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ON THE CHARACTER SPACE OF VECTOR-VALUED LIPSCHITZ ALGEBRAS

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ABSTRACT. We show that the character space of the vector-valued Lipschitz algebra $Lip^\alpha(X, E)$ of order α, is homeomorphic to the cartesian product $X \times M_E$ in the product topology, where X is a compact metric space and E is a unital commutative Banach algebra. We also characterize the form of each character on $Lip^\alpha(X, E)$.

By appealing to the injective tensor product, we then identify the character space of the vector-valued polynomial Lipschitz algebra $Lip_P^\alpha(X, E)$, generated by the polynomials on the compact space $X \subseteq \mathbb{C}^n$. It is also shown that $Lip_P^\alpha(X, E)$ is the injective tensor product $Lip_P^\alpha(X) \widehat{\otimes}_\epsilon E$. Finally, we characterize the form of each character on $Lip_P^\alpha(X, E)$.

Keywords: Vector-valued Lipschitz algebra, character space, injective tensor product, polynomial approximation.

MSC(2010): Primary: 46E40; Secondary: 46J10, 46B28, 46M05, 47B48.

1. Introduction

For a compact Hausdorff space X and a topological vector space E, let $C(X, E)$ denote the space of all continuous maps from X into E. Whenever $(E, \|\cdot\|)$ is a normed algebra over the complex numbers $\mathbb{C}$, we define the uniform norm on $C(X, E)$ by

$$\|f\|_X = \sup_{x \in X} \|f(x)\|. \tag{1.1}$$

For $f, g \in C(X, E)$ and $\lambda \in \mathbb{C}$, the pointwise operations λf, $f + g$ and fg in $C(X, E)$ are defined as usual. It is easy to see that $(C(X, E), \|\cdot\|_X)$

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is a Banach algebra whenever E is a Banach algebra. If $E = \mathbb{C}$ we get the ordinary uniform (function) algebra $C(X, \mathbb{C}) = C(X)$. Let (X, d) be a compact metric space and E be a Banach algebra. For a constant $0 < \alpha \leq 1$ and a map $f : X \to E$, the Lipschitz constant of f and the vector-valued Lipschitz algebra $Lip^\alpha(X, E)$ are defined as follows:

$$p_\alpha(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha},$$

$$Lip^\alpha(X, E) = \{f : X \to E : p_\alpha(f) < \infty\}.$$

We define the norm on $Lip^\alpha(X, E)$ by $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$. It is easy to see that $(Lip^\alpha(X, E), \|\cdot\|_\alpha)$ is complete and it is, in fact, a Banach algebra [1]. Note that whenever $E = \mathbb{C}$ we get the classical complex-valued Lipschitz algebra $Lip^\alpha(X)$, which was first studied by Sherbert in [13] and [14]. For certain properties of vector-valued Lipschitz algebras (spaces) one may refer to [1, 8] and [15].

A character on a Banach algebra A is a nonzero complex homomorphism on A. The set of all characters on A, denoted by M_A, is called the character space of A. The injective tensor product of Banach algebras A and B is denoted by $A \widehat{\otimes}_\epsilon B$ (see Section 2 for the definition of tensor products). For the following well-known result, which is due to J. Tomiyama, see [16, Theorem 2] or [9, Theorem 2.11.2].

Theorem 1.1. *Let A and B be commutative Banach algebras. If $A \widehat{\otimes}_\gamma B$ is a Banach algebra for the cross-norm $\gamma \geq \epsilon$, then $M_{A \widehat{\otimes}_\gamma B}$ is homeomorphic to $M_A \times M_B$.*

It is interesting to note that for a compact Hausdorff space X and a commutative Banach algebra E, $C(X, E)$ is (isometrically) isomorphic to $C(X) \widehat{\otimes}_\epsilon E$ [9, Proposition 1.5.6]. In order to identify the character space of $C(X, E)$ one can apply Theorem 1.1 to show that the character space of $C(X, E)$ is homeomorphic to $X \times M_E$, which has already been proved by Hausner in [2].

To determine the character space of $Lip^\alpha(X, E)$, one may try to adapt the same method as in the case of $C(X, E)$, while applying Theorem 1.1. However, an important question arising here is the following:

Question 1.2. Let X be a compact metric space and E be a commutative Banach algebra. Is $Lip^\alpha(X, E)$ isometrically isomorphic to the Banach algebra $Lip^\alpha(X) \widehat{\otimes}_\gamma E$ for some cross-norm $\gamma \geq \epsilon$?

To the best of our knowledge, the answer to this question is still unknown (see [7] and [8, Lemma 5.10]). Although, in [15, Theorem 2.6] there is a positive answer to this question, but there are some gaps in the proof, which has also been mentioned by Runde in his review on the reference [15]. Regardless of the answer to Question 1.2, in Section 2, we show that the character space of $Lip^\alpha(X, E)$ is homeomorphic to $X \times M_E$. Indeed, our method is based on using a vector-valued version of a theorem due to Honary [3] relating the character space of a Banach function algebra to the character space of its uniform closure. We also characterize the form of each character on $Lip^\alpha(X, E)$.

