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GORENSTEIN PROJECTIVE OBJECTS IN ABELIAN CATEGORIES

H. CHENG* AND X. ZHU

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ABSTRACT. Let $\mathcal{A}$ be an abelian category with enough projective objects and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. We define Gorenstein projective objects with respect to $\mathcal{X}$ and $\mathcal{Y}_{\mathcal{X}}$, respectively, where $\mathcal{Y}_{\mathcal{X}}$=$\{Y \in Ch(\mathcal{A}) | Y$ is acyclic and $Z_nY \in \mathcal{X}\}$. We point out that under certain hypotheses, these two Gorensein projective objects are related in a nice way. In particular, if $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$, we show that $X \in Ch(\mathcal{A})$ is Gorenstein projective with respect to $\mathcal{Y}_{\mathcal{X}}$ if and only if X^i is Gorenstein projective with respect to $\mathcal{X}$ for each i, when $\mathcal{X}$ is a self-orthogonal class or X is $Hom(-, \mathcal{X})$-exact. Subsequently, we consider the relationships of Gorenstein projective dimensions between them. As an application, if $\mathcal{A}$ is of finite left Gorenstein projective global dimension with respect to $\mathcal{X}$ and contains an injective cogenerator, then we find a new model structure on $Ch(\mathcal{A})$ by Hovey's results in [14].

1. **Introduction**

Let $\mathcal{A}$ be an abelian category with enough projective objects and $\mathcal{X}$ a full subcategory of $\mathcal{A}$. By $\mathcal{P}(\mathcal{A})$ we denote the class of all projective objects of $\mathcal{A}$. Let R denote a non-trivial associative ring with identity. All modules are left R-modules which properties are considered with

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*Corresponding author

respect to the left structures, unless stated otherwise. By $\mathcal{P}(R)$ and $\mathcal{GP}(R)$ we denote the classes of all projective modules and all Gorenstein projective modules, respectively.

Recently, Gorenstein projective objects have received some author's attention in the category of R-modules and its chain complex category $Ch(R)$. Enochs, Jenda and Holm ([7], [13]) introduced and studied the Gorenstein projective R-modules. Subsequently, Enochs and García Rozas [4] investigated Gorenstein projective objects (complexes) of $Ch(R)$. By [4], [6], [7] and [13], there are relationships summarized as the following diagram

$$\begin{array}{ccc} \mathcal{P}(R) & \overset{[6]}{\longleftrightarrow} & \mathcal{P}(Ch(R)) \\ {[7][13]}\updownarrow & & [4]\updownarrow \\ \mathcal{GP}(R) & \overset{[4]}{\longleftrightarrow} & \mathcal{GP}(Ch(R)) \end{array}.$$

Lately, Bennis [3] generalized $\mathcal{X}$-Gorenstein projective modules, where $\mathcal{P}(R) \subseteq \mathcal{X}$. Naturally, we denote $\mathcal{Y}_{\mathcal{X}}$=$\{Y \in Ch(\mathcal{A}) | Y$ is acyclic and $Z_nY \in \mathcal{X}\}$ with respect to $\mathcal{X}$ by the structure of projective objects of $Ch(R)$. In particular, if $\mathcal{P}(R) \subseteq \mathcal{X}$, then $\mathcal{P}(Ch(R)) \subseteq \mathcal{Y}_{\mathcal{X}}$ by [6] and [12]. Furthermore, we define $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective objects of $Ch(\mathcal{A})$ (see Definition 3.1). In this paper, the main purpose is to show whether there are similar relationships as the above diagram in an abelian category with enough projective objects.

In Section 3, we give some characterizations of $\mathcal{X}$-Gorenstein projective objects. One of the main proposes of this section is to consider the $\mathcal{X}$-Gorenstein projective dimension. Moreover, if $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$, there are similar results with respect to $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective objects by [17].

In Section 4, we investigate the relationships between $\mathcal{X}$-Gorenstein projective objects of $\mathcal{A}$ and $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective objects of $Ch(\mathcal{A})$ (e.g., the category of R-modules and its chain complex $Ch(R)$). Let $X \in Ch(\mathcal{A})$. We prove that if X is $Hom(-, \mathcal{X})$-exact or $\mathcal{X}$ is a self-orthogonal class, then X is $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective if and only if X^i is $\mathcal{X}$-Gorenstein projective in $\mathcal{A}$ for each $i \in \mathbb{Z}$. Furthermore, if X is acyclic, the above is equivalent to that Z_iX is $\mathcal{X}$-Gorenstein projective for each i. As an important example, let $\mathcal{X}$ be the class of all Gorenstein projective R-modules and X an acyclic and $Hom(-, \mathcal{X})$-exact complex of projective R-modules, then X is projective in $Ch(R)$. We prove that if $\mathcal{X}$ is a self-orthogonal class, then $\mathcal{Y}_{\mathcal{X}}$-$\mathcal{GP}dim(X) \leq n$ if and only if $\mathcal{X}$-$Gpd(X^i) \leq n$ for each $i \in \mathbb{Z}$, i.e., $\mathcal{Y}_{\mathcal{X}}$-$\mathcal{GP}dim(X)$=Sup$\{\mathcal{X}$-$Gpd(X^i)|i \in \mathbb{Z}\}$.

In Section 5, let $\mathcal{A}$ be a bicomplete abelian category with finite left $\mathcal{X}$-Gorenstein projective global dimension (i.e., $l.\mathcal{X}$-$GPD(\mathcal{A}) < \infty$). If $\mathcal{A}$ admits an injective cogenerator, we show that $(Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))$, $Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))^{\perp})$ and $(Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))\cap\mathcal{E}$, $(Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))\cap\mathcal{E})^{\perp})$ are complete hereditary cotorsion pairs, where $\mathcal{E}$ is the class of all acyclic complexes of $Ch(\mathcal{A})$. Furthermore, by the results in [14], these cotorsion pairs induce a model structure on $Ch(\mathcal{A})$, where the class of cofibrant objects is $Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))$.

2. **Preliminaries**

Throughout this paper, $\mathcal{A}$ is an abelian category with enough projective objects and all subcategories are full subcategories of $\mathcal{A}$. We write $\mathcal{P}$=$\mathcal{P}(\mathcal{A})$ for the subcategory of all projective objects of $\mathcal{A}$.

This work contains some general remarks and terminologies, which will be important for our studies. A sequence of objects of $\mathcal{A}$

$$\cdots \to X^{n-1} \xrightarrow{\delta_X^{n-1}} X^n \xrightarrow{\delta_X^{n-1}} \cdots$$

is called a complex if $\delta_X^n\delta_X^{n-1} = 0$ for each $n \in \mathbb{Z}$. We write (X, δ) or X for this complex. A complex X is *right (left) bounded* if $X^n = 0$ for all $n > k$ $(n < k)$ for some $k \in \mathbb{Z}$, and X is *bounded* if it is left and right bounded. In particular, given an object M, we let $S^n(M)$ denote the complex with all entries 0 except M in degree n. We let $D^n(M)$ denote the complex with all entries 0 except in degrees n-1 and n, with all differentials 0 except $\delta_X^{n-1} = 1_M$.

For a complex X and $n \in \mathbb{Z}$, let

$$Z_nX = Ker(\delta_X^n), B_nX = Im(\delta_X^{n-1}), C_nX = Coker(\delta_X^{n-1}).$$

The nth *homology object* of X is $H^n(X) = Ker(\delta_X^n)/Im(\delta_X^{n-1})$. In particular, a complex X is *acyclic* if $H^n(X) = 0$ for each $n \in \mathbb{Z}$.

