

**FIXED POINTS FOR E -ASYMPTOTIC
CONTRACTIONS AND BOYD-WONG TYPE
 E -CONTRACTIONS IN UNIFORM SPACES**

A. AGHANIAN, K. FALLAHI AND K. NOUROUZI*

Communicated by Gholam Hossein Esslamzadeh

ABSTRACT. In this paper we discuss the fixed points of asymptotic contractions and Boyd-Wong type contractions in uniform spaces equipped with an E -distance. A new version of Kirk's fixed point theorem is given for asymptotic contractions and Boyd-Wong type contractions is investigated in uniform spaces.

1. Introduction and preliminaries

In 2003, Kirk [5] discussed the existence of fixed points for (not necessarily continuous) asymptotic contractions in complete metric spaces. Jachymski and Jóźwik [4] constructed an example to show that continuity of the self-mapping is essential in Kirk's theorem. They also established a fixed point result for uniformly continuous asymptotic φ -contractions in complete metric spaces.

MSC(2010): Primary: 47H10; Secondary: 54E15, 47H09.

Keywords: Separated uniform space, E -asymptotic contraction, Boyd-Wong type E -contraction, fixed point.

Received: 1 June 2012, Accepted: 1 December 2012.

*Corresponding author

© 2013 Iranian Mathematical Society.

Motivated by [5, Theorem 2.1] and [4, Example 1], we aim to give a more general form of [5, Theorem 2.1] in uniform spaces where the self-mappings are assumed to be continuous. We also generalize the Boyd-Wong fixed point theorem [3, Theorem 1] to uniform spaces equipped with an E -distance.

We begin with some basics in uniform spaces which are needed in this paper. The reader can find an in-depth discussion in, e.g., [6].

A uniformity on a nonempty set X is a nonempty collection $\mathcal{U}$ of subsets of $X \times X$ (called the entourages of X) satisfying the following conditions:

- (1) Each entourage of X contains the diagonal $\{(x, x) : x \in X\}$;
- (2) $\mathcal{U}$ is closed under finite intersections;
- (3) For each entourage U in $\mathcal{U}$, the set $\{(x, y) : (y, x) \in U\}$ is in $\mathcal{U}$;
- (4) For each $U \in \mathcal{U}$, there exists an entourage V such that $(x, y), (y, z) \in V$ implies $(x, z) \in U$ for all $x, y, z \in X$;
- (5) $\mathcal{U}$ contains the supersets of its elements.

If $\mathcal{U}$ is a uniformity on X , then $(X, \mathcal{U})$ (shortly denoted by X) is called a uniform space.

If d is a metric on a nonempty set X , then it induces a uniformity, called the uniformity induced by the metric d , in which the entourages of X are all the supersets of the sets

$$\{(x, y) \in X \times X : d(x, y) < \varepsilon\},$$

where $\varepsilon > 0$.

It is well-known that a uniformity $\mathcal{U}$ on a nonempty set X is separating if the intersection of all entourages of X coincides with the diagonal $\{(x, x) : x \in X\}$. In this case, X is called a separated uniform space.

We next recall some basic concepts about E -distances. For more details and examples, the reader is referred to [1].

Definition 1.1. [1] Let X be a uniform space. A function $p : X \times X \rightarrow \mathbb{R}^{\geq 0}$ is called an E -distance on X if

- (1) for each entourage U in $\mathcal{U}$, there exists a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $(x, y) \in U$ for all $x, y, z \in X$;
- (2) p satisfies the triangular inequality, i.e.,

$$p(x, y) \leq p(x, z) + p(z, y) \quad (x, y, z \in X).$$

If p is an E -distance on a uniform space X , then a sequence $\{x_n\}$ in X is said to be p -convergent to a point $x \in X$, denoted by $x_n \xrightarrow{p} x$, whenever $p(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, and X is p -Cauchy whenever

$p(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$. The uniform space X is called p -complete if every p -Cauchy sequence in X is p -convergent to some point of X .

The next lemma contains an important property of E -distances on separated uniform spaces. The proof is straightforward and it is omitted here.