In Section 3, for a compact set $X \subseteq \mathbb{C}^n$ and a unital commutative Banach algebra E, we first introduce the vector-valued polynomial Lipschitz algebra $Lip_P^\alpha(X, E)$ and then identify the character space of this algebra as well as characterizing the form of each character on $Lip_P^\alpha(X, E)$. We also show that the answer to Question 1.2 is positive for the vector-valued polynomial Lipschitz algebra $Lip_P^\alpha(X, E)$.

For further results on the character space of the classic Lipschitz algebra $Lip^\alpha(X)$ as well as the extended analytic Lipschitz algebras, one may refer to [5] and [6].

2. Vector-valued Lipschitz algebras

Let a be an element of the Banach algebra A and $\widehat{a}(h) = h(a)$ for all $h \in A^*$, where A^* denotes the dual space of A. If M_A is not empty then for every $a \in A$ the mapping $\widehat{a} : M_A \to \mathbb{C}$ is the Gelfand transform of a. The Gelfand topology of M_A is the relative topology on $M_A \cup \{0\}$, induced by the weak*-topology of A^*.

Let X be a compact Hausdorff space. A subalgebra A of $C(X)$ which contains the constants and separates the points of X is called a function algebra on X. If, moreover, A is a Banach algebra under some norm, then it is called a Banach function algebra on X. T. G. Honary in [3] proved that if A is a Banach function algebra on a compact Hausdorff space X, then $M_{\overline{A}} \cong M_A$ (homeomorphic) if and only if $\|\widehat{f}\|_{M_A} = \|f\|_X$ for every $f \in A$. Note that the condition $\|\widehat{f}\|_{M_A} = \|f\|_X$ is, in fact, equivalent to $\|\widehat{f}\|_{M_A} \leq \|f\|_X$ for all $f \in A$.

We next state a vector-valued version of this result. For the proof one may adapt the same method as in [3].

Theorem 2.1. *Let X be a compact Hausdorff space, E be a unital Banach algebra and A be a Banach subalgebra of $C(X, E)$ such that*

M_A is non-empty and $\overline{A}$ is unital, where $\overline{A}$ is the closure of A with respect to the uniform norm (1.1) on $C(X, E)$. Then the restriction map $F : M_{\overline{A}} \to M_A$ is a homeomorphism if and only if $\|\widehat{f}\|_{M_A} \leq \|f\|_X$ for every $f \in A$.

Corollary 2.2. Let X be a compact metric space and E be a unital commutative Banach algebra. Then $M_{Lip^\alpha(X,E)} \cong M_{\overline{Lip^\alpha(X,E)}}$.

Proof. Let $f \in Lip^\alpha(X, E)$. By induction it is easy to see that

$$p_\alpha(f^n) \leq n \; p_\alpha(f)\|f\|_X^{n-1},$$

for all $n \in \mathbb{N}$. Hence,

$$\|f^n\|_\alpha \leq \|f\|_X^n + np_\alpha(f)\|f\|_X^{n-1} = \|f\|_X^{n-1}(\|f\|_X + np_\alpha(f)),$$

which implies that

$$\|\widehat{f}\|_{M_{Lip^\alpha(X,E)}} = \lim_{n\to\infty} \|f^n\|_\alpha^{\frac{1}{n}} \leq \|f\|_X.$$

Consequently, by Theorem 2.1, $M_{Lip^\alpha(X,E)} \cong M_{\overline{Lip^\alpha(X,E)}}$. $\square$

In [15] it has been shown that the character space of $Lip^\alpha(X, E)$ can be identified with X, which is not correct and has also been mentioned by V. Runde in his report in MathSciNet on the reference [15]. Using Theorem 2.1 and Corollary 2.2, we adapt a different approach from the one given in [15] to show that the character space of $Lip^\alpha(X, E)$ is, in fact, the cartesian product $X \times M_E$, with respect to the product topology. Some more related results will be presented too. We first bring some preliminary results and definitions. The first one, which appeared in [1], is easy to show.

Lemma 2.3. *Let X be a compact metric space and E be a Banach algebra. Then,*

$$Lip^\alpha(X, E) = \{f : X \to E : \sigma \circ f \in Lip^\alpha(X), \; for \; all \; \sigma \in E^*\}.$$

For a subalgebra $A \subseteq C(X, E)$ we define

$$E^* \circ A = \{\sigma \circ f : f \in A, \; \sigma \in E^*\}.$$

Lemma 2.4. *Let X be a compact metric space and E be a Banach algebra. Then,*

$$E^* \circ Lip^\alpha(X, E) = Lip^\alpha(X).$$

Proof. It is easy to see that $E^* \circ Lip^\alpha(X, E) \subseteq Lip^\alpha(X)$. For the converse inclusion let $f \in Lip^\alpha(X)$. For a nonzero (fixed) vector $v \in E$ define $T : \mathbb{C} \to E$ by $T(\lambda) = \lambda v$. Clearly, T is a continuous linear map and $M = T(\mathbb{C})$ is a closed (one dimensional) subspace of E. Since $T^{-1} : M \to \mathbb{C}$ is a continuous linear functional, by the Hahn-Banach theorem, there exists $\sigma \in E^*$ such that $\sigma|_M = T^{-1}$. Since $\sigma \circ T \circ f = f$ and $T \circ f \in Lip^\alpha(X, E)$, it follows that $f \in E^* \circ Lip^\alpha(X, E)$ and hence

$$Lip^\alpha(X) \subseteq E^* \circ Lip^\alpha(X, E).$$

$\square$

It is worth mentioning that, by Lemma 2.3, $Lip^\alpha(X, E)$ is the maximal subalgebra of $C(X, E)$ satisfying $E^* \circ Lip^\alpha(X, E) \subseteq Lip^\alpha(X)$, that is, if A is a subalgebra of $C(X, E)$ with $E^* \circ A \subseteq Lip^\alpha(X)$, then A is contained in $Lip^\alpha(X, E)$. To see this, let A be such a subalgebra and let $f \in A$. Then, for every $\sigma \in E^*$ we have $\sigma \circ f \in Lip^\alpha(X)$ and hence, by Lemma 2.3, $f \in Lip^\alpha(X, E)$, which implies that $A \subseteq Lip^\alpha(X, E)$.