A complex X is called $Hom(-,\mathcal{X})$-*exact* if the complex $Hom(X, N)$ is acyclic for each object $N \in \mathcal{X}$. Dually, it is $Hom(\mathcal{X}, -)$-*exact* if the complex $Hom(M, X)$ is acyclic for each object $M \in \mathcal{X}$.

For convenience, we denote the category of all complexes by $Ch(\mathcal{A})$. In particular, if $\mathcal{A}$ is the category of all R-modules, we denote $Ch(\mathcal{A})$

by $Ch(R)$. Obviously, the category $Ch(\mathcal{A})$ is an abelian category with enough projective objects by [12, Proposition 3.2].

Definition 2.1. *Let X and Y be complexes. The $\mathcal{H}om(X,Y)$ denotes the complex with $\mathcal{H}om(X,Y)^n = \prod_{i\in\mathbb{Z}} Hom_{\mathcal{A}}(X^i, Y^{n+i})$ and with differential given by $\delta^n_{\mathcal{H}om(X,Y)}(f) = (\partial_Y^{i+n} f^i - (-1)^n f^{i+1}\partial_X^i)$ for $f = (f^i)_{i\in\mathbb{Z}} \in \mathcal{H}om(X,Y)^n$. A morphism $f : X \to Y$ is an element of $Ker(\partial^0_{\mathcal{H}om(X,Y)})$, and a morphism f is null-homotopic if it is in $Im(\partial^0_{\mathcal{H}om(X,Y)})$.*

It is easy to check that a complex X is $Hom(-,\mathcal{X})$*-exact* if and only if the complex $\mathcal{H}om(X, S^0(M))$ is exact for each $M \in \mathcal{X}$.

Throughout this paper, $Hom(X,Y)$ denotes the set of all morphisms of complexes from X to Y in the category of complexes, and $Ext^i(X,Y)$'s are computed by using the classical injective resolutions of Y or the classical projective resolutions of X, but these extension functors are not the same as those defined in [2].

Definition 2.2. [10, Definition 3.2.1] *A complex D is called Gorenstein projective if there is an exact sequence of complexes*

$$\cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$$

such that

(1) P^i is a projective complex for each i;

(2) $Ker(P^0 \to P^1) = D$;

(3) the sequence remains exact when $Hom(-,P)$ is applied to it for any projective complex P.

Definition 2.3. *If $\mathcal{F}$ is a class of objects of $\mathcal{A}$, ${}^\perp\mathcal{F}$ will denote the class of objects $M \in \mathcal{A}$ such that $Ext^1_{\mathcal{A}}(M,F) = 0$ for all $F \in \mathcal{F}$ and $\mathcal{F}^\perp$ will denote the class of objects $N \in \mathcal{A}$ such that $Ext^1_{\mathcal{A}}(F,N) = 0$ for all $F \in \mathcal{F}$. In particular, we call the class $\mathcal{F}$ self-orthogonal if $Ext^{\geq 1}_{\mathcal{A}}(M,N) = 0$ for all $M, N \in \mathcal{F}$.*

Definition 2.4. [1, 3.11] *Let $\mathcal{F}$ be a class of objects of $\mathcal{A}$. We call $\mathcal{F}$ projective resolving if the following conditions are satisfied:*

(1) $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{F}$;

(2) Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{A}$ and $X'' \in \mathcal{F}$. Then $X \in \mathcal{F}$ if and only if $X' \in \mathcal{F}$.

Definition 2.5. [10] *Let $\mathcal{F}$ be a class of objects of $\mathcal{A}$ and $X \in \mathcal{A}$. We say that a morphism $\phi : X \to F$ is an $\mathcal{F}$-preenvelope if for any morphism $\psi : X \to F'$ with $F' \in \mathcal{F}$ there is a morphism $\tau : F \to F'$*

such that $\tau\phi = \psi$. Moreover, if any $f : F \to F$ such that $\phi = f\phi$ is an automorphism of F, then $\phi : X \to F$ is called an $\mathcal{F}$-envelope.

Dually, we have the concepts of $\mathcal{F}$-precover and $\mathcal{F}$-cover.

By [11, Lemma 3.1] and [12, Lemma 4.2], there are a few common and useful connection relationships between an abelian category $\mathcal{A}$ and its chain complex category $Ch(\mathcal{A})$ as follow:

Lemma 2.6. *Let $\mathcal{A}$ be an abelian category. For any object $A \in \mathcal{A}$ and chain complex $X \in Ch(\mathcal{A})$, we have monomorphisms and isomorphisms*

$$Ext^1_{\mathcal{A}}(A, Z_nX) \hookrightarrow Ext^1(S^n(A), X), Ext^1_{\mathcal{A}}(A, X^n) \cong Ext^1(D^n(A), X);$$

$$Ext^1_{\mathcal{A}}(X^n/B_nX, A) \hookrightarrow Ext^1(X, S^n(A)), Ext^1_{\mathcal{A}}(X^n, A) \cong Ext^1(X, D^n(A)).$$

In particular, if X is acyclic, then these monomorphisms are actually isomorphic for each $n \in \mathbb{Z}$.

3. Gorenstein projective objects in abelian categories

In this section, let $\mathcal{A}$ be an abelian category with enough projective objects and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. We give a detailed treatment of Gorenstein projective objects with respect to $\mathcal{X}$. As an important example in $Ch(\mathcal{A})$, we can get similar conclusions of Gorenstein projective objects with respect to a class of objects of $Ch(\mathcal{A})$, when this class contains all projective objects of its chain complex category $Ch(\mathcal{A})$.

Recall that a complex P is a *projective object* of $Ch(\mathcal{A})$ if the functor $Hom(P, -)$ is exact. It is also equivalent to that: (1) P is contractible; (2) P^i is projective in $\mathcal{A}$ for each $i \in \mathbb{Z}$. According to [8], we can say P is projective if and only if P is acyclic and Z_nP is projective in $\mathcal{A}$ for each $n \in \mathbb{Z}$. As this point, we denote $\mathcal{Y}_{\mathcal{X}}$=$\{Y \in Ch(\mathcal{A}) | Y$ is acyclic and $Z_nY \in \mathcal{X}\}$. Obviously, $\mathcal{Y}_{\mathcal{X}}$ contains all projective objects of $Ch(\mathcal{A})$ provided that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$. Hence, we can uniformly define the Gorenstein projective objects with respect to a subcategory of $\mathcal{A}$ (e.g., subcategory $\mathcal{X}$ of $\mathcal{A}$ and subcategory $\mathcal{Y}_{\mathcal{X}}$ of $Ch(\mathcal{A})$). For convenience, we only simply deal with the case of $\mathcal{X}$-Gorenstein projective objects of $\mathcal{A}$.