Lemma 1.2. [1] *Let $\{x_n\}$ and $\{y_n\}$ be two arbitrary sequences in a separated uniform space X equipped with an E -distance p . If $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p} y$, then $x = y$. In particular, $p(z, x) = p(z, y) = 0$ for some $z \in X$ implies $x = y$.*

Using E -distances, p -boundedness and p -continuity are defined in uniform spaces.

Definition 1.3. [1] *Let p be an E -distance on a uniform space X . Then*

(1) *X is called p -bounded if*

$$\delta_p(X) = \sup \{p(x, y) : x, y \in X\} < \infty.$$

(2) *A mapping $T : X \rightarrow X$ is called p -continuous on X if $x_n \xrightarrow{p} x$ implies $Tx_n \xrightarrow{p} Tx$ for all sequences $\{x_n\}$ and all points x in X .*

2. E -asymptotic contractions

In this section, we denote by Φ the class of all functions $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ with the following properties:

- φ is continuous on $\mathbb{R}^{\geq 0}$;
- $\varphi(t) < t$ for all $t > 0$.

It is worth mentioning that if $\varphi \in \Phi$, then

$$0 \leq \varphi(0) = \lim_{t \rightarrow 0^+} \varphi(t) \leq \lim_{t \rightarrow 0^+} t = 0,$$

that is, $\varphi(0) = 0$.

Following [5, Definition 2.1], we define E -asymptotic contractions.

Definition 2.1. *Let p be an E -distance on a uniform space X . We say that a mapping $T : X \rightarrow X$ is an E -asymptotic contraction if*

$$(2.1) \quad p(T^n x, T^n y) \leq \varphi_n(p(x, y)) \quad \text{for all } x, y \in X \text{ and } n \geq 1,$$

where $\{\varphi_n\}$ is a sequence of nonnegative functions on $\mathbb{R}^{\geq 0}$ converging uniformly to some $\varphi \in \Phi$ on the range of p .

If (X, d) is a metric space, then replacing the E -distance p by the metric d in Definition 2.1, we get the concept of an asymptotic contraction introduced by Kirk [5, Definition 2.1]. So each asymptotic contraction on a metric space is an E -asymptotic contraction on the uniform space induced by the metric. But in the next example, we see that the converse is not generally true.

Example 2.2. Uniformize the set $X = [0, 1]$ with the uniformity induced from the Euclidean metric and put $p(x, y) = y$ for all $x, y \in X$. It is easily verified that p is an E -distance on X . Define $T : X \rightarrow X$ and $\varphi_1 : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ by

$$Tx = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{8} & x = 1 \end{cases} \quad \text{and} \quad \varphi_1(t) = \begin{cases} \frac{1}{16} & 0 \leq t < 1 \\ \frac{1}{8} & t \geq 1 \end{cases}$$

for all $x \in X$ and all $t \geq 0$, and set $\varphi_n = \varphi$ for $n \geq 2$, where φ is any arbitrary fixed function in Φ . Clearly, $\varphi_n \rightarrow \varphi$ uniformly on $\mathbb{R}^{\geq 0}$ and $T^n = 0$ for all $n \geq 2$. To see that T is an E -asymptotic contraction on X , it suffices to check (2.1) for $n = 1$. To this end, given $x, y \in [0, 1]$, if $y = 1$, then we have

$$p(Tx, T1) = T1 = \frac{1}{8} = \varphi_1(1) = \varphi_1(p(x, 1)),$$

and for $0 \leq y < 1$, we have

$$p(Tx, Ty) = Ty = 0 \leq \frac{1}{16} = \varphi_1(y) = \varphi_1(p(x, y)).$$

But T fails to be an asymptotic contraction on the metric space X with the functions φ_n since

$$\left| T1 - T\frac{1}{2} \right| = \frac{1}{8} > \frac{1}{16} = \varphi_1\left(\frac{1}{2}\right) = \varphi_1\left(\left|1 - \frac{1}{2}\right|\right).$$

In the next example, we see that an E -asymptotic contraction need not be p -continuous.

