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MODIFIED HALPERN ITERATION OF ASYMPTOTICALLY NON-EXPANSIVE MAPPINGS

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Abstract. For an asymptotically non-expansive self-mapping T, we will prove the strong convergence of $\{x_n\}$ defined by $x_{n+1} = (1-\alpha_n-\beta_n)x_n+\alpha_n u+\beta_n T^n x_n,\ x_{n+1} = \alpha_n u+(1-\alpha_n)T^n x_n$, whenever $\{\beta_n\}, \{\alpha_n\} \subset (0, 1)$ satisfy (C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, (C2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, (C3) $\lim\limits_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$ or (C4) $\sum\limits_{n=0}^{\infty} (k_n - 1) < +\infty$. As an application, we also establish the strong convergence of the viscosity approximation schemes with a contraction f given by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n x_n$$

and

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(x_n) + \beta_n T^n x_n.$$

1. Introduction

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^*\rangle$. Let $F(T)$ be the set of

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all fixed points of a mapping T, that is, $F(T) = \{x \in E : Tx = x\}$, and let $\mathbb{N}$ denote the set of all positive integers.

Let K be a nonempty closed convex subset of E. A mapping $T : K \to K$ is said to be *asymptotically non-expansive* if for each $n \geq 1$, there exists a nonnegative real number k_n satisfying $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \ \forall x, y \in K;$$

when $k_n \equiv 1$, T is called *non-expansive*.

The concept of asymptotically non-expansive mapping, a natural generalization of the important class of non-expansive mappings, was introduced by Goebel and Kirk [10] in 1972. Furthermore, Goebel and Kirk also showed the existence of a fixed point of the asymptotically non-expansive mapping in uniformly convex Banach space E. Kirk et al. [16] extended this result to a reflexive Banach space E with the fixed point property for non-expansive mappings. Subsequently, considerable research works have been devoted to the approximation of fixed point of asymptotically non-expansive mapping (see, e.g., [3, 4, 6-8, 10, 16, 25, 34-37, 41-43, 46, 47] and the references contained therein).

In 1967, Halpern [12] ($u = 0$) was the first who introduced the following iteration scheme for a nonexpansive mapping T, referred to as *Halpern iteration*, for $u, x_0 \in K$, $\alpha_n \in [0, 1]$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall\, n \geq 0. \tag{1.1}$$

Subsequently, considerable research efforts, within the past 40 years or so, have been devoted to studying strong convergence of this scheme for approximating fixed points of T with various types of additional conditions. For examples, see Lions [13], Wittmann [51], Reich [19-23] Shioji and Takahashi [40], Song [27], Song and Chen [29-32], Song and Xu [33], Suzuki [38, 39], Takahashi and Ueda [44].

The Halpern iteration is employed by Chang et al. [7] for asymptotically non-expansive mapping T, referred to as *modified Halpern iteration*, for $u, x_0 \in K$, $\alpha_n \in [0, 1]$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T^n x_n, \quad \forall\, n \geq 0. \tag{1.2}$$

They showed the result.

Theorem 1.1. ([7, Theorem 2])*Let E be a real Banach space with a uniformly Gâteaux differentiable norm, K a nonempty closed convex subset of E and $T : K \to K$ an asymptotically non-expansive mapping* *with a sequence* k_n *and* $F(T) \neq \emptyset$. *Let* $\sum_{n=0}^{\infty}(k_n - 1) < +\infty$ *and* $\{\alpha_n\}$ *be a real sequence in* $(0,1)$ *satisfying the conditions:* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}$ *be the iterative sequence defined by (1.2). For a given* $u \in K$, *define a sequence of contractive mappings* $S_n : K \to K$ *by*

$$S_n(z) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n z, \ z \in K, \ n \geq 1,$$

where $\lim_{n\to\infty} t_n = 1$ *and* $k_n^2 - 1 \leq (1 - \frac{t_n}{k_n})^2, \forall n \leq n_0$. *Let* z_n *be the unique fixed point of* S_n*; i.e.,*

$$z_n = S_n(z_n) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n z_n. \tag{1.3}$$

If

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

and $\{z_n\}$ *converges strongly to some* $z \in F(T)$, *then the sequence* $\{x_n\}$ *converges strongly to* z *if and only if* $\{x_n\}$ *is bounded.*

In 2000, Moudafi [15] introduced the following viscosity approximation method with a contraction f for a non-expansive mapping T:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{1.4}$$

and proved that $\{x_n\}$ converged to a fixed point p of T in a Hilbert space. Xu [48] proved that under certain appropriate conditions on $\{\alpha_n\}$, $\{x_n\}$ converged strongly to a fixed point of T, a uniformly smooth Banach space which solved some variational inequality. Several mathematicians studied the strong convergence of the viscosity approximation method to some (common) fixed point of a non-expansive mapping (finite or infinite family). For examples, see [14, 26, 28-31, 33, 38]. Recently, the viscosity approximation method is extended by Shahzad and Udomene [36] to develop new iterative schemes for an asymptotically non-expansive mapping. They proved the following result.

Theorem 1.2. ([36, Theorem 3.3])*Let* E *be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure,* K *a nonempty closed convex subset of* E, $T : K \to K$ *an asymptotically non-expansive mapping with a sequence* k_n, $F(T) \neq \emptyset$, *and* $f : K \to K$ *a contraction with constant* $\alpha \in [0,1)$. *Let* $\sum_{n=0}^{\infty}(k_n - 1) < +\infty$ *and*

$\{t_n\}$ be a real sequence in $(0, \xi_n)$ satisfying the conditions: $\lim_{n\to\infty} t_n = 1$, $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$, where $\xi_n = \min\{\frac{(1-\alpha)k_n}{k_n-\alpha}, \frac{1}{k_n}\}$. For an arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be iteratively defined by

$$x_{n+1} = (1 - \frac{t_n}{k_n})f(x_n) + \frac{t_n}{k_n}T^n x_n. \tag{1.5}$$

Then,

(i) for each $n \geq 0$, there is a unique $z_n \in K$ satisfying (1.3).