Now, we give the definition of the $\mathcal{X}$-Gorenstein projective object in an abelian category $\mathcal{A}$, where $\mathcal{A}$ has enough projective objects, as follow:

Definition 3.1. *Let $\mathcal{X}$ be a full subcategory of an abelian category $\mathcal{A}$. An object M is called $\mathcal{X}$-Gorenstein projective if there exists an exact* *sequence of projective objects*

$$\boldsymbol{P} : \cdots \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$$

such that $M \cong Ker(P^0 \to P^1)$ *and* $\boldsymbol{P}$ *is* $Hom(-, \mathcal{X})$*-exact.*

The sequence $\boldsymbol{P}$ *is called a complete projective resolution of* M*. In particular, we denote by* $\mathcal{X}$*-*$\mathcal{GP}(\mathcal{A})$ *the class of all* $\mathcal{X}$*-Gorenstein projective objects.*

Proposition 3.2. *If* C *is a* $\mathcal{X}$*-Gorenstein projective object, then* $Ext^i(C, X) = 0$ *for each* $X \in \mathcal{X}$ *and* $i \geq 1$*.*

Proof. It is trivial by Definition 3.1. □

Remark 3.3. *(1) Obviously,* $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$*-*$\mathcal{GP}(\mathcal{A})$*.*

(2) Let $\mathcal{A}$ *be the category of* R*-modules and* $\mathcal{X}$=$\mathcal{P}(R)$*. We have that the* $\mathcal{X}$*-Gorenstein projective objects coincide with Gorenstein projective modules and the* $\mathcal{Y}_{\mathcal{X}}$*-Gorenstein projective complexes are Gorenstein projective complexes.*

(3) If $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$*, we have the following relations:*

$\{$*projective objects*$\} \subseteq \{\mathcal{X}$*-Gorenstein projective objects*$\} \subseteq \{$*Gorenstein projective objects*$\}$*.*

(4) If $\boldsymbol{P}$ *is a complete projective resolution of* M*, then kernels and images of all differentials of* $\boldsymbol{P}$ *are* $\mathcal{X}$*-Gorenstein projective by Proposition 3.2.*

(5) By Definition 3.1, if $X \in \mathcal{A}$*, then* X *is* $\mathcal{X}$*-Gorenstein projective if and only if* $D^n(X)$ *is* $\mathcal{Y}_{\mathcal{X}}$*-Gorenstein projective for each* n*. However, if* X *is* $\mathcal{X}$*-Gorenstein projective,* $S^n(X)$ *is not necessarily* $\mathcal{Y}_{\mathcal{X}}$*-Gorenstein projective for any* n*. Next, we show that there exists a Gorenstein projective object which is not* $\mathcal{X}$*-Gorenstein projective, that is,* $\mathcal{GP}(\mathcal{A}) \subsetneqq \mathcal{X}$*-*$\mathcal{GP}(\mathcal{A})$*.*

Example 3.4. *Consider the quasi-Frobenius local ring* $R = k[X]/(X^2)$ *where* k *is a field, and denote by* $\overline{X}$ *the residue class in* R *of* X*. Let* $\mathcal{X}$ *be the class of* R*-modules. Then* $(\overline{X})$ *is Gorenstein projective which is not* $\mathcal{X}$*-Gorenstein projective. Similarly, so is* $D^n((\overline{X}))$ *for each* n*.*

Proof. It is trivial by [16, Example 3.7], Lemma 2.6 and Remark 3.3(5). □

In the following, let $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$ (see 1087, line 3). We set out to investigate $\mathcal{X}$-Gorenstein projective objects in an abelian category $\mathcal{A}$.

Theorem 3.5. *The class of all $\mathcal{X}$-Gorenstein projective objects is projective resolving. Furthermore, it is closed under arbitrary direct sums and under direct summands.*

Proof. The proof is similar to the proof of [13, Theorem 2.5]. □

Lemma 3.6. [17, Lemma 1.1] *Let $\mathcal{B}$ be a projective resolving class (not necessarily closed under direct summands) of $\mathcal{A}$. For any object $C \in \mathcal{A}$, if*

$$0 \to K^n \to G^{n-1} \to \cdots \to G^1 \to G^0 \to C \to 0$$

and

$$0 \to T^n \to T^{n-1} \to \cdots \to T^1 \to T^0 \to C \to 0$$

are exact, where $G^0, \cdots, G^{n-1}$ and $T^0, \cdots, T^{n-1}$ are in $\mathcal{B}$, then K^n is in $\mathcal{B}$ if and only if T^n is in $\mathcal{B}$.

As this point, we introduce the $\mathcal{X}$-Gorenstein projective dimension as follows:

Definition 3.7. *We say that an object X has a $\mathcal{X}$-Gorenstein projective dimension less than or equal to n, denoted by $\mathcal{X}$-$\mathcal{GP}dim(X) \leq n$, if there exists an exact sequence*

$$0 \to G^n \to G^{n-1} \to \cdots \to G^1 \to G^0 \to X \to 0,$$

where G^i is $\mathcal{X}$-Gorenstein projective for each $i = 0, 1, \cdots, n$. If no such finite sequence exists, define $\mathcal{X}$-$\mathcal{GP}dim(X) = \infty$. Otherwise, if n is the least such integer, define $\mathcal{X}$-$\mathcal{GP}dim(X)$=n.

Obviously, if $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$, then $\mathcal{GP}dim(X) \leq \mathcal{X}$-$\mathcal{GP}dim(X) \leq pd(X)$ for each object X of $\mathcal{A}$.

Similarly, we can define $\mathcal{X}$-$pd(X)$. We also find $\mathcal{X}$-$pd(X) \leq pd(X)$.

Lemma 3.8. *Let X be a $\mathcal{X}$-Gorenstein projective object, then $Ext^i(X, F) = 0$ for all $i \geq 1$ and $\mathcal{X}$-$pd(F) < \infty$.*

Proof. Suppose $\mathcal{X}$-pd$(F) \leq n$ for some $n \in \mathbb{N}$. We have the exact sequence $0 \to F^n \to F^{n-1} \to \cdots \to F^1 \to F^0 \to F \to 0$, where $F^i \in \mathcal{X}$ for each $i = 0, 1, \cdots n$. Applying the functor $Hom(X, -)$ to the above sequence, we get $Ext^i(X, F) \cong Ext^{i+n}(X, F^n) = 0$ by dimension shifting and Proposition 3.2 for all $i, n \geq 1$. □

Proposition 3.9. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ with $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$. If X is an object with $\mathcal{X}$-$\mathcal{GP}dim(X) \leq n$, then X admits a surjective $\mathcal{X}$-Gorenstein projective precover, where the projective dimension of its kernels is less than or equal to n-1.*

Proof. By analogy with the proof of [13, Theorem 2.10]. □

Theorem 3.10. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ with $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$, X an object with finite $\mathcal{X}$-Gorenstein projective dimension and n a positive integer. Then the following conditions are equivalent:*

(1) $\mathcal{X}$-$\mathcal{GP}dim(X) \leq n$.

(2) $Ext^i(X, F) = 0$ for all $i > n$, and all F with $\mathcal{X}$-$pd(F) < \infty$.

(3) $Ext^i(X, F) = 0$ for all $i > n$, and all $F \in \mathcal{X}$.

(4) For every exact sequence $0 \to K^n \to G^{n-1} \to \cdots \to G^1 \to G^0 \to X \to 0$ with G^i being $\mathcal{X}$-Gorenstein projective for all $i = 0, 1, \cdots, n-1$, we have that K^n is $\mathcal{X}$-Gorenstein projective.

Consequently, the $\mathcal{X}$-Gorenstein projective dimension of X is determined by the formulas: $\mathcal{X}$-$\mathcal{GP}dim(X)$=$Sup\{i \mid \exists F \in \mathcal{X} : Ext^i(X, F) \neq 0\}$.

Proof. By Theorem 3.5 and [17, Theorem 3.1]. □

Proposition 3.11. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ with $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$ and X an object of $\mathcal{A}$ with finite projective dimension. Then $\mathcal{GP}dim(X)$ $=\mathcal{X}$-$\mathcal{GP}dim(X) = pd(X)$.*

Proof. We only to show that $\mathcal{GP}dim(X)$=$pd(X)$ by Definition 3.7.

Suppose $pd(X) = n$. Then $\mathcal{GP}dim(X) \leq pd(X) = n$.