Example 2.3. Let X and p be as in Example 2.2. Define a mapping $T : X \rightarrow X$ by $Tx = 0$ if $0 < x \leq 1$ and $T0 = 1$. Note that T is fixed point free. Now, let φ_1 be the constant function 1 and $\varphi_2 = \varphi_3 = \cdots = \varphi$, where φ is an arbitrary fixed function in Φ . Then T satisfies (2.1) and since $T0 \neq 0$, it follows that T fails to be p -continuous on X .

Theorem 2.4. *Let p be an E -distance on a separated uniform space X such that X is p -complete and let $T : X \rightarrow X$ be a p -continuous E -asymptotic contraction for which the functions φ_n in Definition 2.1 are all continuous on $\mathbb{R}^{\geq 0}$ for large indices n . Then T has a unique fixed point $u \in X$, and $T^n x \xrightarrow{p} u$ for all $x \in X$.*

Proof. We divide the proof into three steps.

Step 1: $p(T^n x, T^n y) \rightarrow 0$ as $n \rightarrow \infty$ for all $x, y \in X$.

Let $x, y \in X$ be given. Letting $n \rightarrow \infty$ in (2.1), we get

$$0 \leq \limsup_{n \rightarrow \infty} p(T^n x, T^n y) \leq \lim_{n \rightarrow \infty} \varphi_n(p(x, y)) = \varphi(p(x, y)) \leq p(x, y) < \infty.$$

Now, if

$$\limsup_{n \rightarrow \infty} p(T^n x, T^n y) = \varepsilon > 0,$$

then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $p(T^{n_k} x, T^{n_k} y) \rightarrow \varepsilon$, and so by the continuity of φ , one obtains

$$\varphi(p(T^{n_k} x, T^{n_k} y)) \rightarrow \varphi(\varepsilon) < \varepsilon.$$

Therefore, there is an integer $k_0 \geq 1$ such that $\varphi(p(T^{n_{k_0}} x, T^{n_{k_0}} y)) < \varepsilon$. So (2.1) yields

$$\begin{aligned} \varepsilon &= \limsup_{n \rightarrow \infty} p(T^n x, T^n y) \\ &= \limsup_{n \rightarrow \infty} p(T^n(T^{n_{k_0}} x), T^n(T^{n_{k_0}} y)) \\ &\leq \lim_{n \rightarrow \infty} \varphi_n(p(T^{n_{k_0}} x, T^{n_{k_0}} y)) \\ &= \varphi(p(T^{n_{k_0}} x, T^{n_{k_0}} y)) < \varepsilon, \end{aligned}$$

which is a contradiction. Hence

$$\limsup_{n \rightarrow \infty} p(T^n x, T^n y) = 0.$$

Consequently,

$$0 \leq \liminf_{n \rightarrow \infty} p(T^n x, T^n y) \leq \limsup_{n \rightarrow \infty} p(T^n x, T^n y) = 0,$$

that is, $p(T^n x, T^n y) \rightarrow 0$.

Step 2: The sequence $\{T^n x\}$ is p -Cauchy for all $x \in X$.

Suppose that $x \in X$ is arbitrary. If $\{T^n x\}$ is not p -Cauchy, then there exist $\varepsilon > 0$ and positive integers m_k and n_k such that

$$m_k > n_k \geq k \quad \text{and} \quad p(T^{m_k} x, T^{n_k} x) \geq \varepsilon \quad k = 1, 2, \dots$$

Keeping the integer n_k fixed for sufficiently large k , say $k \geq k_0$, and using Step 1, we may assume without loss of generality that $m_k > n_k$ is the smallest integer with $p(T^{m_k}x, T^{n_k}x) \geq \varepsilon$, that is,

$$p(T^{m_k-1}x, T^{n_k}x) < \varepsilon.$$

Hence for each $k \geq k_0$, we have

$$\begin{aligned} \varepsilon &\leq p(T^{m_k}x, T^{n_k}x) \\ &\leq p(T^{m_k}x, T^{m_k-1}x) + p(T^{m_k-1}x, T^{n_k}x) \\ &< p(T^{m_k}x, T^{m_k-1}x) + \varepsilon. \end{aligned}$$

If $k \rightarrow \infty$, since $p(T^{m_k}x, T^{m_k-1}x) \rightarrow 0$, it follows that $p(T^{m_k}x, T^{n_k}x) \rightarrow \varepsilon$.