Before stating the next definition, we recall the concept of tensor product as stated in [12, §1.1].

We write $B(X \times Y, Z)$ to denote the vector space of all bilinear mappings from the cartesian product $X \times Y$ of vector spaces X and Y into a vector space Z. When Z is the scalar field we denote the corresponding space of bilinear maps simply by $B(X \times Y)$. Now, the tensor product $X \otimes Y$ of the vector spaces X and Y can be constructed as a space of linear functionals on $B(X \times Y)$ in the following way:

For $x \in X$ and $y \in Y$ we denote by $x \otimes y$ the linear functional on $B(X \times Y)$ given by evaluation at the point $(x, y) \in X \times Y$. In other words, $(x \otimes y)(T) = T(x, y)$, for each bilinear map $T \in B(X \times Y)$.

For the vector space X let X' denote the algebraic dual of X, consisting of all linear functionals on X. Whenever X is a topological (in particular, normed) vector space then X^* denotes the (topological) dual of X, consisting of all continuous linear functionals on X.

The tensor product $X \otimes Y$ is the subspace of the algebraic dual $B(X \times Y)'$ spanned by the elements $x \otimes y$. Thus, a typical tensor in $X \otimes Y$ has the form $u = \sum_{i=1}^n \lambda_i x_i \otimes y_i$, where n is a natural number, $\lambda_i \in \mathbb{C}, x_i \in X$ and $y_i \in Y$.

By [12, Proposition 1.4], for every bilinear map $T : X \times Y \to Z$ there exists a unique linear map $\tilde{T} : X \otimes Y \to Z$ such that

$$T(x, y) = \tilde{T}(x \otimes y),$$

for all $x \in X$ and $y \in Y$. This fact allows us to consider tensors as bilinear maps on $X' \times Y'$. To see this, consider the bilinear map $B_{x,y}$ on $X' \times Y'$ given by $B_{x,y}(\varphi, \psi) = \varphi(x)\psi(y)$, for $x \in X$, $y \in Y$, $\varphi \in X'$ and $\psi \in Y'$. The map

$$(x, y) \in X \times Y \longmapsto B_{x,y} \in B(X' \times Y')$$

is easily seen to be bilinear and so, by [12, Proposition 1.4], there is a unique linear map from $X \otimes Y$ into $B(X' \times Y')$ that maps $x \otimes y$ to $B_{x,y}$. By [12, Proposition 1.2] this map is injective and hence one can consider the identification

$$x \otimes y \sim B_{x,y} \in B(X' \times Y').$$

Whenever X and Y are Banach spaces, we can replace X' and Y' by X^* and Y^*, respectively, in the argument above. This leads to the canonical embedding $X \otimes Y \subseteq B(X^* \times Y^*)$, where $B(X^* \times Y^*)$ is the space of bounded bilinear mappings on $X^* \times Y^*$. Hence, the norm on $B(X^* \times Y^*)$ induces a norm on the space $X \otimes Y$. This norm is called the injective norm on $X \otimes Y$ and is denoted by $\| \cdot \|_\epsilon$. Thus, for every $u \in X \otimes Y$ we have

$$\|u\|_\epsilon = \sup\{|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)| : \varphi \in X_1^*, \psi \in Y_1^*\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of u, and X_1^* and Y_1^* are the closed unit balls in X^* and Y^*, respectively. The closure of $X \otimes Y$ with the injective norm is called the injective tensor product of X and Y and is denoted by $X \widehat{\otimes}_\epsilon Y$.

Now, let X be a compact Hausdorff space and E be a locally convex topological vector space. Consider the bilinear mapping

$$T : C(X) \times E \to C(X, E)$$

given by $T(f, v) = fv$, for $f \in C(X)$ and $v \in E$. Here, $fv : X \to E$ is the continuous function given by $(fv)(x) = f(x)v$ for every $x \in X$. Hence, applying [12, Proposition 1.4], as we mentioned above, there exists a unique linear mapping

$$(2.1) \qquad\qquad \phi : C(X) \otimes E \to C(X, E),$$

such that $\phi(f \otimes v) = fv$ for every $f \in C(X)$ and $v \in E$. Applying a similar discussion as in [12, page 11] one can see that the linear mapping ϕ in (2.1) is injective. Therefore, in the rest of this paper we use the

(linear) identification

(2.2) $$f \otimes v \sim \phi(f \otimes v) = fv \in C(X, E),$$

for $f \in C(X)$ and $v \in E$ (see also [10]). Hence, the vector-valued function $f \otimes v : X \to E$ is given by

$$(f \otimes v)(x) = f(x)v, \quad (f \in C(X), \ x \in X, \ v \in E).$$

Applying the identification (2.2) one can consider $C(X) \otimes E$ as a subspace of $C(X, E)$. Also, for a subspace A of $C(X)$, let $A \otimes E$ denote the subspace of $C(X, E)$ spanned by $\{f \otimes v : f \in A, v \in E\}$.