Next, we only show that $\mathcal{GP}dim(X) = n$. Assume $\mathcal{GP}dim(X) = m < n$. There are exact sequences

$$0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^{-1} \to P^0 \to X \to 0$$

and

$$0 \to G^{-m} \to G^{-m+1} \to \cdots \to G^{-1} \to G^0 \to X \to 0,$$

where P^{-i} is projective and G^{-j} is $\mathcal{X}$-Gorenstein projective for $i = 0, 1, \cdots, n; j = 0, 1, \cdots, m$. Let $K^{-i} = Ker(P^{-i+1} \to P^{-i+2})$. Then K^{-m} is Gorenstein projective by Lemma 3.6 and $Ext^1(K^{-m-1}, K^{-m}) = 0$, since $\mathcal{X}$-$pd(K^{-m-1}) \leq pd(K^{-m-1}) < \infty$. So K^{-m} is a direct summand of P^{-i}, i.e., K^{-m} is projective. Thus this is contradiction to the assumption. □

Remark 3.12. *Through the above analysis, we can obtain similar properties of $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective objects of $Ch(\mathcal{A})$, since there are enough projective objects in $Ch(\mathcal{A})$. In particular, if $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$, then $\mathcal{Y}_{\mathcal{X}}$ contains all projective objects of $Ch(\mathcal{A})$.*

4. The relationships between Gorenstein projective objects of $\mathcal{A}$ and those of $Ch(\mathcal{A})$

Let $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$. In this section, we consider the relationships between $\mathcal{X}$-Gorenstein projective objects of $\mathcal{A}$ and $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective objects (complexes) of $Ch(\mathcal{A})$. Furthermore, given an object $X \in Ch(\mathcal{A})$, we provide relationships between $\mathcal{Y}_{\mathcal{X}}$-$\mathcal{GP}dim(X)$ and $\mathcal{X}$-$\mathcal{GP}dim(X^i)$ for each $i \in \mathbb{Z}$.

To begin this section, first, we recall that a *continuous chain* of subcomplexes (resp., submodules) of a given complex (resp., module) C is a set of subcomplexes (resp., submodules) of C, $\{C_\alpha | \ \alpha < \lambda\}$ (for some ordinal number λ), such that C_α is a subcomplex (resp., submodule) of C_β for all $\alpha \leq \beta < \lambda$, and that $C_\gamma = \sum_{\alpha<\gamma} C_\alpha$ whenever $\gamma < \lambda$ is a limit ordinal.

Next, we need the following two important lemmas:

Lemma 4.1. [9, Lemma 1] *Suppose that $A = A_\mu$ is the union of a continuous chain of submodules $A = \bigcup_{\alpha<\mu} A_\alpha$, such that $Ext(A_0, C) = 0$ and $Ext(A_{\alpha+1}/A_\alpha, C) = 0$ for all $\alpha + 1 < \mu$. Then $Ext(A, C) = 0$.*

Lemma 4.2. [9, Lemma 17] *Let $(A_\alpha \mid \alpha + 1 \leq \mu)$ be a sequence of R-modules and $(f_{\alpha\beta} \mid \alpha \leq \beta \leq \mu)$ a sequence of monomorphisms such that $\{(A_\alpha, f_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ is a direct system which is continuous. Let C be an R-module such that $Ext(A_{\alpha+1}/f_{\alpha,\alpha+1}(A_\alpha), C) = 0$ for all $\alpha + 1 \leq \mu$. Then $Ext(A_\mu, C) = 0$.*

Notations (1) By the proofs of [9, Lemma 1] and [9, Lemma 17], there are similar results in abelian categories with enough projective objects.

(2) The dual of Lemma 4.2 also holds.

Lemma 4.3. *Let X be a right bounded complex with $X^i \in {}^\perp\mathcal{X}$ for all $i \in \mathbb{Z}$. Then $Ext^1(X, F) = 0$ for all $F \in \mathcal{Y}_{\mathcal{X}}$.*

Proof. Without loss of generality, we may assume that

$$X : \cdots \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to 0.$$

For each $n \in \mathbb{N}$, let

$$X(n) : 0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to 0.$$

There is an exact sequence of complexes $0 \to X(n) \to X(n+1) \to S^{-n-1}(X^{-n-1}) \to 0$, where $X(n) \to X(n+1)$ is a natural inclusion morphism which we denote by $f_{n,n+1}$. Setting $f_{i,j} = f_{j,j-1} f_{j-1,j-2} \cdots f_{i+1,i}$

for all $0 < i \leq j$, we conclude that $\{(X(n), f_{i,j}) \mid i \leq j\}$ is a direct system which is continuous and $X = \varinjlim X(n)$.

Let $F \in \mathcal{Y}_{\mathcal{X}}$. Then $Z_n F \in \mathcal{X}$ for each $n \in \mathbb{Z}$. By Lemma 2.6, we have

$$Ext^1(S^{-n-1}(X^{-n-1}), F) \cong Ext^1(X^{-n-1}, Z_{-n-1}F) = 0,$$

since $X^{-n-1} \in {}^{\perp}\mathcal{X}$. Then $Ext^1(X, F) = 0$ by Lemma 4.2. □

Proposition 4.4. *Let X be a complex of $\mathcal{X}$-Gorenstein projective objects. If X is $Hom(-, \mathcal{X})$-exact, then $Ext^1(X, F) = 0$ for all $F \in \mathcal{Y}_{\mathcal{X}}$.*

Proof. Let $X : \cdots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to \cdots$ be a complex and $F \in \mathcal{Y}_{\mathcal{X}}$, where X^i is $\mathcal{X}$-Gorenstein projective for each i. So $X^i \in {}^{\perp}\mathcal{X}$. For each $n \in \mathbb{N}$, consider

$$X(n) : \cdots \to X^{-1} \to X^0 \to \cdots \to X^n \to B_{n+1}X \to 0.$$

We have $X(n) \subseteq X(n+1)$ and $X(n+1)/X(n) : \cdots \to 0 \to C_{n+1}X \to B_{n+2}X \to 0 \to \cdots$. Since X is $Hom(-, \mathcal{X})$-exact and X^i is $\mathcal{X}$-Gorenstein projective for each i, we have $C_iX \in {}^{\perp}\mathcal{X}$. Note that the sequence $0 \to B_iX \to X^i \to C_iX \to 0$ is exact for each i and ${}^{\perp}\mathcal{X}$ is projective resolving, then $B_iX \in {}^{\perp}\mathcal{X}$. Thus, we have $Ext^1(X(n+1)/X(n), F) = 0$ by Lemma 4.3 for each $n \in \mathbb{N}$. Furthermore, $X(0)$ is a right bounded complex with $X^i \in {}^{\perp}\mathcal{X}$ for all $i \in \mathbb{Z}$, so $Ext^1(X(0), F) = 0$ by Lemma 4.3.
Obviously, X is the union of the continuous chain of complexes $(X(n))_{n\in\mathbb{N}}$. So we conclude $Ext^1(X, F) = 0$ for all $F \in \mathcal{Y}_{\mathcal{X}}$ by Lemma 4.1. □

Proposition 4.5. *Let X be a complex of $\mathcal{X}$-Gorenstein projective objects. If X is $Hom(-, \mathcal{X})$-exact, then X admits a $\mathcal{Y}_{\mathcal{X}}$-preenvelope. In this case, there is an exact sequence $0 \to X \to G \to T \to 0$ such that G is projective and T is a complex of $\mathcal{X}$-Gorenstein projective objects with $Hom(-, \mathcal{X})$-exact.*

Proof. Let $X : \cdots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to \cdots$ be a complex with X^i being $\mathcal{X}$-Gorenstein projective for each i. By Definition 3.1, X^i admits an $\mathcal{X}$-preenvelope. Assume that $f^i : X^i \to P^i$ is a $\mathcal{X}$-preenvelope of X^i, where f^i is a monomorphism and P^i is projective for each i.