We next show by induction that

$$(2.2) \quad \limsup_{k \rightarrow \infty} p(T^{m_k+i}x, T^{n_k+i}x) \geq \varepsilon, \quad i = 1, 2, \dots$$

To this end, note first that from Step 1,

$$\begin{aligned} \varepsilon &= \lim_{k \rightarrow \infty} p(T^{m_k}x, T^{n_k}x) = \limsup_{k \rightarrow \infty} p(T^{m_k}x, T^{n_k}x) \\ &\leq \limsup_{k \rightarrow \infty} \left[p(T^{m_k}x, T^{m_k+1}x) + p(T^{m_k+1}x, T^{n_k+1}x) \right. \\ &\quad \left. + p(T^{n_k+1}x, T^{n_k}x) \right] \\ &\leq \limsup_{k \rightarrow \infty} p(T^{m_k}x, T^{m_k+1}x) + \limsup_{k \rightarrow \infty} p(T^{m_k+1}x, T^{n_k+1}x) \\ &\quad + \limsup_{k \rightarrow \infty} p(T^{n_k+1}x, T^{n_k}x) \\ &= \limsup_{k \rightarrow \infty} p(T^{m_k+1}x, T^{n_k+1}x), \end{aligned}$$

that is, (2.2) holds for $i = 1$. If (2.2) is true for some i , then

$$\begin{aligned} \varepsilon &\leq \limsup_{k \rightarrow \infty} p(T^{m_k+i}x, T^{n_k+i}x) \\ &\leq \limsup_{k \rightarrow \infty} \left[p(T^{m_k+i}x, T^{m_k+i+1}x) + p(T^{m_k+i+1}x, T^{n_k+i+1}x) \right. \\ &\quad \left. + p(T^{n_k+i+1}x, T^{n_k+i}x) \right] \\ &\leq \limsup_{k \rightarrow \infty} p(T^{m_k+i+1}x, T^{n_k+i+1}x). \end{aligned}$$

Consequently, we have

$$\varphi(\varepsilon) = \lim_{k \rightarrow \infty} \varphi(p(T^{m_k}x, T^{n_k}x))$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \lim_{i \rightarrow \infty} \varphi_i(p(T^{m_k}x, T^{n_k}x)) \\
&= \lim_{i \rightarrow \infty} \lim_{k \rightarrow \infty} \varphi_i(p(T^{m_k}x, T^{n_k}x)) \\
&\geq \limsup_{i \rightarrow \infty} \limsup_{k \rightarrow \infty} p(T^{m_k+i}x, T^{n_k+i}x) \\
&\geq \varepsilon,
\end{aligned}$$

where the first equality holds because φ is continuous, the second equality holds because $\{\varphi_i\}$ is pointwise convergent to φ on the range of p , the third equality holds because $\{\varphi_i\}$ is uniformly convergent to φ on the range of p , and the last two inequalities hold by (2.1) and (2.2), respectively. Hence $\varphi(\varepsilon) \geq \varepsilon$, which is a contradiction. Therefore $\{T^n x\}$ is p -Cauchy.

Step 3: T has a unique fixed point.

Because X is p -complete, it is concluded from Steps 1 and 2 that the family $\{\{T^n x\} : x \in X\}$ of Picard iterates of T is p -equiconvergent, that is, there exists $u \in X$ such that $T^n x \xrightarrow{p} u$ for all $x \in X$. In particular, $T^n u \xrightarrow{p} u$. We claim that u is the unique fixed point for T . To this end, first note that since T is p -continuous on X , it follows that $T^{n+1}u \xrightarrow{p} Tu$, and so, by Lemma 1.2, we have $u = Tu$. And if $v \in X$ is a fixed point for T , then

$$p(u, v) = \lim_{n \rightarrow \infty} p(T^n u, T^n v) \leq \lim_{n \rightarrow \infty} \varphi_n(p(u, v)) = \varphi(p(u, v)),$$

which is impossible unless $p(u, v) = 0$. Similarly $p(u, u) = 0$ and using Lemma 1.2 once more, we get $v = u$. $\square$

It is worth mentioning that the boundedness of some orbit of T is not necessary in Theorem 2.4 unlike [5, Theorem 2.1] or [2, Theorem 4.1.15].

