{'}(\|\beta\|_{Y_{t_0}}),$$

which implies:

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq \left(2t_0 + t_0 f^{'}(b_0)\right) \|v - z\|_{Y_{t_0}}.$$

If $t_0 \leq \frac{1}{4+2f'(b_0)}$, then $\|R(v) - R(z)\|_{Y_{t_0}} \leq \frac{1}{2}\|v - z\|_{Y_{t_0}}$. Hence, we see that $R(v)$ is a strict contraction in Y_{t_0} and the proof is complete. $\square$

It follows from the contraction mapping principle that for appropriately chosen t_0, R has a unique fixed point $u \in Y_{t_0}$, which is a solution of (1.1)–(1.3).
If $\|u\|_{Y_{t_0}} < b$, then taking as initial datum $u(\cdot, t_0) \in C(\overline{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0, t_1)$, for certain $t_1 > t_0$.
Now, let us give some results about the maximum principle for nonlocal problems. The following lemma is a version of the maximum principle for nonlocal problems.

Lemma 2.2. *Let $a \in C^0(\overline{\Omega} \times [0,T))$ and let $u \in C^{0,1}(\overline{\Omega} \times [0,T))$ satisfy the following inequalities,*

$$u_t - \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy + a(x,t)u(x,t) \geq 0 \quad in \quad \Omega \times (0,T),$$

$$u(x,t) \geq 0 \quad in \quad (\mathbb{R}^N - \Omega) \times (0,T),$$

$$u(x,0) \geq 0 \quad in \quad \Omega.$$

Then, we have $u(x,t) \geq 0$ in $\overline{\Omega} \times (0,T)$.

Proof. Let $T_0 < T$. Since $a(x,t)$ is bounded in $\overline{\Omega} \times [0,T_0]$, let λ be such that $a(x,t) - \lambda > 0$ in $\overline{\Omega} \times [0,T_0]$. Introduce the function $z(x,t) = e^{\lambda t}u(x,t)$ and let $m = \min_{x \in \overline{\Omega}, t \in [0,T_0]} z(x,t)$. Then, there exists $(x_0,t_0) \in \overline{\Omega} \times [0,T_0]$ such that $m = z(x_0,t_0)$. If $x_0 \in \mathbb{R}^N - \Omega$, then $m \geq 0$. If $x_0 \in \Omega$, then we get $z(x_0,t_0) \leq z(x_0,t)$ for $t \leq t_0$ and $z(x_0,t_0) \leq z(y,t_0)$, for $y \in \Omega$, which implies:

$$z_t(x_0,t_0) \leq 0, \tag{2.2}$$

and

$$\int_{\mathbb{R}^N} J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy \geq 0. \tag{2.3}$$

Using the first inequality of the lemma, it is not hard to see:

$$z_t(x_0,t_0) - \int_{\mathbb{R}^N} J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy$$
$$+(a(x_0,t_0)-\lambda)z(x_0,t_0) \geq 0.$$

It follows from (2.2) and (2.3) that $(a(x_0,t_0)-\lambda)z(x_0,t_0) \geq 0$, which implies $z(x_0,t_0) \geq 0$, because $a(x_0,t_0) - \lambda > 0$. We deduce $u(x,t) \geq 0$ in $\overline{\Omega} \times [0,T_0]$, which leads us to the result. □

A direct consequence of the above result is the following comparison lemma.

Lemma 2.3. *Let $u, v \in C^{0,1}(\overline{\Omega} \times [0,T))$ be such that*

$$u_t - \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy - f(u(x,t)) \geq v_t$$
$$- \int_{\mathbb{R}^N} J(x-y)(v(y,t)-v(x,t))dy - f(v(x,t)) \quad in \quad \Omega \times (0,T),$$

$$u(x,t) \geq v(x,t) \quad in \quad (\mathbb{R}^N - \Omega) \times (0,T),$$

$$u(x,0) \geq v(x,0) \quad in \quad \Omega.$$

Then, we have $u(x,t) \geq v(x,t)$ in $\overline{\Omega} \times (0,T)$.

