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**Domination number of graph fractional powers**

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## DOMINATION NUMBER OF GRAPH FRACTIONAL POWERS

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**ABSTRACT.** For any  $k \in \mathbb{N}$ , the  $k$ -subdivision of a graph  $G$  is a simple graph  $G^{\frac{1}{k}}$ , which is constructed by replacing each edge of  $G$  with a path of length  $k$ . In [Moharram N. Iradmusa, *On colorings of graph fractional powers*, Discrete Math., (310) 2010, No. 10-11, 1551-1556] the  $m$ th power of the  $n$ -subdivision of  $G$  has been introduced as a fractional power of  $G$ , denoted by  $G^{\frac{m}{n}}$ . In this regard, we investigate domination number and independent domination number of fractional powers of graphs.

**Keywords:** Domination number, subdivision of a graph, power of a graph.

**MSC(2010):** Primary: 05C69.

### 1. Introduction and preliminaries

In this paper, we only consider graphs with nonempty edge set that are finite, with no loops or multiple edges. We mention some of the definitions that are used through the paper. We denote the number of vertices and edges in  $G$  by  $p(G)$  and  $q(G)$ ; these two basic parameters are called the order and size of  $G$ , respectively. In addition,  $N_G(v)$ , called the *open neighborhood* of  $v$  in  $G$ , denotes the set of vertices of  $G$  which are adjacent to the vertex  $v$  of  $G$ , and the *closed neighborhood* of  $v$  is  $N_G[v] = N_G(v) \cup \{v\}$ . Also for any set  $S \subseteq V(G)$ , the *open neighborhood of  $S$* ,  $N_G(S)$ , is defined to be  $\bigcup_{v \in S} N_G(v)$ , and the *closed neighborhood of  $S$*  is  $N_G[S] = N_G(S) \cup S$ . Generally, the *open  $k$ -neighborhood* of a vertex  $v \in V(G)$ , denoted by  $N_k(v)$ , is the set  $N_k(v) = \{u : u \neq v, d(u, v) \leq k\}$ ;

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the set  $N_k[v] = N_k(v) \cup \{v\}$  is called the *closed  $k$ -neighborhood* of  $v$  and every vertex  $w \in N_k[v]$  is said to be  *$k$ -adjacent* to  $v$ . Finally, for a set  $S$  of vertices we define  $N_k(S)$  to be the union of the open  $k$ -neighborhoods of vertices in  $S$ , while  $N_k[S]$  is the union of the closed  $k$ -neighborhoods of vertices in  $S$ .

A set  $S \subseteq V$  of vertices in a graph  $G = (V, E)$  is called a *dominating set* if every vertex  $v \in V$  is either an element of  $S$  or adjacent to an element of  $S$ , or equivalently,  $N_G[S] = V(G)$ . The minimum cardinality of a dominating set in  $G$  is called the *domination number* of  $G$  and is denoted by  $\gamma(G)$ . In addition, each dominating set of minimum cardinality is called a  $\gamma$ -*set* of  $G$ . Furthermore, a set  $S$  of vertices in a graph  $G = (V, E)$  is called an *independent dominating set* if  $S$  is both an independent and a dominating set of  $G$ ; the minimum cardinality of an independent dominating set of  $G$  is the *independent domination number*  $i(G)$ . We refer the reader to the textbooks [1, 3] for the necessary definitions and notations.

In [4], the author has introduced the fractional powers of a graph. Let  $G$  be a graph and  $k$  be a positive integer. The  *$k$ -power of  $G$* , denoted by  $G^k$ , is defined on the vertex set  $V(G)$  by joining any two distinct vertices  $x$  and  $y$  with distance at most  $k$  in  $G$  [1], or equivalently,  $E(G^k) = \{xy : 1 \leq d_G(x, y) \leq k\}$ . Also for any mapping  $f : E(G) \rightarrow \mathbb{N}$ , the  *$f$ -subdivision of  $G$* , denoted by  $G^{\frac{1}{f}}$ , is constructed by replacing each edge  $xy$  of  $G$  with a path of length  $f(xy)$ , say  $P_{xy}$ . These paths are called  *$G$ -edges* and any new vertex is called an *internal vertex* or briefly  *$i$ -vertex* and is denoted by  $(xy)_l$ , if it belongs to the  $G$ -edge  $P_{xy}$  and has distance  $l$  from the vertex  $x$ , where  $l \in \{0, 1, 2, \dots, f(xy)\}$ . Note that  $(xy)_l = (yx)_{f(xy)-l}$ ,  $(xy)_0 = x$  and  $(xy)_{f(xy)} = y$ . Also any vertex  $x = (xy)_0$  of  $G^{\frac{1}{f}}$  is a *terminal vertex* or briefly  *$t$ -vertex*. In particular, when  $f(xy) = k$  for each edge  $xy$  of the graph  $G$ , we denote  $G^{\frac{1}{f}}$  by  $G^{\frac{1}{k}}$  and when