Consider the complex $G : \cdots \to P^{i-1} \oplus P^i \to P^i \oplus P^{i+1} \to P^{i+1} \oplus P^{i+2} \to \cdots$. It induces a morphism $g = (g^i)_{i\in\mathbb{Z}}$ from X to G as follows:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^{i-1} & \xrightarrow{d_X^{i-1}} & X^i & \xrightarrow{d_X^i} & X^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle g^{i-1}} & & \downarrow{\scriptstyle g^i} & & \downarrow{\scriptstyle g^{i+1}} & & , \\ \cdots & \longrightarrow & P^{i-1}\oplus P^i & \xrightarrow{d_G^{i-1}} & P^i\oplus P^{i+1} & \xrightarrow{d_G^i} & P^{i+1}\oplus P^{i+2} & \longrightarrow & \cdots \end{array}$$

where $g^i = (f^i, f^{i+1}d_X^i)$ and $d_P^i(x,y) = (y,0)$ for $(x,y) \in P^i \oplus P^{i+1}$. It is not hard to check that g is a monomorphism, $g^i : X^i \to P^i \oplus P^{i+1}$ is a $\mathcal{X}$-preenvelope of X^i for each i and $G \in \mathcal{Y}_{\mathcal{X}}$.

In the following, we conclude that $g : X \to G$ is a $\mathcal{Y}_{\mathcal{X}}$-preenvelope of X. Setting $T = Coker g$, we have that $T^i \in {}^{\perp}\mathcal{X}$ and $\mathcal{X}$-$\mathcal{GP}dim(T^i) \leq 1$ for each $i \in \mathbb{Z}$, since $g^i : X^i \to P^i \oplus P^{i+1}$ is a $\mathcal{X}$-preenvelope of X^i. So there is a split sequence $0 \to P \to M \to T^i \to 0$, where P is projective and M is $\mathcal{X}$-Gorenstein projective. Thus, T^i is $\mathcal{X}$-Gorenstein projective by Theorem 3.5. Furthermore, we have $Ext^1(T^i, F) = 0$ for any $F \in \mathcal{X}$ by Proposition 3.2. If we apply the functor $Hom(-, F)$ to the sequence $0 \to X \to G \to T \to 0$, we have an exact sequence of complexes $0 \to \mathcal{H}om(T, S^0(F)) \to \mathcal{H}om(G, S^0(F)) \to \mathcal{H}om(X, S^0(F)) \to 0$ by Definition 3.1, since Hom is a bifunctor. By hypothesis, $\mathcal{H}om(X, S^0(F))$ and $\mathcal{H}om(G, S^0(F))$ are exact, so is $\mathcal{H}om(T, S^0(F))$ (that is, T is $Hom(-, \mathcal{X})$-exact). Then we have $Ext^1(T, C) = 0$ for all $C \in \mathcal{Y}_{\mathcal{X}}$ by Proposition 4.4. Thus, $g : X \to G$ is a $\mathcal{Y}_{\mathcal{X}}$-preenvelope of X. By the above proof, we have that there is an exact sequence $0 \to X \to G \to T \to 0$ such that G is projective and T is a complex of $\mathcal{X}$-Gorenstein projective objects with $Hom(-, \mathcal{X})$-exact. □

Corollary 4.6. *Let X be a complex of $\mathcal{X}$-Gorenstein projective objects. If X is $Hom(-, \mathcal{X})$-exact, then X admits an exact sequence of complexes $0 \to X \to P^0 \to P^1 \to \cdots \to P^n \to K^{n+1} \to 0$, where P^i is projective and K^{n+1} is a complex of $\mathcal{X}$-Gorenstein projective objects with $Hom(-, \mathcal{X})$-exact for each $n \geq 0$.*

Proof. By Proposition 4.5. □

Theorem 4.7. *Let X be a complex of $Ch(\mathcal{A})$ which is $Hom(-, \mathcal{X})$-exact. Then the following conditions are equivalent:*

(1) X is a $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective complex.

(2) There is an exact sequence $0 \to X \to P^0 \to C^0 \to 0$, where P^0 is a projective complex and C^0 is a complex of $\mathcal{X}$-Gorenstein projective objects which is $Hom(-, \mathcal{X})$-exact.

(3) X^i is $\mathcal{X}$-Gorenstein projective for each $i \in \mathbb{Z}$.

Furthermore, if X is acyclic, then each of the above statements is equivalent to the following:

(4) Z_iX is $\mathcal{X}$-Gorenstein projective for each $i \in \mathbb{Z}$.

Proof. (3)⇒(2) is trivial by Corollary 4.6.

(1)⇒(3) Let X be $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective. Then there is an exact sequence of projective complexes

$$\mathbf{P} : \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

such that $X \cong Ker(P_0 \to P_{-1})$ and $Hom(\mathbf{P}, Y)$ is exact whenever $Y \in \mathcal{Y}_{\mathcal{X}}$. For each $n \in \mathbb{Z}$, we have an exact sequence

$$\mathbf{P}^n : \cdots \to {P_1}^n \to {P_0}^n \to {P_{-1}}^n \to {P_{-2}}^n \to \cdots$$

such that $X^n \cong Ker({P_0}^n \to {P_{-1}}^n)$, where P_i^n denotes the object in the nth row and the ith column of $\mathbf{P}$. Furthermore, if $N \in \mathcal{X}$, then $D^n(N) \in \mathcal{Y}_{\mathcal{X}}$. So we have $Ext^1(X^i, N) \cong Ext^1(X, D^i(N)) = 0$ for each $i \in \mathbb{Z}$. Thus the complex $Hom(\mathbf{P}^n, N)$ is exact. So X^n is $\mathcal{X}$-Gorenstein projective for each $n \in \mathbb{Z}$.

(2)⇒(1) Let $F \in \mathcal{Y}_{\mathcal{X}}$. Applying the functor $Hom(-, F)$ to the sequence $0 \to X \to P^0 \to C^0 \to 0$, we conclude that P^0 is a $\mathcal{Y}_{\mathcal{X}}$-preenvelope of X, since $Ext^1(C^0, F) = 0$ by Proposition 4.4. Moreover, we get the exact sequence $0 \to C^n \to P^{n+1} \to C^{n+1} \to 0$, where P^{n+1} is a projective $\mathcal{Y}_{\mathcal{X}}$-preenvelope of C^n and C^{n+1} is a complex of $\mathcal{X}$-Gorenstein projective objects with $Hom(-, \mathcal{X})$-exact by Propositions 4.4 and 4.5 for each $n \in \mathbb{N}$. Assembling these sequences, we have the long exact sequence $\mathbf{Q} : 0 \to X \to P^0 \to P^1 \to \cdots$, where each P^{n+1} is a $\mathcal{Y}_{\mathcal{X}}$-preenvelope of C^n. So $Hom(\mathbf{Q}, \mathcal{Y}_{\mathcal{X}})$ is exact. However, the class $\mathcal{X}$-$\mathcal{GP}(\mathcal{A})$ is projective resolving by Theorem 3.5. So each term of X is $\mathcal{X}$-Gorenstein projective by the sequence $0 \to X \to P^0 \to C^0 \to 0$, where each term of P^0 and each term of C^0 are $\mathcal{X}$-Gorenstein projective. Note that X is $Hom(-, \mathcal{X})$-exact, since P^0 and C^0 are $Hom(-, \mathcal{X})$-exact. Then $Ext^1(X, F) = 0$ by Proposition 4.4. Furthermore, the category $Ch(\mathcal{A})$ has enough projective objects, there is an exact sequence $0 \to K^{-1} \to P^{-1} \to X \to 0$ with P^{-1} being a projective complex. Similar to the discussion of X, we know that K^{-1} is a complex of $\mathcal{X}$-Gorenstein projective objects. Moreover, since P^{-1} and X are $Hom(-, \mathcal{X})$-exact, K^{-1} is also $Hom(-, \mathcal{X})$-exact. By applying Proposition 4.4 to K^{-1}, we have $Ext^1(K^{-1}, F) = 0$. If we continue in this manner, we have exact sequences $0 \to K^{-i-1} \to P^{-i-1} \to K^{-i} \to 0$