Proof. Let $z(x,t) = u(x,t) - v(x,t)$ in $\mathbb{R}^N \times [0,T)$. Applying the mean value theorem, a routine computation reveals,

$$z_t - \int_{\mathbb{R}^N} J(x-y)(z(y,t) - z(x,t))dy + f^{'}(\xi(x,t))z(x,t) \geq 0$$

$$\text{in} \quad \Omega \times (0,T),$$

$$z(x,t) \geq 0 \quad \text{in} \quad (\mathbb{R}^N - \Omega) \times (0,T),$$

$$z(x,0) \geq 0 \quad \text{in} \quad \Omega,$$

where $\xi(x,t)$ is an intermediate value between $u(x,t)$ and $v(x,t)$. Use Lemma 2.2 to complete the rest of the proof. □

3. Quenching times

Here, we suppose that $f(s) = (b-s)^{-p}$ with p a positive constant. Under some hypotheses, we show that if the L^∞ norm of the initial datum is large enough, then the solution u of (1.1)-(1.3) quenches in a finite time, and its quenching time goes to that of the solution of the differential equation defined by (1.7) as $\|u_0\|_\infty$ tends to b.
Now, let us state our result on the quenching time.

Theorem 3.1. *If $b - \|u_0\|_\infty < \min\{1, (b2^p)^{-1/p}\}$, then the solution u of (1.1)-(1.3) quenches in a finite time T which obeys the following estimates,*

$$0 \leq T - T_{u_0} \leq \left(\frac{b2^p}{p+1} + 1\right)(b - \|u_0\|_\infty)^{2p+1} + o((b - \|u_0\|_\infty)^{2p+1}),$$

where $T_{u_0} = \frac{(b-\|u_0\|_\infty)^{p+1}}{p+1}$ is the quenching time of the solution $\alpha(t)$ of the differential equation defined by (1.7).

Proof. Since $(0, T)$ is the maximal time interval on which u exists, then our goal is to prove that T is finite and obeys the above inequalities. Let $a \in \Omega$ be such that $u_0(a) = \|u_0\|_\infty$. Due to the fact that $u_0 \in C^1(\overline{\Omega})$, using the mean value theorem and the triangle inequality, there exists $\delta > 0$ such that

$$u_0(x) \geq \|u_0\|_\infty - (b - \|u_0\|_\infty)^{p+1} \quad \text{for} \quad x \in B(a, \delta) \subset \Omega, \tag{3.1}$$

where $B(a, \delta) = \{x \in \mathbb{R}^N;\ \|x - a\| < \delta\}$. Here, $\|.\|$ stands for the Euclidean norm in $\mathbb{R}^N$. Consider the following eigenvalue problem,

$$\int_{\mathbb{R}^N} J(x-y)(\varphi(y) - \varphi(x))dy = -\lambda_\delta \varphi(x) \quad \text{in} \quad B(a, \delta), \tag{3.2}$$

$$\varphi(x) = 0 \quad \text{in} \quad \mathbb{R}^N - B(a, \delta), \tag{3.3}$$

$$\varphi(x) > 0 \quad \text{in} \quad B(a, \delta). \tag{3.4}$$

We know that the above problem admits a solution $(\varphi, \lambda_\delta)$ such that $0 < \lambda_\delta < 1$ (see [23] and [24]). We can normalize φ so that $\int_{\mathbb{R}^N} \varphi(x)dx = 1$. Let w be the solution of the following initial-boundary value problem,

$$w_t - \int_{\mathbb{R}^N} J(x-y)(w(y,t) - w(x,t))dy - f(w(x,t)) = 0$$

$$\text{in} \quad B(a, \delta) \times (0, T^*), \tag{3.5}$$

$$w(x,t) = 0 \quad \text{in} \quad (\mathbb{R}^N - B(a, \delta)) \times (0, T^*), \tag{3.6}$$

$$w(x, 0) = u_0(x) \quad \text{in} \quad B(a, \delta), \tag{3.7}$$

where $(0, T^*)$ is the maximal time interval of existence of w. Since the initial datum $u_0(x)$ is nonnegative in $B(a, \delta)$, we deduce from Lemma 2.2 that w is also nonnegative in $\overline{B(a, \delta)} \times (0, T^*)$. Introduce the function $v(t)$ by

$$v(t) = \int_{\mathbb{R}^N} \varphi(x) w(x,t) dx \quad \text{for} \quad t \in [0, T^*).$$

Take the derivative of v in t and use (3.5) to obtain:

$$v'(t) = \int_{\mathbb{R}^N} \varphi(x) \left(\int_{\mathbb{R}^N} J(x-y) w(y,t) dy \right) dx - v(t)$$
$$+ \int_{\mathbb{R}^N} f(w(x,t)) \varphi(x) dx.$$

From Fubini's theorem, we have,

$$\begin{aligned}&\int_{\mathbb{R}^N}\varphi(x)\left(\int_{\mathbb{R}^N}J(x-y)w(y,t)dy\right)dx\\ &=\int_{\mathbb{R}^N}w(y,t)\left(\int_{\mathbb{R}^N}J(x-y)\varphi(x)dx\right)dy.\end{aligned}$$