$$f(xy) = \begin{cases} k & xy = e \\ 1 & xy \neq e, \end{cases}$$

we denote  $G^{\frac{1}{f}}$  with  $S_{e,k}(G)$ . The second one is a useful notation, because for any mapping  $f : E(G) \rightarrow \mathbb{N}$ , we have

$$G^{\frac{1}{f}} = S_{e_1, f(e_1)}(S_{e_2, f(e_2)}(\dots S_{e_q, f(e_q)}(G) \dots)),$$

where  $E(G) = \{e_1, e_2, \dots, e_q\}$ .

Note that for  $k = 1$ , we have  $G^{\frac{1}{1}} = G^1 = G$ , and if the graph  $G$

has  $p$  vertices and  $E(G) = \{e_1, e_2, \dots, e_q\}$ , then the graph  $G^{\frac{1}{f}}$  has  $\sum_{i=1}^q f(e_i) + p - q$  vertices and  $\sum_{i=1}^q f(e_i)$  edges. Especially,  $G^{\frac{1}{k}}$  has  $p + (k-1)q$  vertices and  $kq$  edges. In [4], the fractional power of a graph was defined as follows.

**Definition 1.1.** [4] *Let  $G$  be a graph and  $m, n \in \mathbb{N}$ . The graph  $G^{\frac{m}{n}}$  is defined to be the  $m$ -power of the  $n$ -subdivision of  $G$ . In other words  $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$ .*

We can extend this definition to define  $G^{\frac{m}{f}}$  for any mapping  $f : E(G) \rightarrow \mathbb{N}$  and any  $m \in \mathbb{N}$  as follows.

**Definition 1.2.** *Let  $G$  be a graph,  $f : E(G) \rightarrow \mathbb{N}$  be a mapping and  $m \in \mathbb{N}$ . The graph  $G^{\frac{m}{f}}$  is defined to be the  $m$ -power of the  $f$ -subdivision of  $G$ . In other words  $G^{\frac{m}{f}} = (G^{\frac{1}{f}})^m$ .*

Note that the graphs  $(G^{\frac{1}{f}})^m$  and  $(G^m)^{\frac{1}{f}}$  are different graphs. In this paper, we only consider the graphs  $(G^{\frac{1}{f}})^m$ . Furthermore, the identity  $V(G^{\frac{m}{f}}) = V(G^{\frac{1}{f}})$  follows from Definition 1.2.

For any graph  $G$ , we have  $\gamma(G) \geq \gamma(G^2) \geq \gamma(G^3) \geq \dots \geq \gamma(G^r) = 1$ , where  $r$  is the radius of  $G$ . In fact, because  $G^k$  is a spanning subgraph of  $G^{k+1}$ , any  $\gamma$ -set of  $G^k$  is a dominating set for  $G^{k+1}$  and so  $\gamma(G^k) \geq \gamma(G^{k+1})$ . This shows that the domination number of a graph  $G$  decreases when we replace  $G$  by its powers. In other words, each dominating set of  $G^k$  is equivalent to a distance- $k$  dominating set of  $G$ . One can say that  $S$  is a dominating set if every vertex  $v$  is within a distance of at most one from  $S$ ,  $d(v, S) \leq 1$ . So we can consider domination for the distance greater than one. A set  $S$  is a *distance- $k$  dominating set* if for every vertex  $u \in V(G) \setminus S$ ,  $d(u, S) \leq k$ . The *distance- $k$  domination number*  $\gamma_{\leq k}(G)$  of  $G$  is equal to the minimum cardinality of a distance- $k$  dominating set in  $G$ . Therefore,  $\gamma_{\leq k}(G) = \gamma(G^k)$ . In many ways the distance versions of these concepts are more applicable to modeling real-world problems. For a comprehensive survey of results on distance domination in graphs the reader is referred to [5]. See also [3, 6].

