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COMMON FIXED POINTS OF GENERALIZED CONTRACTIVE MAPS IN CONE METRIC SPACES

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ABSTRACT. We prove a coincidence and common fixed point theorems of three self mappings satisfying a generalized contractive type condition in cone metric spaces. Our results generalize some well-known recent results.

1. Introduction and preliminaries

Huang and Zhang [3] introduced the concept of cone metric space and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [1, 2, 4-7] studied the existence of fixed points of self mappings satisfying a contractive type condition. Here, we obtain points of coincidence and common fixed points for three self mappings satisfying generalized contractive type condition in a complete normal cone metric space. Our results improve and generalize the results in [1, 3].

A subset P of a real Banach space E is called a *cone* if it has the following properties:

(i) P is non-empty, closed and $P \neq \{\mathbf{0}\}$;
(ii) $0 \leq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$;

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(*iii*) $P \cap (-P) = \{\mathbf{0}\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called *normal* if there is a number $\kappa \geq 1$ such that for all $x,\ y, \in E$,

$$\mathbf{0} \leq x \leq y \ \Rightarrow \|x\| \leq \kappa \|y\|. \tag{1.1}$$

The least number $\kappa \geq 1$ satisfying (1.1) is called the *normal constant* of P.

In the following, we always suppose that E is a real Banach space and P is a cone in E with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $\mathbf{0} \leq d(x,y)$, for all $x, y \in X$ and $d(x,y) = \mathbf{0}$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$, for all $x, y \in X$;
(3) $d(x,y) \leq d(x,z) + d(z,y)$,for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X,d) is called a *cone metric space.*

Let x_n be a sequence in X and $x \in X$. If for each $\mathbf{0} \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be *convergent (or* $\{x_n\}$ converges) to x and x is called the *limit* of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \longrightarrow x$, as $n \to \infty$. If for each $\mathbf{0} \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space.* Let us recall [5] that if P is a normal cone, then $x_n \in X$ converges to $x \in X$ if and only if $d(x_n, x) \to \mathbf{0}$, as $n \to \infty$. Furthermore, $x_n \in X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \mathbf{0}$, as $n, m \to \infty$.

A pair (f, T) of self-mappings on X are said to be weakly compatible if they commute at their coincidence point (i.e., $fTx = Tfx$, whenever $fx = Tx$). A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = fx = Tx$.

2. Main results

We start with a lemma that will be required in the sequel.

Lemma 2.1. *Let X be a non-empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and $(T, f)$are weakly compatible, then S, T and f have a unique common fixed point.*

Proof. Let v be the point of coincidence of S, T and f. Then, $v = fu = Su = Tu$, for some $u \in X$. By weakly compatibility of (S, f) and (T, f) we have,

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv.$$

It implies that $Sv = Tv = fv = w$ (say). Thus, w is a point of coincidence of S, T and f. Therefore, $v = w$ by uniqueness. Hence, v is the unique common fixed point of S, T and f. □

Here, by providing the next result, we state the following generalization of some recent results.

Theorem 2.2. *Let $(X,\ d)$ be a cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $T, f : X \to X$ satisfy:*

$$d(Tx, Ty) \leq \alpha\,[d(fx, Ty) + d(fy, Tx)] + \gamma\ d(fx,\ fy)$$

for all $x, y \in X$, where $\alpha, \gamma \in [0, 1)$ with $2\alpha + \gamma < 1$. Also, suppose that $T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X. Then, T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Corollary 2.3. *Let $(X,\ d)$ be a cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $T,\ f : X \to X$ satisfy:*

$$d(Tx, Ty) \leq \alpha d(fx, Ty) + \beta d(fy, Tx) + \gamma\ d(fx,\ fy), \tag{2.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$. Also, suppose that $T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X. Then, T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Proof. In (2.1) interchanging the roles of x and y and adding the resulting inequality to (2.1), we obtain:

$$d(Tx, Ty) \leq \frac{\alpha + \beta}{2} \left[d(fx, Ty) + d(fy, Tx)\right] + \gamma\, d(fx,\ fy).$$