In order to state the vector-valued version of the Stone-Weierstrass Theorem, we need the following definition, which is similar to that of [10, pp. 61-63].

Definition 2.5. Let A be a subspace of $C(X, E)$, where X is a compact Hausdorff space and E is a locally convex topological vector space. Then,

    (i) A is a polynomial algebra if $(E^* \circ A) \otimes E \subseteq A$.
    (ii) A is self-adjoint if $E^* \circ A$ is a self-adjoint subspace of $C(X)$.
    (iii) A vanishes at no points of X if for every $x \in X$ there exists $g \in A$ such that $g(x) \neq 0$.
    (iv) A is separating if for every two distinct elements $x, y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$.

We now state a vector-valued version of Stone-Weierstrass approximation theorem for polynomial algebras [10, Corollary 4.18].

Theorem 2.6. *Let X be a compact Hausdorff space, E be a locally convex topological vector space and $A \subseteq C(X, E)$ be a self-adjoint polynomial algebra. Then, A is dense in $C(X, E)$ if and only if A is separating and vanishes at no points of X.*

Now, as a consequence of Lemma 2.4 and Theorem 2.6, we get the following interesting result on vector-valued Lipschitz algebras. Although, this result has already been proved in [15], but we believe that their proof is not correct.

Theorem 2.7. *Let X be a compact metric space and E be a Banach algebra. Then $Lip^\alpha(X, E)$ is uniformly dense in $C(X, E)$, that is,*

$$\overline{Lip^\alpha(X, E)} = C(X, E).$$

Proof. The set $Lip^\alpha(X) \otimes E$ consists of all finite sums of functions of the form $x \mapsto f(x)v$, where $f \in Lip^\alpha(X)$ and $v \in E$. Clearly, for all $x, y \in X$ with $x \neq y$ we have

$$\frac{\|f(x)v - f(y)v\|}{d(x,y)^\alpha} \leq \|v\| p_\alpha(f).$$

Consequently, $Lip^\alpha(X) \otimes E \subseteq Lip^\alpha(X, E)$ and hence, by Lemma 2.4, $Lip^\alpha(X, E)$ is a polynomial algebra. On the other hand, $Lip^\alpha(X, E)$ contains the constant functions $v \in E$ and so it vanishes at no points of X. To show that $Lip^\alpha(X, E)$ is separating, let $x, y \in X$ such that $x \neq y$. For $z \in X$ define $g(z) = d^\alpha(z, y)v$, where v is a (fixed) nonzero vector in E. Clearly $g \in Lip^\alpha(X, E)$ and $g(y) = 0 \neq g(x)$. By Lemma 2.4, $E^* \circ Lip^\alpha(X, E) = Lip^\alpha(X)$ is self-adjoint. Hence it follows from Theorem 2.6 that $\overline{Lip^\alpha(X, E)} = C(X, E)$. $\square$

Remark 2.8. It is interesting to note that, by [2, Lemma 1], every f in $C(X, E)$ can be approximated by a sequence $\{g_n\}$ in C(X, E) of the form $g_n = \sum_{k=1}^n f_k v_k$, where $f_k \in C(X)$ and $v_k \in E$. Since $Lip^\alpha(X)$ is dense in C(X), by applying partitions of unity and a method similar to the one in [2, Lemma 1], one can see that the E-valued Lipschitz algebra $Lip^\alpha(X, E)$ is dense in C(X, E).

By applying Theorem 2.7 we now determine the character space of $Lip^\alpha(X, E)$.

Theorem 2.9. *Let X be a compact metric space and E be a unital commutative Banach algebra. Then, the character space of $Lip^\alpha(X, E)$ is homeomorphic to the cartesian product $X \times M_E$ in the product topology, i.e., $M_{Lip^\alpha(X,E)} \cong X \times M_E$.*

Proof. By applying Corollary 2.2 and Theorem 2.7, we have

$$M_{Lip^\alpha(X,E)} \cong M_{C(X,E)}.$$

On the other hand, by [2] $M_{C(X,E)}$ is homeomorphic to the cartesian product $X \times M_E$ in the product topology, and hence the result follows. $\square$

In particular, when $E = \mathbb{C}$, we have $M_{Lip^\alpha(X)} \cong X \times M_\mathbb{C} \cong X$, which is a known result. See for example [13, Section 2] or [4, page 64].

In [2, Lemma 2] A. Hausner proved that whenever X is a compact Hausdorff space and E is a commutative Banach algebra, every character ϕ on $C(X, E)$ is of the form $\phi = \psi \circ \delta_x$ for some $\psi \in M_E$ and some $x \in X$,

where δ_x is the evaluation homomorphism on $C(X, E)$ at x, defined by $\delta_x(f) = f(x)$. We conclude this section by characterizing the form of each character on $Lip^\alpha(X, E)$.