On the other hand, it is easy to verify that by replacing any graph by its subdivisions, the domination number of graph increases. In other words,  $\gamma(G) \leq \gamma(G^{\frac{1}{k}}) \leq \gamma(G^{\frac{1}{k+1}})$  for any  $k \in \mathbb{N}$ . Now, the following question arises naturally.

**Question 1.3.** *What happens to the domination number, when we consider the fractional power of a graph, specially for fractions less than one?*

In this paper, we investigate the domination of the fractional powers of a graph and establish some relations between the domination numbers of these graphs.

### 2. Main results

Let  $G$  be a graph and suppose that  $f, r : E(G) \rightarrow \mathbb{N}$  are two functions such that  $f(e) \equiv r(e) \pmod{2m + 1}$  for each  $e \in E(G)$ . We begin this section by obtaining a relation between  $\gamma(G^{\frac{m}{f}})$  and  $\gamma(G^{\frac{m}{r}})$ .

**Theorem 2.1.** *Let  $G$  be a graph,  $f, r : E(G) \rightarrow \mathbb{N}$  be two functions,  $m \in \mathbb{N}$ ,  $f(e_i) = (2m + 1)k_i + r(e_i)$  where  $1 \leq r(e_i) \leq 2m + 1$  and  $e_i \in E(G) = \{e_1, e_2, \dots, e_q\}$ . Then*

- (a)  $\gamma(G^{\frac{m}{f}}) = \sum_{i=1}^q k_i + \gamma(G^{\frac{m}{r}})$ ,
- (b)  $i(G^{\frac{m}{f}}) = \sum_{i=1}^q k_i + i(G^{\frac{m}{r}})$ .

In order to prove the above theorem, we need the following lemmas.

**Lemma 2.2.** *Let  $G$  be a graph,  $S$  be a nonempty subset of  $V(G)$ ,  $x$  and  $y$  be two vertices of  $G$  such that  $d_G(x, S) = l_1$ ,  $d_G(y, S) = l_2$ ,  $P_1$  is a  $xu_1$ -path of length  $l_1$  and  $P_2$  is a  $yu_2$ -path of length  $l_2$  where  $u_1, u_2 \in S$ . If  $P_1$  and  $P_2$  have a common edge say  $zt$ , then*

- (a)  $d_G(y, z) - d_G(y, t) = 1$  if and only if  $d_G(x, z) - d_G(x, t) = 1$ ,
- (b)  $d_G(t, u_1) = d_G(t, u_2)$  and  $d_G(z, u_1) = d_G(z, u_2)$ .

*Proof.* (a) It is easy to see that either  $d_G(y, z) - d_G(y, t) = 1$  or  $-1$  (Similarly  $d_G(x, z) - d_G(x, t) = 1$  or  $-1$ ). Assume that  $d_G(y, z) - d_G(y, t) = 1$  and  $d_G(x, z) - d_G(x, t) = -1$ . Then we have  $d_G(y, z) = d_G(y, t) + 1$  and  $d_G(x, t) = d_G(x, z) + 1$  and so  $d_G(y, t) + d_G(z, u_2) = l_2 - 1$  and  $d_G(x, z) + d_G(t, u_1) = l_1 - 1$ . Therefore,  $d_G(y, t) + d_G(z, u_2) + d_G(x, z) + d_G(t, u_1) = (d_G(x, z) + d_G(z, u_2)) + (d_G(y, t) + d_G(t, u_1)) = (l_1 - 1) + (l_2 - 1)$ . It follows that, either  $d_G(x, S) \leq d_G(x, u_2) \leq d_G(x, z) + d_G(z, u_2) \leq l_1 - 1$  or  $d_G(y, S) \leq d_G(y, u_1) \leq d_G(y, t) + d_G(t, u_1) \leq l_2 - 1$ , a contradiction.

(b) The proof is straightforward. □

**Lemma 2.3.** *Let  $G$  be a graph and  $m \in \mathbb{N}$ . Then for any  $e \in E(G)$*

- (a)  $\gamma((S_{e,2m+2}(G))^m) = \gamma(G^m) + 1$  and
- (b)  $i((S_{e,2m+2}(G))^m) = i(G^m) + 1$ .