such that each P^{-i} is projective and each K^{-i} is a complex of $\mathcal{X}$-Gorenstein projective objects with $Ext^1(K^{-i}, F) = 0$ for all $i > 0$. Connecting these short exact sequences, we can obtain the exact sequence $\cdots \to P^{-2} \to P^{-1} \to X \to 0$ which is $Hom(-, \mathcal{Y}_{\mathcal{X}})$-exact. Hence, X is $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective.

(4)⇒(3) Is obvious.

(1)⇒(4) Let X be a $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective complex. Then there is an exact sequence of projective complexes $\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ such that $X \cong Ker(P^0 \to P^1)$. So $Ker(P^n \to P^{n+1})$ is acyclic and $Hom(-, \mathcal{X})$-exact for each n, since X is acyclic and X^i is $\mathcal{X}$-Gorenstein projective for each i. Thus there is an exact sequence of projective objects for each i

$$\cdots \to Z_iP^{-1} \to Z_iP^0 \to Z_iP^1 \to \cdots$$

such that $Z_iX \cong Ker(Z_iP^0 \to Z_iP^1)$. However, X is $Hom(-, \mathcal{X})$-exact, $Ext^1(Z_iX, M)$=0 for all $M \in \mathcal{X}$. Similarly, we can show that $Ext^1(Ker(Z_iP^n \to Z_iP^{n+1}), M)$=0. Hence, the above sequence is $Hom(-, \mathcal{X})$-exact, i.e., Z_iX is $\mathcal{X}$-Gorenstein projective for each $i \in \mathbb{Z}$. □

Example 4.8. *Let $\mathcal{X}$ be the class of all Gorenstein projective R-modules. By Definition 3.1, every $\mathcal{X}$-Gorenstein projective R-module is projective. Let X be a complex of R-modules with $Hom(-, \mathcal{X})$-exact. Then the following conditions are equivalent:*

(1) X is a $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective complex.

(2) X is a complex of projective R-modules.

In particular, if X is acyclic, then each of the above statements is equivalent to the following:

(3) X is a projective complex.

Remark 4.9. *(1) If we remove the condition "$Hom(-, \mathcal{X})$-exact" in Theorem 4.7, (1)⇒(3) always holds.*

(2) Let $\mathcal{X}$ be the class of all projective objects. Then (1), (2) and (3) are equivalent in Theorem 4.7 without the condition "$Hom(-, \mathcal{X})$-exact". Since every $Y \in \mathcal{Y}_{\mathcal{X}}$ is a direct sum (product) of complexes of the form $\{D^n(P) | P \in \mathcal{P}(\mathcal{A})\}$.

In order to formalize the above property, we have the following results:

Lemma 4.10. *Let $\mathcal{X}$ be a self-orthogonal class of objects of $\mathcal{A}$. Then every complex $F \in \mathcal{Y}_{\mathcal{X}}$ is a direct sum (product) of complexes of the form $D^n(X)$ with $X \in \mathcal{X}$ and $n \in \mathbb{Z}$.*

Proof. Let $F \in \mathcal{Y}_{\mathcal{X}}$. Then there is an exact sequence $0 \to Z_nF \to F^n \to Z_{n+1}F \to 0$ for each $n \in \mathbb{Z}$, where $Z_nF, Z_{n+1}F \in \mathcal{X}$. So the sequence splits by hypothesis, that is, $F^n \cong Z_nF \oplus Z_{n+1}F$. Hence, $F \cong \oplus D^n(Z_n(F))$, as desired. □

Lemma 4.11. *Let $\mathcal{X}$ be a self-orthogonal class of objects of $\mathcal{A}$ and X a complex in $Ch(\mathcal{A})$. Then the following conditions are equivalent:*

(1) X is $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective.

(2) X^i is $\mathcal{X}$-Gorenstein projective for each $i \in \mathbb{Z}$.

In particular, if X is acyclic and $Hom(-, \mathcal{X})$-exact, then each of the above statements is equivalent to the following:

(3) Z_iX is $\mathcal{X}$-Gorenstein projective for each i.

Proof. Since $\mathcal{X}$ is a self-orthogonal class of objects of $\mathcal{A}$, by Lemma 4.10, we have $Ext^1(X, F) = Ext^1(X, \prod D^n(Z_nF)) \cong \prod Ext^1(X, D^n(Z_nF)) \cong \prod Ext^1(X^n, Z_nF)$ for any $F \in \mathcal{Y}_{\mathcal{X}}$. In particular, if we remove the condition $''Hom(-, \mathcal{X})$-exact$''$, then Lemma 4.3, Propositions 4.4 and 4.5 are still true by Lemma 2.6. Thus (1) and (2) are equivalent. Furthermore, when X is acyclic and $Hom(-, \mathcal{X})$-exact, then (1)⇔(3) by Theorem 4.7. □

Example 4.12. *Let $\mathcal{X} = \mathcal{P}(R)$. Obviously, $\mathcal{X}$ is a self-orthogonal class and $\mathcal{Y}_{\mathcal{X}}$ is the class of all projective complexes. By Lemma 4.11, the following conditions are equivalent:*

(1) X is a Gorenstein projective complex.

(2) X^i is a Gorenstein projective R-module for each $i \in \mathbb{Z}$.

Furthermore, if X is acyclic with $Hom(-, \mathcal{X})$-exact, then each of the above statements are equivalent to the following:

(3) Z_iX is a Gorenstein projective R-module for each i.

Next, we consider the relationships between $\mathcal{Y}_{\mathcal{X}}$-Gorenstein homological dimension of complexes of $Ch(\mathcal{A})$ and $\mathcal{X}$-Gorenstein homological dimension of objects of $\mathcal{A}$.

Theorem 4.13. *Let $\mathcal{X}$ be a self-orthogonal class of objects of $\mathcal{A}$ and X a complex of $Ch(\mathcal{A})$ with finite $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective dimension. Then the following conditions are equivalent for $n \geq 0$:*

(1) $\mathcal{X}$-$Gpd(X^i) \leq n$ for all $i \in \mathbb{Z}$.

(2) $Ext^j_{\mathcal{A}}(X^i, G) = 0$ for all $i \in \mathbb{Z}$, $j > n$ and G with $\mathcal{X}$-$pd(G) < \infty$.

(3) $Ext^j_{\mathcal{A}}(X^i, F) = 0$ for all $i \in \mathbb{Z}$, $j > n$ and $F \in \mathcal{X}$.

(4) $\mathcal{Y}_{\mathcal{X}}$-$\mathcal{GP}dim(X) \leq n$.