Since the kernel J is symmetric, then we observe that the term on the right hand side of the above equality is $\int_{\mathbb{R}^N}w(y,t)\left(\int_{\mathbb{R}^N}J(y-x)\varphi(x)dx\right)dy$. It follows from (3.2) that

$$\int_{\mathbb{R}^N}\varphi(x)\left(\int_{\mathbb{R}^N}J(x-y)w(y,t)dy\right)dx=\int_{\mathbb{R}^N}w(y,t)(\varphi(y)-\lambda_\delta\varphi(y))dy,$$

which implies:

$$\int_{\mathbb{R}^N}\varphi(x)\left(\int_{\mathbb{R}^N}J(x-y)w(y,t)dy\right)dx=v(t)-\lambda_\delta v(t).$$

Hence, we deduce:

$$v'(t)=-\lambda_\delta v(t)+\int_{\mathbb{R}^N}f(w(x,t))\varphi(x)dx \quad \text{for} \quad t\in(0,T^*).$$

Due to Jensen's inequality, we arrive at

$$v'(t)\geq-\lambda_\delta v(t)+f(v(t)) \quad \text{for} \quad t\in(0,T^*).$$

Obviously, we have,

$$v'(t)\geq(b-v(t))^{-p}\left(1-b(b-v(t))^p\right) \quad \text{for} \quad t\in(0,T^*),$$

because $0<\lambda_\delta<1$ and $0\leq v(t)\leq b$, for $t\in(0,T^*)$. Since $v(0)\geq\|u_0\|_\infty-(b-\|u_0\|_\infty)^{p+1}$, then we see:

$$b-v(0)\leq b-\|u_0\|_\infty+(b-\|u_0\|_\infty)^{p+1}\leq 2(b-\|u_0\|_\infty),$$

which implies $1-b(b-v(0))^p\geq 1-b2^p(b-\|u_0\|_\infty)^p>0$. Therefore, we have,

$$v'(0)\geq(b-v(0))^{-p}(1-b(b-v(0))^p)>0. \tag{3.8}$$

We deduce that $v'(t)>0$, for $t\in(0,T^*)$. In fact, suppose that there exists $t_0\in(0,T^*)$ such that $v'(t)>0$, for $t\in(0,t_0)$, but $v'(t_0)=0$. We observe that $v(t_0)\geq v(0)$. Hence, we have,

$$0=v'(t_0)\geq(b-v(t_0))^{-p}\left(1-b(b-v(0))^p\right)>0,$$

which is a contradiction. Consequently, we get

$$v'(t)\geq(b-v(t))^{-p}\left(1-b(b-v(0))^p\right) \quad \text{for} \quad t\in(0,T^*),$$

because $v(t) \geq v(0)$, for $t \in (0, T^*)$. Due to the fact that $(b - v(0))^p \leq 2^p(b - \|u_0\|_\infty)^p$, we arrive at

$$v'(t) \geq (b - v(t))^{-p} \left(1 - b2^p(b - \|u_0\|_\infty)^p\right) \quad \text{for} \quad t \in (0, T^*).$$

This estimate may be rewritten in the following manner,

$$(b - v)^p dv \geq \left(1 - b2^p(b - \|u_0\|_\infty)^p\right) dt \quad \text{for} \quad t \in (0, T^*).$$

Integrate the above inequality over $(0, T^*)$ to obtain:

$$\frac{(b - v(0))^{p+1}}{p+1} \geq \left(1 - b2^p(b - \|u_0\|_\infty)^p\right) T^*,$$

which implies:

$$T^* \leq \frac{(b - v(0))^{p+1}}{(p+1)(1 - b2^p(b - \|u_0\|_\infty)^p)}.$$

Since $v(0) \geq \|u_0\|_\infty - (b - \|u_0\|_\infty)^{p+1}$, then we get

$$T^* \leq \frac{(b - \|u_0\|_\infty + (b - \|u_0\|_\infty)^{p+1})^{p+1}}{(p+1)(1 - b2^p(b - \|u_0\|_\infty)^p)}.$$