Now, by using Theorem 2.2 we obtain the required result. □

Corollary 2.4. [1] *Let (X, d) be a cone metric space, P be a normal cone with normal constant κ and the mappings $T,\ f : X \to X$ satisfy:*

$$d(Tx, Ty) \leq \gamma\, d(fx, fy),$$

for all $x, y \in X$, where $0 \leq \gamma < 1$. If $T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Corollary 2.5. [1] *Let (X, d) be a cone metric space, P be a normal cone with normal constant κ and the mappings $T, f : X \to X$ satisfy:*

$$d(Tx, Ty) \leq \alpha \left[d(fx, Ty) + d(fy, Tx)\right],$$

for all $x, y \in X$, where $0 \leq \alpha < \frac{1}{2}$. Also, suppose that $T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X. Then, T and f have a unique point of coincidence. Moreover, if (T, f) are weakly compatible, then T and f have a unique common fixed point.

Here, we further improve Theorem 2.2 as follows.

Theorem 2.6. *Let $(X,\ d)$ be a cone metric space, P be a normal cone with normal constant κ. Suppose the mappings $S, T, f : X \to X$ satisfy:*

(2.2) $$d(Sx, Ty) \leq \alpha d(fx, Ty) + \beta d(fy, Sx) + \gamma d(fx, fy),$$

for all $x, y \in X$, where α, β, γ are non-negative real numbers with

$$\alpha + \beta + \gamma < 1.$$

If $S(X) \cup T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $fx_1 = Sx_0$. Similarly, choose a point x_2 in X such that $fx_2 = Tx_1$.

Continuing this process till having chosen x_n in X, we obtain x_{n+1} in X such that

$$\begin{aligned} fx_{2k+1} &= Sx_{2k} \\ fx_{2k+2} &= Tx_{2k+1}, \ (k \geq 0). \end{aligned}$$

Then,

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\leq \alpha d(fx_{2k}, Tx_{2k+1}) + \beta d(fx_{2k+1}, Sx_{2k}) \\ &\quad + \gamma d(fx_{2k}, fx_{2k+1}) \\ &\leq [\alpha + \gamma] d(fx_{2k}, fx_{2k+1}) + \alpha d(fx_{2k+1}, fx_{2k+2}). \end{aligned}$$

This implies:

$$[1 - \alpha] d(fx_{2k+1}, fx_{2k+2}) \leq [\alpha + \gamma] \ d(fx_{2k}, fx_{2k+1}).$$

Thus,

$$d(fx_{2k+1}, \ fx_{2k+2}) \leq \left[\frac{\alpha + \gamma}{1 - \alpha}\right] d(fx_{2k}, \ fx_{2k+1}).$$

Similarly,

$$\begin{aligned} d(fx_{2k+2}, \ fx_{2k+3}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\leq \alpha d(fx_{2k+2}, Tx_{2k+1}) + \beta d(fx_{2k+1}, Sx_{2k+2}) \\ &\quad + \gamma d(fx_{2k+2}, fx_{2k+1}) \\ &\leq \alpha d(fx_{2k+2}, fx_{2k+2}) + \beta d(fx_{2k+1}, fx_{2k+3}) \\ &\quad + \gamma d(fx_{2k+2}, fx_{2k+1}) \\ &\leq [\beta + \gamma] d(fx_{2k+1}, fx_{2k+2}) + \beta d(fx_{2k+2}, fx_{2k+3}). \end{aligned}$$

Hence,

$$d(fx_{2k+2}, fx_{2k+3}) \leq \left[\frac{\beta + \gamma}{1 - \beta}\right] d(fx_{2k+1}, fx_{2k+2}).$$

Now, by induction, we obtain:

$$\begin{aligned} d(fx_{2k+1}, fx_{2k+2}) &\leq \left[\frac{\alpha+\gamma}{1-\alpha}\right] d(fx_{2k},\ fx_{2k+1}) \\ &\leq \left[\frac{\alpha+\gamma}{1-\alpha}\right]\left[\frac{\beta+\gamma}{1-\beta}\right] d(fx_{2k-1},\ fx_{2k}) \\ &\leq \left[\frac{\alpha+\gamma}{1-\alpha}\right]\left[\frac{\beta+\gamma}{1-\beta}\right]\left[\frac{\alpha+\gamma}{1-\alpha}\right] d(fx_{2k-2},\ fx_{2k-1}) \\ &\leq \ldots \leq \left[\frac{\alpha+\gamma}{1-\alpha}\right]\left(\left[\frac{\beta+\gamma}{1-\beta}\right]\left[\frac{\alpha+\gamma}{1-\alpha}\right]\right)^{k} d(fx_0, fx_1) \end{aligned}$$