Theorem 2.10. *Let X be a compact metric space and E be a unital commutative Banach algebra. Then, every character ϕ on $Lip^\alpha(X, E)$ is of the form $\phi = \psi \circ \delta_x$ for some $\psi \in M_E$ and some $x \in X$, where δ_x is the evaluation homomorphism on $Lip^\alpha(X, E)$ at x.*

Proof. Let $\phi \in M_{Lip^\alpha(X,E)}$ and $f \in C(X, E)$. By Theorem 2.7, there exists a sequence $\{f_n\}$ in $Lip^\alpha(X, E)$ such that $\|f_n - f\|_X \to 0$. By Corollary 2.2, $\|\widehat{f}\|_{M_{Lip^\alpha(X,E)}} \leq \|f\|_X$ for every $f \in Lip^\alpha(X, E)$ and so

$$|\phi(f_n) - \phi(f_m)| \leq \|f_n - f_m\|_X \to 0 \quad (n, m \to \infty).$$

Thus, $\lim_{n\to\infty} \phi(f_n)$ exists. If there is another sequence $\{g_n\}$ in $Lip^\alpha(X, E)$ such that $\|g_n - f\|_X \to 0$, then it is easy to see that

$$\lim_{n\to\infty} \phi(g_n) = \lim_{n\to\infty} \phi(f_n).$$

Hence we can define Φ on $C(X, E) = \overline{Lip^\alpha(X, E)}$ by $\Phi(f) = \lim_{n\to\infty} \phi(f_n)$. It is obvious that $\Phi \in M_{C(X,E)}$, and in fact, Φ is an extension of ϕ to $C(X, E)$. By [2, Lemma 2], there exist $\psi \in M_E$ and $x \in X$ such that $\Phi = \psi \circ \delta_x$ on $C(X, E)$ and so $\phi = \psi \circ \delta_x$ on $Lip^\alpha(X, E)$. $\qquad\square$

3. Vector-valued polynomial Lipschitz algebras

Throughout this section, X is a compact set in the complex n-space $\mathbb{C}^n$ and E is a unital commutative Banach algebra.

The algebra of all polynomials on X in the coordinate functions $z_1, ..., z_n$ with coefficients in $\mathbb{C}$ is denoted by $P_0(X)$ and its uniform closure in $C(X)$ is denoted by $P(X)$. Similarly, the algebra of all polynomials in the coordinate functions $z_1, ..., z_n$ with coefficients in E is denoted by $P_0(X, E)$ and its uniform closure in $C(X, E)$ is denoted by $P(X, E)$. Note that $P_0(X, E) \subseteq Lip^\alpha(X, E)$. Next, we define the concept of a vector-valued polynomial Lipschitz algebra.

Definition 3.1. The closed subalgebra of $Lip^\alpha(X, E)$ generated by $P_0(X, E)$ is called the vector-valued polynomial Lipschitz algebra and is denoted by $Lip_P^\alpha(X, E)$.

Whenever $E = \mathbb{C}$, we get the ordinary polynomial Lipschitz algebra $Lip_P^\alpha(X)$, which was studied in [4] and [5].

Theorem 3.2. *The character space of $Lip_P^\alpha(X, E)$ is homeomorphic to the character space of $P(X, E)$, that is, $M_{Lip_P^\alpha(X,E)} \cong M_{P(X,E)}$.*

Proof. As we mentioned before, $P_0(X, E) \subseteq Lip_P^\alpha(X, E)$. Since the norm of a vector-valued Lipschitz algebra is stronger than the uniform norm on X, $Lip_P^\alpha(X, E) \subseteq P(X, E)$. This implies that

$$\overline{Lip_P^\alpha(X, E)} = P(X, E).$$

By the same argument as in Corollary 2.2, we can show that

$$\|\widehat{f}\|_{M_{Lip_P^\alpha(X,E)}} \leq \|f\|_X,$$

for every $f \in Lip_P^\alpha(X, E)$. Hence by Theorem 2.1, the result follows. $\square$

To determine the character space of $Lip_P^\alpha(X, E)$, we apply Theorem 3.2. Hence, it is enough to identify the character space of $P(X, E)$. We first prove the following result, by adapting a classical discussion as in [9, Proposition 1.5.6] (see also [11, Theorem B.2.5]).

Consider the bilinear mapping $T : P(X) \times E \to P(X, E)$ given by $T(f, v) = fv$ for $f \in P(X)$ and $v \in E$. Note that T is the same bilinear mapping given in the previous section, leading to the inclusion $C(X) \otimes E \subseteq C(X, E)$. By the same method as in the previous section, by applying [12, Proposition 1.4] to $T : P(X) \times E \to P(X, E)$, we find the unique linear mapping

$$(3.1) \qquad\qquad \phi : P(X) \otimes E \to P(X, E),$$

such that $\phi(f \otimes v) = fv$ for $f \in P(X)$ and $v \in E$. By [12, page 11] ϕ in (3.1) is injective and hence we have the (linear) identification

$$(3.2) \qquad\qquad f \otimes v \sim \phi(f \otimes v) = fv \in P(X, E),$$

for every $f \in P(X)$ and $v \in E$. Considering the identification (3.2), the space $P(X) \otimes E$ may be regarded as a subspace of $P(X, E)$. Also note that, by [9, Proposition 1.5.1] there exists a unique multiplication on $P(X) \otimes E$, given by

$$(3.3) \qquad\qquad (f \otimes v)(g \otimes w) = fg \otimes vw,$$

for every $f, g \in P(X)$ and $v, w \in E$, with respect to which $P(X) \otimes E$ is an algebra. Before stating the next interesting result, we recall that the closed unit ball of a Banach space A is denoted by A_1.