If in addition, $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then

(ii) the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality,

$$\langle (I-f)p, J(p-x^*)\rangle \leq 0, \quad \forall x^* \in F(T). \tag{1.6}$$

Very recently, Ceng et al. [5] and Ceng et al. [6] considered the following viscosity approximation scheme for a finite family of asymptotically non-expansive mappings $\{T_i\}_{i=1}^N$:

$$x_{n+1} = (1 - \frac{1}{k_n})x_n + \frac{1-t_n}{k_n}f(x_n) + \frac{t_n}{k_n}T_{r_n}^n x_n, \tag{1.7}$$

where $\sum_{n=0}^{\infty}(k_n - 1) < +\infty$ and $\{t_n\}$ is a real sequence in $(0, \xi_n)(\xi_n = \min\{\frac{(1-\alpha)k_n}{k_n-\alpha}, \frac{1}{k_n}\})$ satisfying the conditions:

$\lim_{n\to\infty} t_n = 1$, $\sum_{n=0}^{\infty}(1-t_n) = \infty$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$,

and $r_n = n \mod N$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$. Ceng et al. [5] and Ceng et al. [6] showed that if $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i \in \{1, 2, \ldots, N\}$, then $\{x_n\}$ converged strongly to the unique solution of the variational inequality (1.6) in a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure and in a real Banach space with a weakly continuous duality mapping J_φ, respectively.

Here, for an asymptotically non-expansive self-mapping T, we will prove that the sequence $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T, whenever $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, either (C3) $\lim\limits_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$ or (C4) $\sum\limits_{n=0}^{\infty} (k_n - 1) < +\infty$.

Furthermore, we show the strong convergence of $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n u + \beta_n T^n x_n, \tag{1.8}$$

when $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a real sequence in $(0, 1)$ satisfying the conditions (C1), (C2) and (C3) or (C4).

As an application, we also establish the strong convergence of the viscosity approximation scheme with a contraction given by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n x_n \tag{1.9}$$

and

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(x_n) + \beta_n T^n x_n \tag{1.10}$$

where f is a contractive self-mapping.

2. Preliminaries and basic results

In the proof of our main results, we need the following definitions and results.

By a gauge function we mean a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\lim\limits_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^*; \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\}, \forall x \in E\}$$

is called the duality mapping with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Browder [1] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ.

Following Browder [1], we say that a Banach space E has *a weakly continuous duality mapping* if there exists a gauge φ for which the duality map J_φ is single-valued and weak-weak* sequentially continuous; that is, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weak* to $J_\varphi(x)$. It is known that l^p $(1 < p <$

∞) has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, t \geq 0.$$

Then,

$$J_\varphi(x) = \partial \Phi(\|x\|), x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis.

The first part of the following lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [2, Theorem 5]; see also [9, 34].

Lemma 2.1. *Assume E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) For all $x, y \in E$, there holds the inequality,

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle.$$

(ii) If a sequence $\{x_n\}$ in E is weakly convergent to a point x, then there holds the identity,

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|),$$

for all $x, y \in E$.

If C is a nonempty convex subset of a Banach space E and D is a nonempty subset of C, then a mapping $P : C \to D$ is called a *retraction* if P is continuous with $F(P) = D$. A mapping $P : C \to D$ is called *sunny* if

$$P(Px + t(x - Px)) = Px, \ \forall x \in C,$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is said to be a *sunny non-expansive retract* of C if there exists a sunny non-expansive retraction of C onto D; for more details, see [19, 24, 45]. The following lemma is well known [24, 45].

Lemma 2.2. *Let C be nonempty convex subset of a smooth Banach space E, $\emptyset \neq D \subset C$ and $P : C \to D$ a retraction. Then, P is both sunny and non-expansive if and only if there holds the inequality,*

$$\langle x - Px, J(y - Px) \rangle \leq 0, \ \text{for all } x \in C \text{ and } y \in D, \tag{2.1}$$

where J is the normalized duality mapping associated with the gauge $\varphi(t) = t$. Hence, there is at most one sunny non-expansive retraction from C onto D.

Note that the inequality (2.1) is equivalent to the inequality,

$$\langle x - Px, J_\varphi(y - Px)\rangle \le 0, \text{ for all } x \in C \text{ and } y \in D, \tag{2.2}$$

where φ is an arbitrary gauge. This is because there holds the relation,

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x), \forall x \neq 0.$$

Let $S(E) := \{x \in E; \|x\| = 1\}$ denote the unit sphere of a Banach space E. E is said to have: *(i) a uniformly Gâteaux differentiable norm*, if for each y in $S(E)$, $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ is uniformly attained for $x \in S(E)$; *(ii) a uniformly Fréchet differentiable norm* (we also say that E is *uniformly smooth*) if the above limit is attained uniformly for $(x, y) \in S(E) \times S(E)$. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2};\ \|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon\},$$

for each $\varepsilon \in (0, 2]$. A Banach space E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$. If E is uniformly convex, then

$$\|\frac{x + y}{2}\| \le r[1 - \delta_E(\frac{\varepsilon}{r})], \tag{2.3}$$

for every $x, y \in E$ with $\|x\| \le r,\ \|y\| \le r \le R$, and $\|x-y\| \ge \varepsilon > 0$ ([45, Theorem 4.1.4, pp. 93-98]). We also need the demiclosedness principle for asymptotically non-expansive mappings.

Lemma 2.3. (Cho et al. [4, Theorem 1.6]) *Let K a nonempty closed convex subset of a uniformly convex Banach space E, and $T : K \to K$ an asymptotically non-expansive mapping. Then, $I - T$ is demiclosed at 0; i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x = Tx$.*

Quite recently, Song [25] essentially proved the following result.

Lemma 2.4. (Song [25, Proposition 2.4])*Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Suppose that $T : K \to K$ is an asymptotically non-expansive mapping with k_n. Suppose that for the bounded sequence $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ satisfying one of the two conditions:*

(i) $\lim\limits_{k\to\infty} \|x_{n_k} - \frac{1}{n_k + 1} \sum\limits_{j=0}^{n_k} T^j x_{n_k}\| = 0$ *and* $h(z) = \limsup\limits_{k\to\infty} \|x_{n_k} - z\|^2,\ \forall z \in K$.