and

$$\begin{aligned} d(fx_{2k+2}, fx_{2k+3}) &\leq \left[\frac{\beta+\gamma}{1-\beta}\right]\ d(fx_{2k+1}, fx_{2k+2}) \\ &\leq \ldots \leq \left(\left[\frac{\beta+\gamma}{1-\beta}\right]\ \left[\frac{\alpha+\gamma}{1-\alpha}\right]\right)^{k+1} d(fx_0, fx_1), \end{aligned}$$

for each $k \geq 0$. Let

$$\lambda = \left[\frac{\alpha+\gamma}{1-\alpha}\right],\ \mu = \left[\frac{\beta+\gamma}{1-\beta}\right].$$

Then, $\lambda\mu < 1$. Now, for $p < q$ we have,

$$\begin{aligned} d(fx_{2p+1}, fx_{2q+1}) &\leq d(fx_{2p+1}, fx_{2p+2}) + d(fx_{2p+2}, fx_{2p+3}) \\ &\quad + d(fx_{2p+3}, fx_{2p+4}) + \ \ldots + d(fx_{2q}, fx_{2q+1}) \\ &\leq \left[\lambda \sum_{i=p}^{q-1} (\lambda\mu)^i + \sum_{i=p+1}^{q} (\lambda\mu)^i\right] d(fx_0, fx_1) \\ &\leq \left[\frac{\lambda(\lambda\mu)^p[1-(\lambda\mu)^{q-p}]}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}[1-(\lambda\mu)^{q-p}]}{1-\lambda\mu}\right] d(fx_0, fx_1) \\ &\leq \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1-\lambda\mu}\right] d(fx_0, fx_1) \\ &\leq (1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1) \end{aligned}$$

$$d(fx_{2p}, fx_{2q+1}) \leq (1+\lambda)\left[\tfrac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1),$$

$$d(fx_{2p},\ fx_{2q}) \leq (1+\lambda)\left[\tfrac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1),$$

and

$$d(fx_{2p+1}, fx_{2q}) \leq (1+\mu)\left[\tfrac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right] d(fx_0, fx_1).$$

Hence, for $0 < n < m$, there exists $p < n < m$ such that $p \to \infty$ as $n \to \infty$, and

$$d(fx_n, fx_m) \leq Max\left\{(1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right], (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right]\right\} d(fx_0, fx_1).$$

Since P is a normal cone with normal constant κ, we have,

$$\|d(fx_n, fx_m)\| \leq$$
$$\kappa\left[Max\left\{(1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right], (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right]\right\}\right] \|d(fx_0, fx_1)\|.$$

Thus, if $m, n \to \infty$, then

$$Max\left\{(1+\mu)\left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu}\right], (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right]\right\} \to 0,$$

and so $d(fx_n, fx_m) \to 0$. Hence, $\{fx_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exist $u, v \in X$ such that $fx_n \to v = fu$. Since

$$\begin{aligned} d(fu,\ Su) &\leq d(fu, fx_{2n}) + d(fx_{2n}\ Su) \\ &\leq d(vfx_{2n}) + d(Tx_{2n-1}, Su) \\ &\leq d(v, fx_{2n}) + \alpha d(fu, Tx_{2n-1}) \\ &+ \beta\left[d(fx_{2n-1}, fu) + d(fu, Su)\right] + \gamma d(fu, fx_{2n-1}), \end{aligned}$$

it implies that

$$\begin{aligned} d(fu, Su) &\leq \frac{1}{1-\beta}[d(v, fx_{2n}) + \alpha d(v, fx_{2n}) + \beta d(fx_{2n-1}, v) \\ &\quad + \gamma d(v, fx_{2n-1})] \\ &\leq \frac{1}{1-\beta}\left[(1+\alpha)\, d(v, fx_{2n}) + \beta d(fx_{2n-1}, v) + \gamma d(v, fx_{2n-1})\right]. \end{aligned}$$