We deduce that w quenches in a finite time because the quantity on the right hand side of the above inequality is finite. On the other hand, since the initial datum u_0 is nonnegative in Ω, from Lemma 2.2 we know that u is also nonnegative in $\overline{\Omega} \times (0, T)$, which implies:

$$u_t - \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy - (b - u(x,t))^p$$

$$\geq w_t - \int_{\mathbb{R}^N} J(x-y)(w(y,t) - w(x,t))dy$$

$$-(b - w(x,t))^p \quad \text{in} \quad B(a, \delta) \times (0, T_*),$$

$$u(x,t) \geq w(x,t) \quad \text{in} \quad (\mathbb{R}^N - B(a,\delta)) \times (0, T_*),$$

$$u(x,0) \geq w(x,0) \quad \text{in} \quad B(a,\delta),$$

where $T_* = \min\{T, T^*\}$. We deduce from Lemma 2.3 that

$$u(x,t) \geq w(x,t) \quad \text{in} \quad \overline{B(a,\delta)} \times (0, T_*).$$

This implies that $T \leq T^*$. Indeed, suppose that $T > T^*$. We find,

$$\|u(\cdot, T^*)\|_\infty \geq \|w(\cdot, T^*)\|_\infty = b,$$

which is a contradiction because $(0,T)$ is the maximal interval time of existence of the solution u. We conclude:

$$T \leq T^* \leq \frac{(b-\|u_0\|_\infty + (b-\|u_0\|_\infty)^{p+1})^{p+1}}{(p+1)(1-b2^p(b-\|u_0\|_\infty)^p)}. \tag{3.9}$$

On the other hand, setting

$$z(x,t) = \alpha(t) \quad \text{in} \quad \mathbb{R}^N \times [0,T_*),$$

it is not hard to see:

$$z_t - \int_{\mathbb{R}^N} J(x-y)(z(y,t)-z(x,t))dy - f(z(x,t)) = 0 \quad \text{in} \quad \Omega \times (0,T_{u_0}),$$

$$z(x,t) \geq 0 \quad \text{in} \quad (\mathbb{R}^N - \Omega) \times (0,T_{u_0}),$$

$$z(x,0) \geq u(x,0) \quad \text{in} \quad \Omega.$$

Lemma 2.3 implies:

$$0 \leq u(x,t) \leq z(x,t) = \alpha(t) \quad \text{in} \quad \overline{\Omega} \times (0,T_*^*),$$

where $T_*^* = \min\{T, T_{u_0}\}$. It follows that $T \geq T_{u_0}$. Indeed, suppose that $T < T_{u_0}$, which implies:

$$\|u(\cdot,T)\|_\infty \leq \alpha(T) < b.$$

But, this is a contradiction because $(0,T)$ is the maximal time interval of existence of the solution u. Hence, we have,

$$T \geq T_{u_0} = \frac{(b-\|u_0\|_\infty)^{p+1}}{(p+1)}. \tag{3.10}$$

Apply Taylor's expansion to obtain:

$$(b-\|u_0\|_\infty + (b-\|u_0\|_\infty)^{p+1})^{p+1}$$

$$= (b-\|u_0\|_\infty)^{p+1} + (p+1)(b-\|u_0\|_\infty)^{2p+1} + o((b-\|u_0\|_\infty)^{2p+1}),$$

and

$$\frac{1}{1-b2^p(b-\|u_0\|_\infty)^p} = 1 + b2^p(b-\|u_0\|_\infty)^p + o((b-\|u_0\|_\infty)^p).$$

Use (3.9), (3.10) and the above equalities to complete the rest of the proof. $\square$

Remark 3.2. If Theorem 3.1 holds, then we observe:

$$0 \leq \frac{T}{T_{u_0}} - 1 \leq (b2^p + p + 1)(b - \|u_0\|_\infty)^p + o((b - \|u_0\|_\infty)^p).$$

We deduce that $\frac{T}{T_{u_0}}$ goes to one when $\|u_0\|_\infty$ goes to b.

4. **Numerical results**

Here, we give some computational results to confirm the theory established in the previous section. We consider the problem (1.1)–(1.3) in the case where $\Omega = (-3, 3)$, $f(u) = (1 - u)^{-p}$, $u_0(x) = a\sin(\pi x)$ with $a \in (0, 1)$, $p > 0$,

$$J(x) = \begin{cases} \frac{1}{4} & \text{if} \quad |x| < 2, \\ 0 & \text{if} \quad |x| \geq 2. \end{cases}$$

We start by the construction of an adaptive scheme as follows. Let I be a positive integer and $h = \frac{3}{I}$. Define the grid $x_i = ih$, $-I \leq i \leq I$, and approximate the solution $u(x, t)$ of the problem (1.1)–(1.3) by the solution $U_h^{(n)} = (U_{-I}^{(n)}, \ldots, U_I^{(n)})^T$ of the following discrete equations,