(ii) $\lim_{k\to\infty} \|x_{n_k+1} - \frac{1}{n_k+1}\sum_{j=0}^{n_k} T^j x_{n_k}\| = 0$ *and* $h(z) = \limsup_{k\to\infty} \|x_{n_k+1} - z\|^2, \ \forall z \in K$.
Then, there exists a unique $x \in K$ *such that*

$$h(x) = \inf_{z\in K} h(z) \ \textit{and}\ x = Tx.$$

Lemma 2.5. (Song [25, Lemma 3.1]) *Let* K *be a nonempty closed convex subset of a smooth Banach space* E. *Suppose that* $T : K \to K$ *is an asymptotically non-expansive mapping with a coefficient* $k_n \in [1, +\infty)$ *and* $F(T) \neq \emptyset$. *If* $t_m \in (0, 1)$ *satisfies the condition* $\lim_{m\to\infty} \frac{b_m}{t_m} = 0$, *where* $b_m = \frac{1}{m+1}\sum_{j=0}^{m}(k_j - 1)$. *Then,*

(i) *for each* $t_m \in (0, 1)$, *there exactly exists one* $z_m \in K$ *such that for sufficiently large nonnegative integer* N *and an any anchor point* $u \in K$,

$$z_m = \begin{cases} x \in K, \ m = 0, 1, 2, \cdots, N-1, \\ t_m u + (1-t_m)\frac{1}{m+1}\sum_{j=0}^{m} T^j z_m, \ m \geq N. \end{cases} \tag{2.4}$$

(ii) *For any fixed* $y \in F(T)$, $\exists \alpha > 0$ *such that*

$$\|z_m - y\|^2 \leq \frac{1}{\alpha}\langle u - y, J(z_m - y)\rangle.$$

(iii) $\{z_m\}$ *is bounded.*

(iv) $\langle u - z_m, J(y - z_m)\rangle \leq \frac{b_m}{t_m} M$ *for all* $y \in F(T)$ *and some constant* $M > 0$.

Lemma 2.6. (Song [25, Theorem 3.2]) *Let* K *be a nonempty closed convex subset of a uniformly convex Banach space* E *with a weakly continuous duality mapping* J_φ. *Suppose that* $T : K \to K$ *is an asymptotically non-expansive mapping with* $k_n \in [1, +\infty)$ *and* $F(T) \neq \emptyset$. *Then,* $F(T)$ *is a sunny non-expansive retract of* K. *Furthermore, let* $\{z_m\}$ *be defined by (2.4) and* $t_m \in (0, 1)$ *satisfy* $\lim_{m\to\infty} t_m = 0$ *and* $\lim_{m\to\infty} \frac{b_m}{t_m} = 0$. *Then,* $Pu = \lim_{m\to\infty} z_m$ *defines a unique sunny non-expansive retraction from* K *into* $F(T)$; *i.e.* Pu *is the unique solution in* $F(T)$ *of the following variational inequality,*

$$\langle u - Pu, J_\varphi(y - Pu)\rangle \leq 0, \quad \textit{for all } y \in F(T). \tag{2.5}$$

Lemma 2.7. (Liu [17] and Xu [49, 50]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the property,*

$$a_{n+1} \leq (1-t_n)a_n + t_n c_n + b_n, \quad \forall\ n \geq 0,$$

where $\{t_n\}$*,* $\{b_n\}$ *and* $\{c_n\}$ *satisfy the restrictions:* $\sum\limits_{n=0}^{\infty} t_n = \infty$*,* $\sum\limits_{n=0}^{\infty} |b_n| < +\infty$ *and* $\limsup\limits_{n\to\infty} c_n \leq 0$*. Then,* $\{a_n\}$ *converges to zero as* $n \to \infty$*.*

3. Strong convergence of modified Halpern iteration

With the help of Lemma 2.5, we first show the strong convergence of $\{z_m\}$ defined by (2.4) in a uniformly convex Banach space with a Gâteaux differentiable norm, extending and complementing Lemma 2.6.

Lemma 3.1. *Let* K *be a nonempty closed convex subset of a uniformly convex Banach space* E *with a uniformly Gâteaux differentiable norm. Suppose* $T : K \to K$ *is an asymptotically nonexpansive mapping with* k_n*. Then, (1)* $F(T)$ *is a sunny non-expansive retract of* K*.*
Furthermore, let $\{z_m\}$ *be defined by* (2.4) *and* $t_m \in (0,1)$ *satisfy the following conditions:* $\lim\limits_{m\to\infty} t_m = 0$ *and* $\lim\limits_{m\to\infty} \dfrac{b_m}{t_m} = 0$*. Then,*
(2) $Pu = \lim\limits_{m\to\infty} z_m$ *defines a unique sunny non-expansive retraction from* K *into* $F(T)$*.*

Proof. Lemma 2.5 *(iii)* implies the boundedness of $\{z_m\}$ and $\{Tz_m\}$. Let $A_m = \frac{1}{m+1}\sum_{j=0}^{m} T^j$. Then, we have,

$$\|A_m z_m - p\| \leq \frac{1}{m+1}\sum_{j=0}^{m} \|T^j z_m - p\| \leq (b_m + 1)\|z_m - p\|.$$

Thus, $\{A_m z_m\}$ is bounded. It follows from (2.4) that

$$\lim_{m\to\infty} \|z_m - A_m z_m\| = \lim_{m\to\infty} t_m \|u - A_m z_m\| = 0.$$

Now, we claim that the set $\{z_m\}$ is relatively sequentially compact. In fact, for each subsequence $\{z_{m_k}\}$ of $\{z_m\}$, let

$$h(y) = \limsup_{k\to\infty} \|z_{m_k} - y\|^2, \quad \forall y \in K.$$

It follows from Lemma 2.4*(i)* that there exists a unique $x^* \in K$ such that

$$h(x^*) = \inf_{y \in K} h(y), \quad x^* = Tx^*.$$

For any given $t \in (0,1)$, if $x_t = x^* + t(u - x^*) = (1-t)x^* + tu$, then $x_t = x^* + t(u - x^*) = (1-t)x^* + tu \in K$ by the convexity of K and hence $h(x^*) \leq h(x_t)$. Since $z_{m_k} - x_t = (z_{m_k} - x^*) - t(u - x^*)$, then we have,

$$\begin{aligned}\|z_{m_k} - x_t\|^2 &= \langle z_{m_k} - x^*, J(z_{m_k} - x_t)\rangle - t\langle u - x^*, J(z_{m_k} - x_t)\rangle \\ &\leq \frac{\|z_{m_k} - x^*\|^2 + \|z_{m_k} - x_t\|^2}{2} - t\langle u - x^*, J(z_{m_k} - x_t)\rangle,\end{aligned}$$

implying

$$\|z_{m_k} - x_t\|^2 \leq \|z_{m_k} - x^*\|^2 - 2t\langle u - x^*, J(z_{m_k} - x_t)\rangle.$$

Thus, we have,

$$h(x_t) \leq h(x^*) - 2t \liminf_{k\to\infty} \langle u - x^*, J(z_{m_k} - x_t)\rangle,$$

that is,

$$\liminf_{k\to\infty} \langle u - x^*, J(z_{m_k} - x_t)\rangle \leq \frac{h(x^*) - h(x_t)}{2t} \leq 0. \tag{3.1}$$