Hence,

$$\|d(fu, Su)\| \leq \frac{\kappa}{1-\beta}\left\|(1+\alpha)\, d(v, fx_{2n}) + (\beta+\gamma)\, d(v, fx_{2n-1})\right\|.$$

If $n \to \infty$, then we obtain $\|d(fu, Su)\| = 0$. Hence, $fu = Su$. Similarly, by using the inequality, we have,

$$d(fu, Tu) \leq d(fu, fx_{2n+1}) + d(fx_{2n+1}, Tu).$$

We can show that $fu = Tu$, implying that v is a common point of coincidence of S, T and f; that is,

$$v = fu = Su = Tu.$$

Now, we show that f, S and T have unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^* = Tu^*$, for some u^* in X. Now,

$$\begin{aligned} d(v, v^*) &= d(Su, Tu^*) \\ &\leq \alpha d(fu, Tu^*) + \beta d(fu^*, Su) + \gamma d(fu, fu^*) \\ &\leq (\alpha + \beta + \gamma)\, d(v, v^*). \end{aligned}$$

Hence, $v = v^*$. If (S, f) and (T, f) are weakly compatible, then

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv.$$

It implies that $Sv = Tv = fv = w$ (say). Hence, w is a point of coincidence of S, T and f, and so $v = w$ by uniqueness. Thus, v is the unique common fixed point of S, T and f. □

Example 2.7. Let $X = \{1, 2, 3\}, E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Define $d : X \times X \to E$ as follows:

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y \\ (\frac{5}{7}, 5) & \text{if } x \neq y \text{ and } x, y \in X - \{2\} \\ (1, 7) & \text{if } x \neq y \text{ and } x, y \in X - \{3\} \\ (\frac{4}{7}, 4) & \text{if } x \neq y \text{ and } x, y \in X - \{1\}. \end{cases}$$

Define the mappings $T, f : X \to X$ as follows:

$$T(x) = \begin{cases} 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases} \text{ and } fx = x.$$

Then, $d(T(3), T(2)) = (\frac{5}{7}, 5)$. Now, for $2\alpha + \gamma < 1$, we have,

$$\begin{aligned} &\alpha\,[d(f(3), T(2)) + d(f(2), T(3))] + \gamma d(f(3), f(2)) \\ &= \alpha\,[d(3, T(2)) + d(2, T(3))] + \gamma d(3, 2) \\ &= \gamma(\frac{4}{7}, 4) + \alpha\,[d(3, 3) + d(2, 1)] \\ &= \alpha\,[0 + (1, 7)] + \gamma(\frac{4}{7}, 4) = (\frac{7\alpha + 4\gamma}{7}, 7\alpha + 4\gamma) \\ &< (\frac{8\alpha + 4\gamma}{7}, 8\alpha + 4\gamma) = (\frac{4\,(2\alpha + \gamma)}{7}, 4\,(2\alpha + \gamma)) \\ &< (\frac{4}{7}, 4) < (\frac{5}{7}, 5) = d(T(3), T(2)). \end{aligned}$$

It follows that the mappings T and f do not satisfy the conditions of Theorem 2.2. Hence, Theorem 2.2 and its corellaries 2.3, 2.4 and 2.5 are not applicable here. Now, define the mapping $S : X \to X$ by $Sx = 1$ for all $x \in X$. Then,

$$d(Sx, Ty) = \begin{cases} (0,0) & if\ y \neq 2 \\ (\frac{5}{7}, 5) & if\ y = 2 \end{cases}$$

and

$$\alpha d(fx, Ty) + \beta d(fy, Sx) + \gamma d(fx, fy) = (\frac{5}{7}, 5)$$

if $y = 2$, $\alpha = \gamma = 0$ and $\beta = \frac{5}{7}$. It follows that all conditions of Theorem 2.6 are satisfied for $\alpha = \gamma = 0$, $\beta = \frac{5}{7}$ and one can obtain the unique common fixed point 1 for S, T and f.

3. **Conclusion**

Our results generalized theorems 1 and 4 in [2] and theorems 2.3 and 2.7 in [1].

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