As a consequence of Theorem 2.4, we have the following version of [1, Theorem 3.1].

Corollary 2.5. *Let p be an E -distance on a separated uniform space X such that X is p -complete and p -bounded and let a mapping $T : X \rightarrow X$ satisfy*

$$(2.3) \quad p(Tx, Ty) \leq \varphi(p(x, y)) \quad \text{for all } x, y \in X,$$

where $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ is nondecreasing and continuous with $\varphi^n(t) \rightarrow 0$ for all $t > 0$. Then T has a unique fixed point $u \in X$, and $T^n x \xrightarrow{p} u$ for all $x \in X$.

Proof. Note first that $\varphi(0) = 0$; for if $0 < t < \varphi(0)$ for some t , then the monotonicity of φ implies that $0 < t < \varphi(0) \leq \varphi^n(t)$ for all $n \geq 1$, which contradicts the fact that $\varphi^n(t) \rightarrow 0$.

Next, since φ is nondecreasing, it follows that T satisfies

$$p(T^n x, T^n y) \leq \varphi^n(p(x, y)) \quad \text{for all } x, y \in X \text{ and } n \geq 1.$$

Setting $\varphi_n = \varphi^n$ for each $n \geq 1$ in Definition 2.1, it is seen that $\{\varphi_n\}$ converges pointwise to the constant function 0 on $[0, +\infty)$, and since

$$\sup \left\{ \varphi^n(p(x, y)) : x, y \in X \right\} = \varphi^n(\delta_p(X)) \rightarrow 0,$$

it follows that $\{\varphi_n\}$ converges uniformly to 0 on the range of p . Because the constant function 0 belongs to Φ , it is concluded that T is an E -asymptotic contraction on X . Moreover, φ_n 's are all continuous on $\mathbb{R}^{\geq 0}$ and (2.3) ensures that T is p -continuous on X . Consequently, the result follows immediately from Theorem 2.4. $\square$

The next corollary is a partial modification of Kirk's theorem [5, Theorem 2.1] in uniform spaces. One can find it with an additional assumption, e.g., in [2, Theorem 4.1.15].

Corollary 2.6. *Let X be a complete metric space and let $T : X \rightarrow X$ be a continuous asymptotic contraction for which the functions φ_n in Definition 2.1 are all continuous on $\mathbb{R}^{\geq 0}$ for large indices n . Then T has a unique fixed point $u \in X$, and $T^n x \rightarrow u$ for all $x \in X$.*

3. Boyd-Wong type E -contractions

In this section, we denote by Ψ the class of all functions $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ with the following properties:

- ψ is upper semicontinuous on $\mathbb{R}^{\geq 0}$ from the right, i.e.,

$$t_n \downarrow t \geq 0 \quad \text{implies} \quad \limsup_{n \rightarrow \infty} \psi(t_n) \leq \psi(t);$$

- $\psi(t) < t$ for all $t > 0$, and $\psi(0) = 0$.

It might be interesting for the reader to be mentioned that the family Φ defined and used in Section 2 is contained in the family Ψ but these two families do not coincide. To see this, consider the function $\psi(t) = 0$ if $0 \leq t < 1$, and $\psi(t) = \frac{1}{2}$ if $t \geq 1$. Then ψ is upper semicontinuous from the right but it is not continuous on $\mathbb{R}^{\geq 0}$. Furthermore, the upper semicontinuity of ψ on $\mathbb{R}^{\geq 0}$ from the right and the condition that $\psi(t) <$

t for all $t > 0$, do not imply that ψ vanishes at zero in general. In fact, the function $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ defined by the rule

$$\psi(t) = \begin{cases} a & t = 0 \\ \frac{t}{2} & 0 < t < 1 \\ \frac{1}{2t} & t \geq 1 \end{cases}$$

for all $t \geq 0$, where a is an arbitrary positive real number, confirms this claim.