(5) For each exact sequence $\cdots \to G_j \to G_{j-1} \to \cdots \to G_0 \to X \to 0$, *where* G_j *is* $\mathcal{Y}_{\mathcal{X}}$*-Gorenstein projective for each* $j \in \mathbb{N}$, *we have that* $Ker(G_j \to G_{j-1})$ *is* $\mathcal{Y}_{\mathcal{X}}$*-Gorenstein projective for each* $j \geq n-1$.

(6) $Ext^j(X, \widehat{G}) = 0$ *for all* $j > n$, *and all* $\widehat{G}$ *with* $\mathcal{Y}_{\mathcal{X}}$*-pd*$(\widehat{G}) < \infty$.

(7) $Ext^j(X, \widehat{F}) = 0$ *for all* $j > n$, *and all* $\widehat{F} \in \mathcal{Y}_{\mathcal{X}}$.

(8) X *admits a surjective* $\mathcal{Y}_{\mathcal{X}}$*-Gorenstein projective precover* $\varphi\colon C \to X$ *with* $K = Ker(\varphi)$ *satisfying* $\mathcal{Y}_{\mathcal{X}}$*-pd*$(K) \leq n-1$.

In particular, $\mathcal{Y}_{\mathcal{X}}$*-*$\mathcal{GP}dim(X)$=$Sup\{\mathcal{X}\text{-}Gpd(X^i) | i \in \mathbb{Z}\}$.

Proof. (1)⇔(2)⇔(3) by Theorem 3.10.

(1)⇒(4) Fix an exact sequence

$$0 \to K \to G_{n-1} \to \cdots \to G_1 \to G_0 \to X \to 0$$

of complexes, where each G_i is a $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective complex for $i = 0, 1, \cdots, n-1$. The sequence always exists as every projective complex is $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective. Then there is an exact sequences

$$0 \to K^i \to G^i_{n-1} \to \cdots \to G^i_1 \to G^i_0 \to X^i \to 0$$

of objects in $\mathcal{A}$ for each i. Since $\mathcal{X}$-$\mathcal{GP}dim(X^i) \leq n$, K^i is $\mathcal{X}$-Gorenstein projective. Thus K is $\mathcal{Y}$-Gorenstein projective complex by Lemma 4.11.

(4)⇒(1) Let $\mathcal{Y}_{\mathcal{X}}$-$\mathcal{GP}dim(X) \leq n$. By Theorem 3.10, there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to X \to 0,$$

where G_i is a $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective complex for each $i = 0, 1, \cdots, n$. So it induces an exact sequence

$$0 \to G^i_n \to G^i_{n-1} \to \cdots \to G^i_1 \to G^i_0 \to X^i \to 0$$

of objects of $\mathcal{A}$ for each i. By Theorem 4.7, G^i_j is $\mathcal{X}$-Gorenstein projective for each j and i. Hence, $\mathcal{X}$-$Gpd(X^i) \leq n$.

(4)⇔(5)⇔(6)⇔(7)⇔(8) are true by [17, Theorem 3.1]. Since the class of all $\mathcal{Y}_{\mathcal{X}}$-Gorenstein projective objects is projective resolving by Theorem 3.5 and Lemma 4.11, $\mathcal{Y}_{\mathcal{X}}$ is closed under direct summands and $Ext^{\geq 1}(T, H) = 0$ for all $H \in \mathcal{Y}_{\mathcal{X}}$ and $T \in \mathcal{Y}_{\mathcal{X}}$-$\mathcal{GP}(Ch(\mathcal{A}))$ by Proposition 3.2. □

Remark 4.14. *Let* $\mathcal{A}$ *be the category of R-modules. If* $\mathcal{X} = \mathcal{P}(R)$, *then* $\mathcal{Y}_{\mathcal{X}}$=$\mathcal{P}(Ch(R))$. *By* [10], [13] *and Theorem 4.13, there are also relationships between the objects with finite Gorenstein projective dimension in the category of R-modules and those objects in its chain complex category* $Ch(R)$ *as above.*

5. **Cotorsion pairs in** $Ch(\mathcal{A})$

In this section, let $\mathcal{A}$ be a bicomplete abelian category with enough projective objects and $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{X}$. For convenience, we denote by $\widehat{\mathcal{X}}$-$pd(\mathcal{A})$ the class of all objects with finite $\mathcal{X}$-projective dimension and by $l.\mathcal{X}$-$GPD(\mathcal{A})$ the left $\mathcal{X}$-Gorenstein projective global dimension of $\mathcal{A}$.

Suppose that $\mathcal{C}$ and $\mathcal{F}$ are two classes of complexes of $Ch(\mathcal{A})$ and $\mathcal{W}$ is a thick subcategory of $Ch(\mathcal{A})$, where $\mathcal{W}$ is *thick* provided it is closed under direct summands, and if two out of three of terms in a short exact sequence are in $\mathcal{W}$, so is the third. In [14], Hovey proved that two complete cotorsion pairs $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$ and $(\mathcal{C}\cap\mathcal{W}, \mathcal{F})$ give rise to a model structure on $Ch(\mathcal{A})$ such that $\mathcal{C}$ is the class of all cofibrant objects, $\mathcal{F}$ is the class of all fibrant objects and $\mathcal{W}$ is the class of all trivial objects. In view of this, we discuss cotorsion pairs in $Ch(\mathcal{A})$ with respect to the class $\mathcal{X}$-$\mathcal{GP}(\mathcal{A})$.

Lemma 5.1. *If* $l.\mathcal{X}$-$GPD(\mathcal{A}) < \infty$, *then* ($\mathcal{X}$-$\mathcal{GP}(\mathcal{A})$, $\widehat{\mathcal{X}}$-$pd(\mathcal{A})$) *is a complete hereditary cotorsion pair which is cogenerated by* $\mathcal{X}$-$\mathcal{GP}(\mathcal{A})$.

Proof. By analogy with the proofs of [5, Corollary 11.5.3] and [5, Proposition 11.5.9], we can prove that ($\mathcal{X}$-$\mathcal{GP}(\mathcal{A})$, $\mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$) is a complete hereditary cotorsion pair.

In the following, we only prove $\mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$=$\widehat{\mathcal{X}}$-$pd(\mathcal{A})$.

Clearly, $\widehat{\mathcal{X}}$-$pd(\mathcal{A}) \subseteq \mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$. Next, we will prove that $\mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp} \subseteq \widehat{\mathcal{X}}$-$pd(\mathcal{A})$. That is, we shall show that $\mathcal{X}$-$pd(M) < \infty$ for each $M \in \mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$.

Firstly, we will show the following result: if $0 \to X \to X' \to X'' \to 0$ is an exact sequence with $X', X'' \in \mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$, then $X \in \mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$. In fact, for each $G \in \mathcal{X}$-$\mathcal{GP}(\mathcal{A})$, we have the exact sequence

$$0 = Ext^1(G', X'') \to Ext^2(G', X) \to Ext^2(G', X') \to \cdots . \qquad (1)$$

By the Definition 3.1, there is an exact sequence $0 \to G \to P \to G' \to 0$ with $P \in \mathcal{P}(\mathcal{A})$ and $G' \in \mathcal{X}$-$\mathcal{GP}(\mathcal{A})$. From this sequence, we get other exact sequences

$$0 = Ext^1(G, X') \to Ext^2(G', X') \to Ext^2(P, X') = 0 \qquad (2)$$

and

$$0 = Ext^1(P, X) \to Ext^1(G, X) \to Ext^2(G', X) \to Ext^2(P, X) = 0. \qquad (3)$$

By (1),(2) and (3), we can obtain $Ext^1(G, X) = 0$, i.e., $X \in \mathcal{X}$-$\mathcal{GP}(\mathcal{A})^{\perp}$.