On the other hand, since J is uniformly continuous on the bounded set from norm topology to weak star topology and $\lim_{t\to 0} x_t = x^*$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u - x^*, J(z_{m_k} - x^*)\rangle < \langle u - x^*, J(z_{m_k} - x_t)\rangle + \varepsilon, \quad \forall t \in (0,\delta),\ k \in \mathbb{N}.$$

Thus, by (3.1), we have,

$$\liminf_{k\to\infty} \langle u - x^*, J(z_{m_k} - x^*)\rangle \leq \liminf_{k\to\infty} \langle u - x^*, J(z_{m_k} - x_t)\rangle + \varepsilon \leq \varepsilon.$$

Since ε is arbitrary, then we obtain:

$$\liminf_{k\to\infty} \langle u - x^*, J(z_{m_k} - x^*)\rangle \leq 0. \tag{3.2}$$

Therefore, by Lemma 2.5*(ii)*, we have,

$$\liminf_{k\to\infty} \|z_{m_k} - x^*\|^2 \leq \frac{1}{\alpha} \liminf_{k\to\infty} \langle u - x^*, J(z_{m_k} - x^*)\rangle \leq 0,$$

that is,

$$\liminf_{k\to\infty} \|z_{m_k} - x^*\| = 0.$$

Hence, $\{z_{m_k}\}$ has a subsequence converging strongly to $x^* \in F(T)$. Since then $\{z_{m_k}\}$ is an arbitrary subsequentially, $\{z_m\}$ is relatively sequently compact.

Finally, we prove that the sequence $\{z_m\}$ converges strongly. For this, we assume that two subsequences $\{z_{m_k}\}$ and $\{z_{m_i}\}$ of $\{z_m\}$ are such that $z_{m_k} \to x^*$ and $z_{m_i} \to x$, respectively. Then, we need to show $x^* = x$. Indeed, from Lemma 2.4*(i)*, we obtain $x^*, x \in F(T)$, because $\limsup_{k\to\infty} \|z_{m_k} - x^*\|^2 = 0$ and $\limsup_{i\to\infty} \|z_{m_i} - x\|^2 = 0$. Therefore, noting Lemma 2.5*(iv)*, for any $p \in F(T)$, we obtain:

$$(3.3) \quad \langle u - z_{m_k}, J(p - z_{m_k})\rangle \le \frac{b_{m_k}}{t_{m_k}} M, \quad \langle u - z_{m_i}, J(p - z_{m_i})\rangle \le \frac{b_{m_i}}{t_{m_i}} M.$$

Since the duality mapping J is norm-weak* uniformly continuous, $\lim_{k\to\infty} z_{m_k} = x^*$ and $\lim_{i\to\infty} z_{m_i} = x$, then letting $k \to \infty$ and $i \to \infty$ in (3.3), respectively, we obtain (noting the condition $\lim\limits_{m\to\infty} \dfrac{b_m}{t_m} = 0$),

$$(3.4) \quad \langle u - x^*, J(p - x^*)\rangle \le 0, \quad \langle u - x, J(p - x)\rangle \le 0, \quad \forall p \in F(T).$$

In particular,

$$\langle u - x^*, J(x - x^*)\rangle \le 0, \quad \langle u - x, J(x^* - x)\rangle \le 0.$$

Thus, by the addition of the above two inequalities, we have,

$$\|x - x^*\|^2 = \langle x - x^*, J(x - x^*)\rangle \le 0$$

and so we have $x = x^*$. Thus, we proved that the sequence $\{z_m\}$ was sequentially compact and each cluster point of $\{z_m\}$ equaled to the point x^*. Therefore, from [18, Proposition 2.1.31], we obtain that $z_m \to x^*$ as $m \to \infty$.

Let $Pu = \lim\limits_{m\to\infty} z_m = x^*$. Then, from (3.4), we have,

$$\langle u - Pu, J(p - Pu)\rangle \le 0 \quad \forall p \in F(T).$$

It follows from Lemma 2.2 that P is a unique sunny non-expansive retraction from K into $F(T)$. This completes the proof. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm. Suppose that $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$. Let $\{x_n\}$ be defined by*

$$(3.5) \qquad x_{n+1} = \alpha_n u + (1 - \alpha_n)T^n x_n, \quad n \ge 0.$$

Assume that $\{\alpha_n\}$ is a real sequence in $(0, 1)$ satisfying the conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, *(C2)* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and (C3)* $\lim_{n\to\infty} \dfrac{\theta_n}{\alpha_n} = 0$.

If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *then as* $n \to \infty$, $\{x_n\}$ *converges strongly to* Pu, *where* P *is a unique sunny non-expansive retraction from* K *into* $F(T)$.

Proof. Take $p \in F(T)$. Since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, then there exists $N \in \mathbb{N}$, for all $n \geq N$, $\frac{\theta_n}{\alpha_n} \leq \frac{1}{2}$. Choose a constant $M > 0$ sufficiently large such that

$$\|x_N - p\| \leq M \text{ and } \|u - p\| \leq \frac{M}{2}.$$

We proceed by induction to show $\|x_n - p\| \leq M, \forall n \geq 1$. Assume $\|x_n - p\| \leq M$, for some $n \geq N$. We show $\|x_{n+1} - p\| \leq M$. From the iteration process (3.5), we estimate as follows:

$$\begin{aligned}
\|x_{n+1} - p\| \leq& \alpha_n\|u - p\| + (1 - \alpha_n)\|T^n x_n - p\| \\
\leq& \alpha_n\|u - p\| + (1 - \alpha_n)(1 + \theta_n)\|x_n - p\| \\
\leq& \frac{M}{2}\alpha_n + \frac{M}{2}\alpha_n(1 - \alpha_n) + (1 - \alpha_n)M \\
\leq& \frac{M}{2}\alpha_n + \frac{M}{2}\alpha_n + (1 - \alpha_n)M = M.
\end{aligned}$$

This proves the boundedness of the sequence $\{x_n\}$.