Theorem 3.1. *Let p be an E-distance on a separated uniform space X such that X is p -complete and let $T : X \rightarrow X$ satisfy*

$$(3.1) \quad p(Tx, Ty) \leq \psi(p(x, y)) \quad \text{for all } x, y \in X,$$

where $\psi \in \Psi$. Then T has a unique fixed point $u \in X$, and $T^n x \xrightarrow{p} u$ for all $x \in X$.

Proof. We divide the proof into three steps as Theorem 2.4.

Step 1: $p(T^n x, T^n y) \rightarrow 0$ as $n \rightarrow \infty$ for all $x, y \in X$.

Let $x, y \in X$ be given. Then for each nonnegative integer n , by the contractive condition (3.1) we have

$$(3.2) \quad p(T^{n+1}x, T^{n+1}y) \leq \psi(p(T^n x, T^n y)) \leq p(T^n x, T^n y).$$

Thus, $\{p(T^n x, T^n y)\}$ is a nonincreasing sequence of nonnegative numbers and so it converges decreasingly to some $\alpha \geq 0$. Letting $n \rightarrow \infty$ in (3.2), by the upper semicontinuity of ψ from the right, we get

$$\alpha = \lim_{n \rightarrow \infty} p(T^{n+1}x, T^{n+1}y) \leq \limsup_{n \rightarrow \infty} \psi(p(T^n x, T^n y)) \leq \psi(\alpha),$$

which is a contradiction unless $\alpha = 0$. Consequently, $p(T^n x, T^n y) \rightarrow 0$.

Step 2: The sequence $\{T^n x\}$ is p -Cauchy for all $x \in X$.

Let $x \in X$ be arbitrary and suppose on the contrary that $\{T^n x\}$ is not p -Cauchy. Then similar to the proof of Step 2 of Theorem 2.4, it is seen that there exist an $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k$ for each k and $p(T^{m_k}x, T^{n_k}x) \rightarrow \varepsilon$. On the other hand, for each k , by (3.1) we have

$$\begin{aligned} p(T^{m_k}x, T^{n_k}x) &\leq p(T^{m_k}x, T^{m_k+1}x) + p(T^{m_k+1}x, T^{n_k+1}x) \\ &\quad + p(T^{n_k+1}x, T^{n_k}x) \end{aligned}$$

$$\begin{aligned} &\leq p(T^{m_k}x, T^{m_k+1}x) + \psi(p(T^{m_k}x, T^{n_k}x)) \\ &\quad + p(T^{n_k+1}x, T^{n_k}x). \end{aligned}$$

Letting $k \rightarrow \infty$ and using Step 1 and the upper semicontinuity of ψ from the right, we obtain

$$\begin{aligned} \varepsilon &= \lim_{k \rightarrow \infty} p(T^{m_k}x, T^{n_k}x) = \limsup_{k \rightarrow \infty} p(T^{m_k}x, T^{n_k}x) \\ &\leq \limsup_{k \rightarrow \infty} \left[p(T^{m_k}x, T^{m_k+1}x) + \psi(p(T^{m_k}x, T^{n_k}x)) \right. \\ &\quad \left. + p(T^{n_k+1}x, T^{n_k}x) \right] \\ &\leq \limsup_{k \rightarrow \infty} p(T^{m_k}x, T^{m_k+1}x) + \limsup_{k \rightarrow \infty} \psi(p(T^{m_k}x, T^{n_k}x)) \\ &\quad + \limsup_{k \rightarrow \infty} p(T^{n_k+1}x, T^{n_k}x) \\ &= \limsup_{k \rightarrow \infty} \psi(p(T^{m_k}x, T^{n_k}x)) \\ &\leq \psi(\varepsilon), \end{aligned}$$

which is a contradiction. Therefore, $\{T^n x\}$ is p -Cauchy.

Step 3: T has a unique fixed point.