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=-I}^{I} hJ(x_i - x_j)(U_j^{(n)} - U_i^{(n)})$$

$$+(1 - U_i^{(n)})^{-p}, \quad -(I - 1) \leq i \leq I - 1,$$

$$U_{-I}^{(n)} = 0, \quad U_I^{(n)} = 0,$$

$$U_i^{(0)} = a\sin(\pi x_i), \quad -I \leq i \leq I,$$

where $n \geq 0$. It is easy to check that the condition of stability of the above scheme is given by $\Delta t_n \leq \frac{1}{(6+h)\|J\|_\infty}$. Due to the fact that the kernel J is bounded, one sees that this condition is not restrictive in comparison with the condition of stability of local parabolic problems given generally by $\Delta t_n \leq \frac{h^2}{2}$ (see [33] and [34]). In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the step so that we take

$$\Delta t_n = \min\{\frac{4}{6 + h}, h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}\},$$

where $\|U_h^{(n)}\|_\infty = \sup_{-I\leq i\leq I}|U_i^{(n)}|$. Note that the restriction on the time step ensures the nonnegativity of the discrete solution.
It is important to point out that the first condition on the time step is obtained by noting that $\|J\|_\infty = \frac{1}{4}$. Due to this condition, which is not restrictive, one may weaken the factor h^2 on the second condition taking h^α with α a nonnegative constant. The use of this kind of restriction may provoke instabilities in the case of local diffusion problems.
We need the following definition.

Definition 4.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme quenches in a finite time if $\lim_{n\to\infty}\|U_h^{(n)}\|_\infty = 1$ and the series $\sum_{n=0}^{\infty}\Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty}\Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of size 16, 32, 64, 128. We take, for the numerical quenching time, $t_n = \sum_{j=0}^{n-1}\Delta t_j$, which is computed at the first time when

$$|t_{n+1} - t_n| \leq 10^{-16}.$$

The order (s) of the method is computed by

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical results for $a = 0.9$, $p = 1$ are given in Table 1.

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained by the explicit Euler method.

I	t_n	n	CPU time	s
16	0.005044	1648	52	-
32	0.005033	5890	733	-
64	0.005023	10014	4189	0.14
128	0.005014	122120	11356	0.15

Numerical results for $a = 0.95$, $p = 1$ are given in Table 2.

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained by the explicit Euler method.

I	t_n	n	CPU time	s
16	0.001257	1470	47	-
32	0.001255	5176	582	-
64	0.001254	16206	4114	0.99
128	0.001253	58582	11273	0.01

Remark 4.2. If we consider the problem as defined by (1.1)–(1.3) in the case with the initial datum $u_0(x) = 0.9\sin(\pi x)$ and the reaction term $f(u) = (1-u)^{-1}$, it is not hard to see that the quenching time of the solution of the differential equation defined by (1.7) equals 0.005. We observe from Table 1 that the numerical quenching time is approximately 0.005. with the initial datum $u_0(x) = 0.95\sin(x\pi)$ and the reaction term $f(u) = (1-u)^{-1}$, we find that the quenching time of the solution of the differential equation defined by (1.7) equals 0.00125. We discover from Table 2 that the numerical quenching time is approximately 0.00125. These results have been proved in Theorem 3.1.

Remark 4.3. Consider the following initial-boundary value problem,

$$u_t = u_{xx} + (1-u)^{-p} \quad \text{in} \quad (-3,3)\times(0,T),$$

$$u(-3,t) = 0, \quad u(3,t) = 0, \quad t \in (0,T),$$

$$u(x,0) = a\sin(\pi x), \quad x \in (-3,3),$$

where $a = 0.95 > 0$. The discretization of the above problem leads us to the scheme below,

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + (1 - U_i^{(n)})^{-p},$$

$$, -(I-1) \le i \le I-1,$$

$$U_{-I}^{(n)} = 0, \quad U_I^{(n)} = 0,$$

$$U_i^{(0)} = a\sin(\pi i h), \quad -I \le i \le I,$$

where $\Delta t_n = \min\{\frac{h^2}{2}, h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}\}$.
We find the following results as shown in Table 3.

Table 3: Values of the exponent, Numerical quenching times, Numerical quenching times for ODE defined in (1.7) .

p	Numerical quenching time	Quenching time of the ODE in (1.7)
1	2.3780e-3	1.2500e-3
2	4.2516e-5	4.1666e-5
3	1.5647e-6	1.5625e-6
4	6.2510e-8	6.2500e-8

In view of the above results, we note that when $p = 1$ or 2, the quenching time of the above problem does not approach that of the solution of the differential equation defined by (1.7).

Acknowledgments

The authors thank the anonymous referee for the through reading of the manuscript and giving several suggestions that helped us improve the presentation of the paper.

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