From the hypothesis $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we have,

$$\begin{aligned}
\|T^j x_n - x_n\| \leq& \|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots + \|T^{j-1}x_n - T^j x_n\| \\
\leq& (1 + k_1 + k_2 + \cdots + k_{j-1})\|x_n - Tx_n\| \to 0 (n \to \infty).
\end{aligned}$$

Let $A_m = \frac{1}{m+1}\sum_{j=0}^{m} T^j$. Then,

$$\|A_m x_n - x_n\| \leq \frac{1}{m+1}\sum_{j=0}^{m} \|T^j x_n - x_n\| \to 0 (n \to \infty),$$

and hence,

$$\lim_{n\to\infty} \|A_m x_n - x_n\| = 0. \tag{3.6}$$

Let $b_n = \dfrac{1}{n+1}\sum_{j=0}^{n} \theta_j$. We may choose $t_m \in (0, 1)$ such that

$$\lim_{m\to\infty} \frac{b_m}{t_m} = 0, \quad \lim_{m\to\infty} t_m = 0. \tag{3.7}$$

Then, it follows from Lemma 3.1 that $Pu = \lim_{m\to\infty} z_m$, where $\{z_m\}$, defined by (2.4). For each $m > M_0$, using the equalities,

$$(1 - t_m)(A_m z_m - z_m) + t_m(u - Pu + Pu - z_m) = 0$$

and

$$\langle A_m x - A_m y, J(x - y)\rangle \le \|A_m x - A_m y\|\|x - y\| \le (b_m + 1)\|x - y\|^2,$$

we have

$$\begin{aligned}
&t_m\langle u - Pu, J(x_n - z_m)\rangle \\
&= (1 - t_m)\langle z_m - A_m z_m, J(x_n - z_m)\rangle + t_m\langle z_m - Pu, J(x_n - z_m)\rangle \\
&\le (1 - t_m)\langle z_m - x_n + x_n - A_m x_n + A_m x_n - A_m z_m, J(x_n - z_m)\rangle \\
&+ t_m\|z_m - Pu\|\|x_n - z_m\| \\
&\le (1 - t_m)[\langle z_m - x_n, J(x_n - z_m)\rangle + \langle A_m x_n - A_m z_m, J(x_n - z_m)\rangle] \\
&\quad + (1 - t_m)\langle x_n - A_m x_n, J(x_n - z_m)\rangle + t_m\|z_m - Pu\|L \\
&\le (1 - t_m)[-\|x_n - z_m\|^2 + (b_m + 1)\|x_n - z_m\|^2] \\
&+ \|x_n - A_m x_n\|L + t_m\|z_m - Pu\|L \\
&\le b_m L^2 + \|x_n - A_m x_n\|L + t_m\|z_m - Pu\|L,
\end{aligned}$$

where L is a constant such that $L \ge \|x_n - z_m\|$. Thus, it follows,

$$\langle u - Pu, J(x_n - z_m)\rangle \le \frac{b_m}{t_m}L^2 + \frac{\|x_n - A_m x_n\|}{t_m}L + \|z_m - Pu\|L,$$

and so it follows from (3.6) and (3.7),

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle u - Pu, J(x_n - z_m)\rangle \le 0. \tag{3.8}$$

On the other hand, the fact that $z_m \to Pu$, as $m \to \infty$, together with the fact that the duality mapping J is norm-weak* uniformly continuous, would result in:

$$\lim_{m\to\infty}\langle u - Pu, J(x_n - z_m)\rangle = \langle u - Pu, J(x_n - Pu)\rangle, \quad \forall n \in \mathbb{N}.$$

Thus, for all $\varepsilon > 0$, there exists $m_1 \in \mathbb{N}$ such that

$$\langle u - Pu, J(x_n - Pu)\rangle \le \langle u - Pu, J(x_n - z_m)\rangle + \varepsilon, \quad \forall m \ge m_1,\ n \in \mathbb{N}$$

and so, using (3.8),

$$\limsup_{n\to\infty}\langle u - Pu, J(x_n - Pu)\rangle \le \limsup_{m\to\infty}\limsup_{n\to\infty}\langle u - Pu, J(x_n - z_m)\rangle + \varepsilon \le \varepsilon.$$

Since ε is arbitrary, then we obtain,

$$\limsup_{n\to\infty}\langle u - Pu, J(x_n - Pu)\rangle \le 0. \tag{3.9}$$

We next show $x_n \to Pu$. It follows from the equality (3.5) that

$$\begin{aligned}
&\|x_{n+1} - Pu\|^2 \\
=&\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle + (1-\alpha_n)\langle T^n x_n - Pu, J(x_{n+1} - Pu)\rangle \\
\le&\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle + (1-\alpha_n)\|T^n x_n - Pu\|\|x_{n+1} - Pu\| \\
\le&(1-\alpha_n)(\theta_n + 1)\|x_n - Pu\|\|x_{n+1} - Pu\| \\
&\quad + \alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle \\
\le&(1-\alpha_n)\frac{(\theta_n + 1)^2\|x_n - Pu\|^2 + \|x_{n+1} - Pu\|^2}{2} \\
&\quad + \alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - Pu\|^2 \le&(1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle \\
&+ (1-\alpha_n)[(\theta_n + 1)^2 - 1]\|x_n - Pu\|^2 \\
\le&(1-\alpha_n)\|x_n - Pu\|^2 + \theta_n(\theta_n + 2)\|x_n - Pu\|^2 \\
&+ 2\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle,
\end{aligned}$$

that is,

$$\|x_{n+1} - Pu\|^2 \le (1-\alpha_n)\|x_n - Pu\|^2 + \gamma_n\alpha_n, \tag{3.10}$$

where $\gamma_n = \frac{\theta_n}{\alpha_n}(\theta_n + 2)\|x_n - Pu\|^2 + 2\langle u - Pu, J(x_{n+1} - Pu)\rangle$.

It follows from the condition $\lim\limits_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$ and boundedness of $\{x_n\}$ along with the inequality (3.9) that

$$\limsup_{n\to\infty}\gamma_n \le 0.$$

Applying Lemma 2.7 to the inequality (3.10), we conclude $x_n \to Pu$. This completes the proof. $\square$

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm. Suppose $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$. Let $\{x_n\}$ be defined by (3.5) and $\{\alpha_n\}$ be a real sequence in $(0, 1)$ satisfying the conditions:*

(C1) $\lim\limits_{n\to\infty}\alpha_n = 0$, *(C2)* $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, *and (C4)* $\sum\limits_{n=0}^{\infty}\theta_n < +\infty$.

If $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then as $n \to \infty$, $\{x_n\}$ converges strongly to Pu, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Proof.. Following Theorem 3.2, we have,

$$\limsup_{n\to\infty}\langle u - Pu, J(x_n - Pu)\rangle \leq 0$$

and

$$\begin{aligned}\|x_{n+1} - Pu\|^2 \leq &(1-\alpha_n)\|x_n - Pu\|^2 + \theta_n(\theta_n + 2)\|x_n - Pu\|^2 \\ &+ 2\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle.\end{aligned}$$

That is,

$$\|x_{n+1} - Pu\|^2 \leq (1-\alpha_n)\|x_n - Pu\|^2 + \lambda_n\alpha_n + \theta_n M,$$

where $\lambda_n = 2\langle u - Pu, J(x_{n+1} - Pu)\rangle$. It follows:

$$\sum_{n=0}^{\infty}\theta_n M < +\infty \text{ and } \limsup_{n\to\infty}\lambda_n \leq 0.$$

Apply Lemma 2.7 to yield the desired result. This completes the proof. □

Theorem 3.4. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E with a weakly continuous duality mapping J_φ. Suppose that $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$. Let $\{x_n\}$ be defined by (3.5) and $\{\alpha_n\}$ be a real sequence in $(0, 1)$ satisfying the conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, (C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$, and (C3) $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$.