Since X is p -complete, it follows from Steps 1 and 2 that the family $\{\{T^n x\} : x \in X\}$ is p -equiconvergent to some $u \in X$. In particular, $T^n u \xrightarrow{p} u$. Since (3.1) implies the p -continuity of T on X , it follows that $T^{n+1}u \xrightarrow{p} Tu$ and so, by Lemma 1.2, we have $u = Tu$, that is, u is a fixed point for T . If $v \in X$ is a fixed point for T , then

$$p(u, v) = p(Tu, Tv) \leq \psi(p(u, v)),$$

which is impossible unless $p(u, v) = 0$. Similarly $p(u, u) = 0$. Therefore, using Lemma 1.2 once more, one gets $v = u$. $\square$

As an immediate consequence of Theorem 3.1, we have the Boyd-Wong's theorem [3] in metric spaces:

Corollary 3.2. *Let X be a complete metric space and let a mapping $T : X \rightarrow X$ satisfy*

$$(3.3) \quad d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all } x, y \in X,$$

where $\psi \in \Psi$. Then T has a unique fixed point $u \in X$, and $T^n x \rightarrow u$ for all $x \in X$.

In the following example, we see that Theorem 3.1 guarantees the existence and uniqueness of a fixed point while Corollary 3.2 cannot be applied.

Example 3.3. Let the set $X = [0, 1]$ be endowed with the uniformity induced by the Euclidean metric and define a mapping $T : X \rightarrow X$ by $Tx = 0$ if $0 \leq x < 1$, and $T1 = \frac{1}{4}$. Then T does not satisfy (3.3) for any $\psi \in \Psi$ since it is not continuous on X . In fact, if $\psi \in \Psi$ is arbitrary, then

$$\left| T1 - T\frac{3}{4} \right| = \frac{1}{4} > \psi\left(\frac{1}{4}\right) = \psi\left(\left|1 - \frac{3}{4}\right|\right).$$

Now set $p(x, y) = \max\{x, y\}$. Then p is an E -distance on X and T satisfies (3.1) for the function $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ defined by the rule $\psi(t) = \frac{t}{4}$ for all $t \geq 0$. It is easy to check that this ψ belongs to Ψ , and the hypotheses of Theorem 3.1 are fulfilled.

Remark 3.4. In Theorem 2.4 (Corollary 2.6), assume that for some index k the function φ_k belongs to Φ . Then Theorem 3.1 (Corollary 3.2) implies that T^k and so T has a unique fixed point u and $T^{kn}x \xrightarrow{p} u$ for all $x \in X$. So, it is concluded by the p -continuity of T that the family $\{\{T^n x\} : x \in X\}$ is p -equiconvergent to u . Hence the significance of Theorem 2.4 (Corollary 2.6) is whenever none of φ_n 's satisfy $\varphi_n(t) < t$ for all $t > 0$, that is, whenever for each $n \geq 1$ there exists a $t_n > 0$ such that $\varphi_n(t_n) \geq t_n$.

REFERENCES

- [1] M. Aamri and D. El Moutawakil, Common fixed point theorems for E -contractive or E -expansive maps in uniform spaces, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **20** (2004), no. 1, 83–91.
- [2] R. P. Agarwal, D. O'Regan and D. R. Sahu, Fixed Point Theory for Lipschitzian-Type Mappings with Applications, Springer, New York, 2009.
- [3] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.* **20** (1969) 458–464.
- [4] J. Jachymski and I. Józwik, On Kirk's asymptotic contractions, *J. Math. Anal. Appl.* **300** (2004), no. 1, 147–159.
- [5] W. A. Kirk, Fixed points of asymptotic contractions, *J. Math. Anal. Appl.* **277** (2003), no. 2, 645–650.
- [6] S. Willard, General Topology, Addison-Wesley Publishing Co., Mass.-London-Don Mills, Ont., 1970.

Aris Aghanians

Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

Email: a.aghanians@dena.kntu.ac.ir

Kamal Fallahi

Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

Email: k.fallahi@dena.kntu.ac.ir

Kourosh Nourouzi

Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

Email: nourouzi@kntu.ac.ir