Secondly, we will show that for each $M \in \mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})^{\perp}$, $\mathcal{X}\text{-}pd(M) < \infty$. In fact, since $l.\mathcal{X}\text{-}GPD(\mathcal{A}) < \infty$, there is a nonnegative integer n such that $\mathcal{X}\text{-}\mathcal{GP}dim(M) = n$. Then there is an exact sequence

$$0 \to K_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \to 0, \tag{4}$$

where K_n is $\mathcal{X}$-Gorenstein projective and Q_i is projective for each i. Let $K_i = Ker(Q_{i-1} \to Q_{i-2})$ for $1 \leq i \leq n-1$, where $Q_{-1} = M$. Note that $M, Q_i \in \mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})^{\perp}$, then $K_i \in \mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})^{\perp}$ by the result of the first step. For K_n, there is an exact sequence $0 \to K_n \to P \to K_n{}' \to 0$, where P is projective and $K_n{}'$ is $\mathcal{X}$-Gorenstein projective. Thus the sequence splits since $K_n \in \mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})^{\perp}$. That is, K_n is projective. From the sequence (4), we obtain $pd(M) \leq n$, i.e., $\widehat{\mathcal{X}}\text{-}pd(M) \leq n$. Hence, $\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})^{\perp} \subseteq \widehat{\mathcal{X}}\text{-}pd(\mathcal{A})$. □

Theorem 5.2. *Let $\mathcal{A}$ be an abelian category containing an injective cogenerator. If $l.\mathcal{X}\text{-}GPD(\mathcal{A}) < \infty$, then $(Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})), Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A}))^{\perp})$ and $(Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})) \cap \mathcal{E}, (Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})) \cap \mathcal{E})^{\perp})$ are cotorsion pairs, where $\mathcal{E}$ is the class of all acyclic complexes of $Ch(\mathcal{A})$.*

Proof. First, we prove that $(Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A})), Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A}))^{\perp})$ is a cotorsion pair.

Let $S = Ch(\mathcal{X}\text{-}\mathcal{GP}(\mathcal{A}))$. Clearly, $S \subseteq {}^{\perp}(S^{\perp})$. Now, we only need to prove ${}^{\perp}(S^{\perp}) \subseteq S$. Given $X \in {}^{\perp}(S^{\perp})$, we have $Ext^1(X, D^n(B)) = 0$ for each $B \in \widehat{\mathcal{X}}\text{-}pd(\mathcal{A})$. In fact, $Ext^1(G, D^n(B)) \cong Ext^1(G^n, B) = 0$ by Lemma 2.6 for all n and $G \in S$, as G^n is $\mathcal{X}$-Gorenstein projective. So $D^n(B) \in S^{\perp}$. But $0 = Ext^1(X, D^n(B)) \cong Ext^1(X^n, B)$, we have $X^n \in^{\perp} \widehat{\mathcal{X}}\text{-}pd(\mathcal{A})$. Thus, X^n is $\mathcal{X}$-Gorenstein projective for each n by Lemma 5.1, that is, $X \in S$. So we have $S =^{\perp} (S^{\perp})$.

Next, we consider the pair $(S \cap \mathcal{E}, (S \cap \mathcal{E})^{\perp})$. Similarly, we only need to show $S \cap \mathcal{E} =^{\perp}((S \cap \mathcal{E})^{\perp})$. Clearly, $S \cap \mathcal{E} \subseteq^{\perp} ((S \cap \mathcal{E})^{\perp})$. Conversely, if we assume $X \in^{\perp} ((S \cap \mathcal{E})^{\perp})$, then we have $X \in S$ by the proof of the above. In the following, we show that $X \in \mathcal{E}$. Let $Y \in S \cap \mathcal{E}$, then $Ext^1(Y, S^n(E)) \cong Ext^1(Y^n/B_nY, E) = 0$ for an injective cogenerator E. So $S^n(E) \in (S \cap \mathcal{E})^{\perp}$. Thus $Ext^1(X, S^n(E))=0$. Applying the functor $Hom(X, -)$ to the exact sequence of complexes $0 \to S^{n+1}(E) \to D^{n+1}(E) \to S^n(E) \to 0$, we have the exact sequence

$$0 \to Hom(X, S^{n+1}(E)) \to Hom(X, D^{n+1}(E)) \to Hom(X, S^n(E)) \to 0$$

of complexes. That is, the complex

$$0 \to Hom(X^{n+1}/B_{n+1}X, E) \to Hom(X^{n+1}, E) \to Hom(X^n/B_nX, E) \to 0$$

is exact. So the sequence $0 \to X^n/B_nX \to X^{n+1} \to X^{n+1}/B_{n+1}X \to 0$ is exact. Thus X is exact, i.e., $X \in S \cap \mathcal{E}$. $\square$

Corollary 5.3. *Suppose that $\mathcal{A}$ is a bicomplete abelian category containing an injective cogenerator. If $l.\mathcal{X}$-$GPD(\mathcal{A}) < \infty$, then we have a model structure on $Ch(\mathcal{A})$, where the cofibrant objects is in $Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))$.*

Proof. By [14, Theorem 2.2], we take $\mathcal{W}$=$\mathcal{E}$, where $\mathcal{E}$ is the class of all acyclic complexes of $Ch(\mathcal{A})$, $\mathcal{C}$=$Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))$ and $\mathcal{F}$=$(Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))\cap \mathcal{E})^{\perp}$. Then $\mathcal{W}$ is a thick subcategory of $Ch(\mathcal{A})$. Now, we only need to show that $(Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))\cap\mathcal{E})^{\perp}\cap\mathcal{E}$=$Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))^{\perp}$. Let $S = Ch(\mathcal{X}$-$\mathcal{GP}(\mathcal{A}))$. If $X \in S^{\perp}$, then $X \in (S\cap\mathcal{E})^{\perp}$. However, by the proof of Theorem 5.2, X is acyclic as there is an injective cogenerator in $\mathcal{A}$. That is, $S \subseteq (S\cap\mathcal{E})^{\perp}\cap\mathcal{E}$.

Conversely, let $X \in (S\cap\mathcal{E})^{\perp}\cap\mathcal{E}$, then $X \in S^{\perp}$.

Indeed, let $Y \in S$, there is, an exact sequence $0 \to Y \to A \to B \to 0$, where $Y \to A$ is an $\mathcal{E}$-preenvelope and B is a dg-projective complex (i.e., B is a complex of projective objects of $\mathcal{A}$ and $\mathcal{H}om(B,X)$ is exact for each exact complex $X \in Ch(\mathcal{A})$) by [8] or [15]. So $B \in S$. Since S is closed under extensions, we have $A \in S\cap\mathcal{E}$. Applying the functor $Hom(-,X)$ to this, we have the exact sequence

$$\cdots \to Ext^1(A,X) \to Ext^1(Y,X) \to Ext^2(B,X) \to \cdots.$$

It induces $Ext^1(A,X)$=0 and $Ext^2(B,X)$=0, since $A \in S\cap\mathcal{E}$ and B is dg-projective. Thus $Ext^1(Y,X)$=0, i.e., $X \in S^{\perp}$. $\square$

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H. Cheng
Department of Mathematics, University of Nanjing, 210093, Nanjing, P. R. China
Email: xiangyun23@sina.com

X. Zhur
Department of Mathematics, University of Nanjing, 210093, Nanjing, P. R. China
Email: zhuxs@nju.edu.cn