If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then as $n \to \infty$, $\{x_n\}$ converges strongly to Pu, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Proof. It follows from the same argument as made in the proof of Theorem 3.2 that the sequence $\{x_n\}$ is bounded. By Lemma 2.6, $F(T)$ is a sunny non-expansive retract of K and Pu is the unique solution in $F(T)$ to the following variational inequality,

$$\langle u - Pu, J_\varphi(y - Pu)\rangle \leq 0 \quad \text{for all } y \in F(T). \tag{3.11}$$

We next show that

$$\limsup_{n\to\infty}\langle u - Pu, J_\varphi(x_{n+1} - Pu)\rangle \leq 0.$$

Indeed, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-p, J_\varphi(x_{n+1}-Pu)\rangle = \lim_{k\to\infty}\langle u-p, J_\varphi(x_{n_k+1}-Pu)\rangle.$$

We may assume that $x_{n_k+1} \rightharpoonup x^*$, by the reflexivity of E and the boundedness of $\{x_n\}$. It follows from Lemma 2.3 and the assumption $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ that $x^* \in F(T)$. From the weak continuity of the duality mapping J_φ and (3.11), we have,

$$\limsup_{n\to\infty}\langle u-Pu, J_\varphi(x_n-Pu)\rangle = \langle u-Pu, J_\varphi(x^*-Pu)\rangle \le 0.$$

Now, we show $x_n \to Pu$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, t \ge 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 > k > 0$, $\varphi(kx) \le \varphi(x)$, and

$$\Phi(kt) = \int_0^{kt}\varphi(\tau)d\tau \overset{\tau=kx}{=} k\int_0^t \varphi(kx)dx \le k\int_0^t \varphi(x)dx = k\Phi(t).$$

Since $\lim_{n\to\infty}\dfrac{\theta_n}{\alpha_n} = 0$, then there exists $M_0 \in \mathbb{N}$ such that such that $\forall n \ge M_0$,

$$\frac{\theta_n}{\alpha_n} < \frac{1}{2},$$
$$\theta_n < \frac{1}{2}\alpha_n.$$

Since $(1-\alpha_n)(\theta_n+1) = 1-\alpha_n+\frac{1}{2}\alpha_n(1-\alpha_n) \le 1-\alpha_n+\frac{1}{2}\alpha_n < 1$, then from Lemma 2.1*(i)*, we have,

$$\begin{aligned}
&\Phi(\|x_{n+1}-Pu\|)\\
\le&\Phi((1-\alpha_n)\|T^n x_n - Pu\|) + \alpha_n\langle u-Pu, J_\varphi(x_{n+1}-Pu)\rangle\\
\le&\Phi((1-\alpha_n)(\theta_n+1)\|x_n-Pu\|) + \alpha_n\langle u-Pu, J_\varphi(x_{n+1}-Pu)\rangle\\
\le&(1-\alpha_n)(\theta_n+1)\Phi(\|x_n-Pu\|) + \alpha_n\langle u-Pu, J_\varphi(x_{n+1}-Pu)\rangle\\
\le&(1-\alpha_n)\Phi(\|x_n-Pu\|) + \alpha_n\gamma_n,
\end{aligned}$$

where $\gamma_n = \frac{\theta_n}{\alpha_n}M + \langle u-Pu, J_\varphi(x_n-Pu)\rangle$, for some constant $M > \Phi(\|x_n-p\|)$. An application of Lemma 2.7 yields $\Phi(\|x_n-Pu\|) \to 0 (n\to\infty)$, and hence $\|x_n-Pu\| \to 0(n\to\infty)$. This completes the proof. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm.*

Suppose that $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$. Let $\{x_n\}$ be defined by:

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n u + \beta_n T^n x_n, \tag{3.12}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ satisfying the conditions (C1), (C2) and (C3). If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then as $n \to \infty$, $\{x_n\}$ converges strongly to Pu, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Proof. It follows from the same argument as given in the proof of Theorem 3.2 that the sequence $\{x_n\}$ is bounded and

$$\limsup_{n\to\infty} \langle u - Pu, J(x_{n+1} - Pu)\rangle \le 0.$$

We next show $x_n \to Pu$. It follows from the equality (3.12) that

$$\begin{aligned}
&\|x_{n+1} - Pu\|^2 \\
=&\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle + (1 - \alpha_n - \beta_n)\langle x_n - Pu, J(x_{n+1} - Pu)\rangle \\
&+ \beta_n\langle T^n x_n - Pu, J(x_{n+1} - Pu)\rangle \\
\le&\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle + (1 - \alpha_n - \beta_n)\|x_n - Pu\|\|x_{n+1} - Pu\| \\
&+ \beta_n\|T^n x_n - Pu\|\|x_{n+1} - Pu\| \\
\le&\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle \\
&+ (1 - \alpha_n - \beta_n)\frac{\|x_n - Pu\|^2 + \|x_{n+1} - Pu\|^2}{2} \\
&+ \beta_n\frac{(\theta_n + 1)^2\|x_n - Pu\|^2 + \|x_{n+1} - Pu\|^2}{2} \\
\le&\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle + [1 - \alpha_n + \beta_n((\theta_n + 1)^2 - 1)]\frac{\|x_n - Pu\|^2}{2} \\
&+ \frac{\|x_{n+1} - Pu\|^2}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - Pu\|^2 \le&(1 - \alpha_n)\|x_n - Pu\|^2 + \beta_n[(\theta_n + 1)^2 - 1]\|x_n - Pu\|^2 \\
&+ 2\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle \\
\le&(1 - \alpha_n)\|x_n - Pu\|^2 + \theta_n(\theta_n + 2)\|x_n - Pu\|^2 \\
&+ 2\alpha_n\langle u - Pu, J(x_{n+1} - Pu)\rangle.
\end{aligned}$$

The rest of the proof is the same as given for Theorem 3.2. This completes the proof. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E with a weakly continuous duality mapping J_φ. Suppose $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$. Let $\{x_n\}$ be defined by (3.12), $\{\beta_n\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ satisfying the conditions (C1), (C2) and (C3). If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then, as $n \to \infty$, $\{x_n\}$ converges strongly to Pu, where P is a unique sunny non-expansive retraction from K into $F(T)$.*