Theorem 3.3. *The algebra $P(X, E)$ is the injective tensor product of $P(X)$ and E, that is, $P(X, E) = P(X)\widehat{\otimes}_\epsilon E$.*

Proof. Consider the identification (linear) mapping

$$\phi : P(X) \otimes E \to P(X, E),$$

given in (3.1) and (3.2). We show that this identification map is indeed a homomorphism. See, for example, [9, Proposition 1.5.6]. Let

$$u = \sum_{i=1}^{n} f_i \otimes a_i, \ v = \sum_{j=1}^{m} g_j \otimes b_j,$$

be arbitrary elements in $P(X) \otimes E$. Then

$$
\begin{aligned}
\phi(uv) &= \phi((\sum_{i=1}^{n} f_i \otimes a_i)(\sum_{j=1}^{m} g_j \otimes b_j)) \\
&= \phi(\sum_{i=1}^{n} \sum_{j=1}^{m} (f_i \otimes a_i)(g_j \otimes b_j)) \\
&= \phi(\sum_{i=1}^{n} \sum_{j=1}^{m} f_i g_j \otimes a_i b_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} f_i g_j a_i b_j = (\sum_{i=1}^{n} f_i a_i)(\sum_{j=1}^{m} g_j b_j) \\
&= (\sum_{i=1}^{n} \phi(f_i \otimes a_i))(\sum_{j=1}^{m} \phi(g_j \otimes b_j)) \\
&= \phi(\sum_{i=1}^{n} f_i \otimes a_i)\phi(\sum_{j=1}^{m} g_j \otimes b_j) = \phi(u)\phi(v),
\end{aligned}
$$

and hence, ϕ is a homomorphism.

Next, we show that ϕ is an isometry if we consider the injective norm on $P(X) \otimes E$. For $u = \sum_{i=1}^{n} f_i \otimes a_i$ in $P(X) \otimes E$, we have

$$
\begin{aligned}
\|\phi(u)\|_X &= \sup\{\|\phi(u)(x)\| : x \in X\} \\
&= \sup\{\|\sum_{i=1}^{n} f_i(x)a_i\| : x \in X\} \\
&= \sup\{|g(\sum_{i=1}^{n} f_i(x)a_i)| : g \in E_1^*, x \in X\} \\
&= \sup\{\|\sum_{i=1}^{n} g(a_i)f_i\| : g \in E_1^*\} \\
&= \sup\{|\sum_{i=1}^{n} g(a_i)\lambda(f_i)| : g \in E_1^*, \lambda \in P(X)_1^*\} \\
&= \|\sum_{i=1}^{n} f_i \otimes a_i\|_\epsilon = \|u\|_\epsilon.
\end{aligned}
$$

Thus, $\phi : P(X) \otimes E \to P(X, E)$ is an isometry and hence the injective norm is an algebra norm on $P(X) \otimes E$. Therefore, the multiplication $P(X) \otimes E$ given in (3.3) can be uniquely extended to $P(X)\widehat{\otimes}_\epsilon E$. Moreover, $\phi : P(X) \otimes E \to P(X, E)$ can be uniquely extended to an isometric homomorphism

$$\phi : P(X)\widehat{\otimes}_\epsilon E \to P(X, E).$$

Now, it only remains to prove the surjectivity of ϕ. For this, it is enough to show that $\phi(P(X)\otimes E)$ is dense in $P(X, E)$. Let $f \in P(X, E)$ and $\varepsilon > 0$. Then, there exists a polynomial $g = \sum_{i=0}^{n} x^i a_i$ in $P_0(X, E)$ such that $\|f - g\|_X < \varepsilon$. If we take p_i to be the monomial function

$p_i(x) = x^i$, then

$$g = \sum_{i=0}^{n} \phi(p_i \otimes a_i) = \phi(\sum_{i=0}^{n} p_i \otimes a_i),$$

and hence

$$\|f - \phi(\sum_{i=1}^{n} p_i \otimes a_i)\|_X < \varepsilon.$$

This implies that $\phi : P(X)\widehat{\otimes}_\epsilon E \to P(X, E)$, is an isometric isomorphism. $\square$

Now, by applying the theorems above, we determine the character space of $Lip_P^\alpha(X, E)$.

Theorem 3.4. *The character space of the polynomial Lipschitz algebra $Lip_P^\alpha(X, E)$ is homeomorphic to the cartesian product $\widehat{X} \times M_E$ in the product topology, that is, $M_{Lip_P^\alpha(X,E)} \cong \widehat{X} \times M_E$, where $\widehat{X}$ is the polynomial convex hull of X.*

Proof. By [16, Theorem 2] or [9, Theorem 2.11.2], the character space of the injective tensor product $P(X)\widehat{\otimes}_\epsilon E$ is homeomorphic to $\widehat{X} \times M_E$, with the product topology. Hence, by Theorem 3.3, $M_{P(X,E)}$ is also homeomorphic to $\widehat{X} \times M_E$. Therefore, Theorem 3.2 implies the desired result, i.e., $M_{Lip_P^\alpha(X,E)} \cong \widehat{X} \times M_E$. $\square$

We now adapt a similar method as in Theorem 3.3, to show that $Lip_P^\alpha(X, E)$ is, in fact, the injective tensor product $Lip_P^\alpha(X)\widehat{\otimes}_\epsilon E$.

In the following theorem we consider the Lipschitz algebra $Lip^\alpha(X, E)$ with the norm $\|f\| = \max(\|f\|_X, \ p_\alpha(f))$, which is equivalent to the ordinary norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$.

Theorem 3.5. *Let $X \subseteq \mathbb{C}^n$ be a compact set and E be a unital commutative Banach algebra. Then, the algebra $Lip_P^\alpha(X, E)$ is the injective tensor product of $Lip_P^\alpha(X)$ and E, that is, $Lip_P^\alpha(X, E) = Lip_P^\alpha(X)\widehat{\otimes}_\epsilon E$.*

Proof. It is enough to show that the identification (linear) map

$$\phi : Lip_P^\alpha(X) \otimes E \to Lip_P^\alpha(X, E)$$

is an isometry if we consider the injective norm on $Lip_P^\alpha(X) \otimes E$. Let $u = \sum_{i=1}^{n} f_i \otimes a_i$ be an element of $Lip_P^\alpha(X) \otimes E$. Then,

$$\|\phi(u)\|_\alpha = \|\textstyle\sum_{i=1}^n f_i a_i\|_\alpha = \max(\|\textstyle\sum_{i=1}^n f_i a_i\|_X,\ p_\alpha(\textstyle\sum_{i=1}^n f_i a_i))$$

$$= \max(\sup_{x\in X}\|\textstyle\sum_{i=1}^n f_i(x)a_i\|,\ \sup_{\substack{x,y\in X\\x\neq y}}\frac{\|\sum_{i=1}^n f_i(x)a_i - \sum_{i=1}^n f_i(y)a_i\|}{d(x,y)^\alpha})$$

$$= \max(\sup\{|\textstyle\sum_{i=1}^n f_i(x)g(a_i)| : x\in X,\ g\in E_1^*\},$$

$$\sup\{\frac{|\sum_{i=1}^n f_i(x)g(a_i) - \sum_{i=1}^n f_i(y)g(a_i)|}{d(x,y)^\alpha} : x,y\in X, x\neq y,\ g\in E_1^*\})$$

$$= \max\big(\sup\{\|\textstyle\sum_{i=1}^n f_i g(a_i)\|_X : g\in E_1^*\}, sup\{p_\alpha(\textstyle\sum_{i=1}^n f_i g(a_i)) : g\in E_1^*\}\big)$$
$$= \sup\{\max(\|\textstyle\sum_{i=1}^n f_i g(a_i)\|_X,\ p_\alpha(\textstyle\sum_{i=1}^n f_i g(a_i))) : g\in E_1^*\}$$
$$= \sup\{\|\textstyle\sum_{i=1}^n f_i g(a_i)\|_\alpha : g\in E_1^*\}$$
$$= \sup\{|\textstyle\sum_{i=1}^n \lambda(f_i)g(a_i)| : g\in E_1^*, \lambda\in Lip_P^\alpha(X)_1^*\} = \|u\|_\epsilon.$$

Therefore, the map ϕ is an isometry and hence, it can be uniquely extended to an isometric homomorphism $\phi : Lip_P^\alpha(X)\widehat{\otimes}_\epsilon E \to Lip_P^\alpha(X,E)$. For this, we show that $\phi(Lip_P^\alpha(X)\otimes E)$ is dense in $Lip_P^\alpha(X,E)$. The rest of the assertions are immediate. Let $f \in Lip_P^\alpha(X,E)$ and $\epsilon > 0$. Then, there exists a polynomial $g = \sum_{i=0}^n x^i a_i$ in $P_0(X,E)$ such that $\|f-g\|_\alpha < \epsilon$. If we take p_i to be the monomial function $p_i(x) = x^i$, then $g = \sum_{i=0}^n \phi(p_i\otimes a_i) = \phi(\sum_{i=0}^n p_i\otimes a_i)$ and so

$$\|f - \phi(\sum_{i=1}^n p_i\otimes a_i)\|_\alpha < \epsilon.$$

Therefore, $\phi : Lip_P^\alpha(X)\widehat{\otimes}_\epsilon E \to Lip_P^\alpha(X,E)$ is an isometric isomorphism. $\square$

As an application of the theorem above we get the following result, which was already proved in Theorem 3.4 by another method.

Corollary 3.6. The character space of the polynomial Lipschitz algebra $Lip_P^\alpha(X,E)$ is homeomorphic to the cartesian product $\widehat{X}\times M_E$, with the product topology, that is, $M_{Lip_P^\alpha(X,E)}\cong \widehat{X}\times M_E$, where $\widehat{X}$ is the polynomial convex hull of X.

Proof. By [16, Theorem 2], the character space of the injective tensor product $Lip_P^\alpha(X)\widehat{\otimes}_\epsilon E$ is homeomorphic to $\widehat{X}\times M_E$ with the product topology. Hence by Theorem 3.5, $M_{Lip_P^\alpha(X,E)}$ is also homeomorphic to $\widehat{X}\times M_E$. $\square$

It is interesting to note that the character spaces of the Lipschitz algebras $Lip^\alpha(X, E)$ and $Lip_P^\alpha(X, E)$ do not change, up to homeomorphism, if we equip them with the equivalent norm $\|f\| = \max(\|f\|_X, \ p_\alpha(f))$.