Proof. It follows from the same argument as given for the proof of Theorem 3.4 that the sequence $\{x_n\}$ is bounded and

$$\limsup_{n\to\infty}\langle u - Pu, J_\varphi(x_{n+1} - Pu)\rangle \leq 0.$$

Now we show $x_n \to Pu$. Since $\lim_{n\to\infty} \dfrac{\theta_n}{\alpha_n} = 0$, then there exists $M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$,

$$\frac{\theta_n}{\alpha_n} < 1,$$
$$\theta_n < \alpha_n.$$

From Lemma 2.1*(i)*, we have,

$$\begin{aligned}
&\Phi(\|x_{n+1} - Pu\|)\\
\leq&\Phi((1 - \alpha_n - \beta_n)\|x_n - Pu\| + \beta_n\|T^n x_n - Pu\|)\\
&+ \alpha_n\langle u - Pu, J_\varphi(x_{n+1} - Pu)\rangle\\
\leq&\Phi((1 - \alpha_n + \theta_n\beta_n)\|x_n - Pu\|) + \alpha_n\langle u - Pu, J_\varphi(x_{n+1} - Pu)\rangle\\
\leq&(1 - \alpha_n + \theta_n\beta_n)\Phi(\|x_n - Pu\|) + \alpha_n\langle u - Pu, J_\varphi(x_{n+1} - Pu)\rangle\\
\leq&(1 - \alpha_n)\Phi(\|x_n - Pu\|) + \theta_n\beta_n\Phi(\|x_n - Pu\|)\\
&+ \alpha_n\langle u - Pu, J_\varphi(x_{n+1} - Pu)\rangle.
\end{aligned}$$

The rest of the proof is the same as given for Theorem 3.4. This completes the proof. □

Similarly, we have the following result.

Theorem 3.7. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E either with a uniformly Gâteaux differentiable norm or with a weakly continuous duality mapping J_φ. Suppose T :*

$K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$. Let $\{x_n\}$ be defined by (3.12), $\{\beta_n\}$ and $\{\alpha_n\}$ be real sequences in $(0,1)$ satisfying the conditions (C1), (C2) and (C4). If $\{x_n\}$ is bounded with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then, as $n \to \infty$, $\{x_n\}$ converges strongly to Pu, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Remark 1.(1) If $\alpha_n = \dfrac{t_n}{k_n}$ in (3.5), then (3.5) becomes the iteration of Chidume et al. [8, Theorem 3.3] and Shahzad and Udomene [36]. In particular, our proof is simpler than theirs as well as Chang et al.'s [7, Theorem 2].

(2) If $\alpha_n = \dfrac{1-t_n}{k_n}$ and $\beta_n = \dfrac{t_n}{k_n}$ in (3.12), then (3.12) turns into the iteration of Ceng et al. [5, Corollary 3.5] and Ceng et al. [6, Corollary 3.7]. Our proof and iteration coefficient are simpler and more general.

(3) Using our proof technique, for an asymptotically non-expansive mappings $\{T_i\}_{i=1}^N$ defined on a uniformly convex Banach space E with a weakly continuous duality mapping J_φ, the strong convergence of $\{x_n\}$ defined by (3.13) or (3.14) is proved easily:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T_{r_n}^n x_n, \tag{3.13}$$

$$x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n u + \beta_n T_{r_n}^n x_n \tag{3.14}$$

whenever $\{\beta_n\}$ and $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying the conditions (C1), (C2) and (C3) or (C4) along with $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for each i, where $r_n = n \mod N$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$.

4. Some applications to the viscosity approximation methods

Theorem 4.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E either with a uniformly Gâteaux differentiable norm or with a weakly continuous duality mapping J_φ. Suppose $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$ and $f : K \to K$ is a contraction with a constant $\alpha \in [0,1)$. Let $\{x_n\}$ be defined by:*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T^n x_n, \tag{4.1}$$

and $\{\alpha_n\}$ be a real sequence in $(0,1)$ satisfying the conditions (C1), (C2) and (C3). If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then, as $n \to \infty$, $\{x_n\}$ converges strongly to $x^* = P(fx^*)$, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Proof. Take $p \in F(T)$. Since $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{\theta_n}{\alpha_n} \leq \frac{1-\alpha}{2}$. Choose a constant $M > 0$ sufficiently large such that

$$\|x_N - p\| \leq M \text{ and } \|f(p) - p\| \leq \frac{(1-\alpha)M}{2}.$$

We proceed by induction to show that $\|x_n - p\| \leq M, \forall n \geq 1$. Assume $\|x_n - p\| \leq M$, for some $n \geq N$. We show that $\|x_{n+1} - p\| \leq M$. From the iterative process (4.1), we estimate as follows:

$$\begin{aligned}
&\|x_{n+1} - p\| \\
\leq&\alpha_n\|f(x_n) - f(p) + f(p) - p\| + (1-\alpha_n)\|T^n x_n - p\| \\
\leq&(1-\alpha_n(1-\alpha))\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\theta_n\|x_n - p\| \\
\leq&(1-\alpha_n(1-\alpha))M + \frac{M(1-\alpha)}{2}\alpha_n + \frac{1-\alpha}{2}\alpha_n(1-\alpha_n)M \\
\leq&(1-\alpha_n(1-\alpha))M + \frac{M(1-\alpha)}{2}\alpha_n + \frac{1-\alpha}{2}\alpha_n M = M.
\end{aligned}$$

This proves the boundedness of the sequence $\{x_n\}$.
It follows from Lemma 3.1 or Lemma 2.6 that $F(T)$ is the sunny non-expansive retract of K. Denote by P a sunny non-expansive retraction of K onto $F(T)$. Then, Pf is a contractive mapping of K into itself. In fact,

$$\|P(fx) - P(fy)\| \leq \|fx - fy\| \leq \alpha\|x - y\|, \text{ for all } x, y \in K.$$