Remark 3.7. One may think that with the same method as in the proof of $Lip_P^\alpha(X, E) = Lip_P^\alpha(X)\widehat{\otimes}_\epsilon E$, we can show that

$$Lip^\alpha(X, E) = Lip^\alpha(X)\widehat{\otimes}_\epsilon E.$$

However, we could not succeed in proving this last equality by applying the same method, and to the best of our knowledge, it has not been proved yet.

We conclude this section by characterizing the form of each character on $Lip_P^\alpha(X, E)$.

Theorem 3.8. *Every character ϕ on $Lip_P^\alpha(X, E)$ turns out to be of the form $\phi = \psi \circ \delta_x$, for some $\psi \in M_E$ and some $x \in \widehat{X}$, where δ_x is the evaluation homomorphism on $Lip_P^\alpha(X, E)$ at x.*

Proof. Since $\overline{Lip_P^\alpha(X, E)} = P(X, E)$, for every $f \in P(X, E)$ there exists a sequence $\{f_n\} \subseteq Lip_P^\alpha(X, E)$ such that $\|f_n - f\|_X \to 0$. As we mentioned before, $\|\widehat{f}\|_{M_{Lip_P^\alpha(X,E)}} \leq \|f\|_X$ for every $f \in Lip_P^\alpha(X, E)$. Hence, for every $\phi \in M_{Lip_P^\alpha(X,E)}$ we have

$$|\phi(f_n) - \phi(f_m)| \leq \|f_n - f_m\|_X \to 0,$$

as $n, m \to \infty$ and hence $\lim_{n\to\infty} \phi(f_n)$ exists. Similar to the proof of Theorem 2.1, we can show that if there exists another sequence $\{g_n\}$ in $Lip_P^\alpha(X, E)$ such that $\|g_n - f\|_X \to 0$, then $\lim_{n\to\infty} \phi(g_n) = \lim_{n\to\infty} \phi(f_n)$ and hence we can define Φ on $P(X, E)$ by $\Phi(f) = \lim_{n\to\infty} \phi(f_n)$. It is easy to see that $\Phi \in M_{P(X,E)}$ and, in fact, Φ is the extension of ϕ to $P(X, E)$. We now show that Φ is a character on

$$\{e\} \otimes P(X) = \{e \otimes f \in P(X, E) : f \in P(X)\},$$

where e is the unit element of E and, moreover, Φ is, in fact, a character on $E \subseteq P(X, E)$. Since $\Phi \in M_{P(X,E)}$, it is enough to show that ϕ is not identically zero on $\{e\} \otimes P(X)$. If Φ is identically zero on $\{e\} \otimes P(X)$ or E, then $\Phi(vg) = \Phi(eg)\Phi(v) = 0$, for all $g \in P(X)$ and $v \in E$. Since $P_0(X, E) = P_0(X) \otimes E$, linear combinations of functions of the type vg ($g \in P_0(X)$ and $v \in E$) are dense in $P(X, E)$, and so Φ is identically zero on $P(X, E)$, which is a contradiction. It is easy to see that $\{e\} \otimes P(X)$

is isometrically isomorphic to $P(X)$. Hence, there exists $x \in \widehat{X}$ such that

$$\Phi(e \otimes g) = \delta_x(g) \quad (g \in P(X)).$$

If ψ is the restriction of Φ to E, then ψ is a character on E. Therefore, for every $G = \sum_{j=1}^m g_j \otimes a_j \in P_0(X) \otimes E$, we have

$$\begin{aligned}
\Phi(G) &= \sum_{j=1}^m \Phi(eg_j)\Phi(a_j) = \sum_{j=1}^m g_j(x)\psi(a_j) \\
&= \psi\left(\sum_{j=1}^m a_j g_j(x)\right) = \psi \circ \delta_x\left(\sum_{j=1}^m a_j g_j\right) \\
&= \psi \circ \delta_x\left(\sum_{j=1}^m g_j \otimes a_j\right) = \psi \circ \delta_x(G),
\end{aligned}$$

which implies that $\Phi = \psi \circ \delta_x$ on $P_0(X) \otimes E = P_0(X, E)$. Consequently, by the continuity of $\Phi : P(X, E) \to \mathbb{C}$, we get $\Phi = \psi \circ \delta_x$ on $P(X, E)$ and since $\phi = \Phi|_{Lip_P^\alpha(X,E)}$, it follows that $\phi = \psi \circ \delta_x$ on $Lip_P^\alpha(X, E)$. $\square$

Remark 3.9. Let (X, d) be a compact metric space and E be a unital commutative Banach algebra. Then, for every $0 < \alpha < 1$, the little vector-valued Lipschitz algebra $\ell ip^\alpha(X, E)$ is defined as a subalgebra of $Lip^\alpha(X, E)$, consisting of those elements f such that

$$\lim_{d(x,y) \to 0} \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha} = 0.$$

The little vector-valued polynomial Lipschitz algebra $\ell ip_P^\alpha(X, E)$ is defined in a similar way. It is worth mentioning that all of the results in Sections 2 and 3, except Lemma 2.3, are also valid for little vector-valued (polynomial) Lipschitz algebras. We also note that

$$\ell ip_P^\alpha(X, E) = Lip_P^\alpha(X, E),$$

for every α, $0 < \alpha < 1$.

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