Banach contraction principle assures that there exist a unique element $x^* \in K$ such that $x^* = P(fx^*)$. Such an $x^* \in K$ is an element of $F(T)$. Thus, we may define a sequence $\{y_n\}$ in K by:

$$y_{n+1} = \alpha_n f(x^*) + (1-\alpha_n)T^n y_n$$

and assume that $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$. Then, Theorem 3.2 or Theorem 3.4 assures that $y_n \to P(f(x^*)) = x^*$, as $n \to \infty$. For every n, we have,

$$\begin{aligned}
&\|x_{n+1} - y_{n+1}\|\\
\le&\alpha_n\|f(x_n) - f(x^*)\| + (1-\alpha_n)\|T^n x_n - T^n y_n\|\\
\le&\alpha_n\|f(x_n) - f(y_n)\| + \alpha_n\|f(y_n) - f(x^*)\| + (1-\alpha_n)k_n\|x_n - y_n\|\\
\le&(1-(1-\alpha)\alpha_n)\|x_n - y_n\| + \alpha\alpha_n\|y_n - x^*\| + (1-\alpha_n)\theta_n\|x_n - y_n\|\\
\le&(1-(1-\alpha)\alpha_n)\|x_n - y_n\| + \alpha\alpha_n\|y_n - x^*\| + \theta_n\|x_n - y_n\|.
\end{aligned}$$

Thus, we obtain, for some constant $M > 0$, the following recursive inequality,

$$\|x_{n+1} - y_{n+1}\| \le (1-(1-\alpha)\alpha_n)\|x_n - y_n\| + \alpha\alpha_n\|y_n - x^*\| + \theta_n M.$$

Since $\|y_n - x^*\| \to 0$, then an application of Lemma 2.7 yields:

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

Hence,

$$\lim_{n\to\infty}\|x_n - x^*\| \le \lim_{n\to\infty}(\|x_n - y_n\| + \|y_n - x^*\|) = 0.$$

Consequently, we obtain the strong convergence of $\{x_n\}$ to $x^* = P(fx^*)$. This completes the proof. $\square$

Similarly, we have the following result.

Theorem 4.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E either with a uniformly Gâteaux differentiable norm or with a weakly continuous duality mapping J_φ. Suppose $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$ and $f : K \to K$ is a contraction with a constant $\alpha \in [0, 1)$. Let $\{x_n\}$ be defined by (4.1) and $\{\alpha_n\}$ be a real sequence in $(0, 1)$ satisfying the conditions (C1), (C2) and (C4). If $\{x_n\}$ is bounded with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then, as $n \to \infty$, $\{x_n\}$ converges strongly to $x^* = P(fx^*)$, where P is a unique sunny non-expansive retraction from K into $F(T)$.*

Theorem 4.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E either with a uniformly Gâteaux differentiable norm or with a weakly continuous duality mapping J_φ. Suppose $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$*

and $f : K \to K$ is a contraction with a constant $\alpha \in [0, 1)$. Let $\{x_n\}$ be defined by:

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(x_n) + \beta_n T^n x_n, \tag{4.2}$$

$\{\beta_n\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ satisfying the conditions (C1), (C2) and (C3). If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then, as $n \to \infty$, $\{x_n\}$ converges strongly to $x^* = P(fx^*)$, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Proof. Take $p \in F(T)$. Since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{\theta_n}{\alpha_n} \leq \frac{1-\alpha}{2}$. Choose a constant $M > 0$ sufficiently large such that

$$\|x_N - p\| \leq M \text{ and } \|f(p) - p\| \leq \frac{(1 - \alpha)M}{2}.$$

We proceed by induction to show that $\|x_n - p\| \leq M, \forall n \geq 1$. Assume $\|x_n - p\| \leq M$, for some $n \geq N$. We show $\|x_{n+1} - p\| \leq M$. From the iterative process (4.2), we estimate as follows:

$$\begin{aligned}
&\|x_{n+1} - p\| \\
\leq &(1 - \alpha_n - \beta_n)\|x_n - p\| + \alpha_n\|f(x_n) - f(p) + f(p) - p\| \\
&+ \beta_n\|T^n x_n - p\| \\
\leq &(1 - \alpha_n(1 - \alpha))\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\theta_n\|x_n - p\| \\
\leq &(1 - \alpha_n(1 - \alpha))M + \frac{M(1 - \alpha)}{2}\alpha_n + \frac{1 - \alpha}{2}\alpha_n\beta_n M \\
\leq &(1 - \alpha_n(1 - \alpha))M + \frac{(1 - \alpha)}{2}\alpha_n M + \frac{1 - \alpha}{2}\alpha_n M = M.
\end{aligned}$$

This proves the boundedness of the sequence $\{x_n\}$.
The rest of the proof is similar to the one given for Theorem 3.4. The only difference is that usage of Theorem 3.2 or Theorem 3.4 is replaced with Theorem 3.5 or Theorem 3.6. This completes the proof. □

Now, using Theorem 3.7, we obtain the following result.

Theorem 4.4. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E either with a uniformly Gâteaux differentiable norm or with a weakly continuous duality mapping J_φ. Suppose $T : K \to K$ is an asymptotically non-expansive mapping with $k_n = 1 + \theta_n$ and $f : K \to K$ is a contraction with a constant $\alpha \in [0, 1)$. Let $\{x_n\}$*

be defined by (4.2), $\{\beta_n\}$ and $\{\alpha_n\}$ be real sequences in $(0,1)$ satisfying the conditions (C1), (C2) and (C4). If $\{x_n\}$ is bounded with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then, as $n \to \infty$, $\{x_n\}$ converges strongly to $x^* = P(fx^*)$, where P is a unique sunny non-expansive retraction from K into $F(T)$.

Remark 2.(1) If $\alpha_n = \dfrac{t_n}{k_n}$ in (4.1), then (4.1) becomes the iteration of Shahzad and Udomene [36, Theorem 3.3]. In particular, our proof and iteration coefficient are simpler.

(2) If $\alpha_n = \dfrac{1-t_n}{k_n}$ and $\beta_n = \dfrac{t_n}{k_n}$ in (4.2), then we obtain the iteration of Ceng et al. [5, Corollary 3.6.] and Ceng et al. [6, Corollary 3.6.].

(3) Using our proof technique, for a finite family of asymptotically non-expansive mappings $\{T_i\}_{i=1}^N$ defined on a uniformly convex Banach space E with a weakly continuous duality mapping J_φ, the strong convergence of $\{x_n\}$ defined by (4.3) or (4.4) is shown easily:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_{r_n}^n x_n, \tag{4.3}$$

$$x_{n+1} = (1-\alpha_n-\beta_n) x_n + \alpha_n f(x_n) + \beta_n T_{r_n}^n x_n, \tag{4.4}$$

whenever $\{\beta_n\}$ and $\{\alpha_n\}$ are real sequences in $(0,1)$ satisfying the conditions (C1), (C2) and (C3) or (C4) along with $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for each i, where $r_n = n \mod N$. The proof and iteration coefficient are simpler and more general than those of Ceng et al. [5, Theorem 3.4] and Ceng et al. [6, Theorem 3.5].

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