*Proof.* Suppose that  $S$  is a  $\gamma$ -set in  $G^m$ , we subdivide the edge  $e = xy$  to a  $(2m + 2)$ -path  $P_{xy}$  and  $H = S_{e,2m+2}(G)$ . Consider two cases:

**Case1.** For any vertex  $v$  of  $G^m$  we have  $d_{H^m}(v, S) \leq m$ . In this case,  $S' = S \cup \{(xy)_{m+1}\}$  is a dominating set for  $H^m$  where  $(xy)_{m+1}$  is the central vertex of  $P_{xy}$  which dominates all  $i$ -vertices of  $P_{xy}$ .

**Case2.** There is some vertex  $v$  in  $G^m$  for which  $d_{H^m}(v, S) > m$  and  $d_{G^m}(v, S) \leq m$ . Assume that  $V_1 = \{v_1, v_2, \dots, v_k\}$  is the set of all vertices which are similar to  $v$ . So  $xy$  is a common edge of the paths  $P_{v_1u_1}, P_{v_2u_2}, \dots, P_{v_ku_k}$  of minimum lengths between the elements of  $V_1$  and  $S$ . Thus, Lemma 2.2 implies that  $d_G(x, u_1) = d_G(x, u_2) = \dots = d_G(x, u_k) = l$  and  $d_G(y, u_1) = d_G(y, u_2) = \dots = d_G(y, u_k) = l - 1$  (suppose that  $d_G(x, S) = d_G(y, S) + 1$ ). To define a dominating set  $S'$  for  $H^m$ , we add  $i$ -vertex  $(xy)_l$  to  $S$ . This vertex dominates all elements of  $V_1$  and the  $i$ -vertices  $(xy)_1, (xy)_2, \dots, (xy)_{l+m}$  of  $P_{xy}$ . In addition, because  $d_G((xy)_{l+m+1}, S) = 2m + 2 - (l + m + 1) + l - 1 = m$  so  $S$  dominates all  $i$ -vertices  $(xy)_{l+m+1}, (xy)_{l+m+2}, \dots, (xy)_{2m+2}$ . Hence,  $S'$  is a dominating set of  $H^m$ .

It follows that,  $\gamma(H^m) \leq |S'| = \gamma(G^m) + 1$ .

Now we prove that  $\gamma(H^m) \geq \gamma(G^m) + 1$ . Let  $S'$  be a  $\gamma$ -set of  $H^m$ . Since the  $i$ -vertex  $(xy)_{m+1}$  is only dominated by  $i$ -vertices of  $P_{xy}$ , so  $S_1 = S' \cap \{(xy)_j | 1 \leq j \leq 2m + 1\} \neq \emptyset$ . If  $|S_1| \geq 2$ , then we can easily show that  $S = (S' - S_1) \cup \{x\}$  is a dominating set of  $G^m$ . Therefore,  $\gamma(H^m) - 1 = |S'| - 2 + 1 \geq |S| \geq \gamma(G^m)$ . Now assume that  $|S_1| = 1$ .

We consider two cases:

**Case1.**  $S_1 = \{(xy)_{m+1}\}$ . In this case, this element of  $\gamma$ -set  $S'$  only dominates  $i$ -vertices of  $P_{xy}$  and so  $S = S' \setminus \{(xy)_{m+1}\}$  must be a dominating set of  $G^m$ . Hence,  $\gamma(H^m) - 1 = |S'| - 1 = |S| \geq \gamma(G^m)$ .

**Case2.**  $S_1 = \{(xy)_i\}$ , where  $i \neq m + 1$ . Without loss of generality, suppose that  $1 \leq i \leq m$ . Because  $N_m[(xy)_i] = \{(xy)_j | 1 \leq j \leq i + m\} \cup N_{m-i}[x]$ ,  $i$ -vertices  $(xy)_{i+m+1}, (xy)_{i+m+2}, \dots, (xy)_{2m+1}$  must be dominated by a vertex of  $G$  such as  $u \in S - S_1$ . Thus,  $d_H((xy)_{i+m+1}, u) \leq m$  and  $d_H(y, u) \leq m - (2m + 2 - (i + m + 1)) = i - 1$ . Hence, we have  $N_{m-i}[x] \subseteq N_m[u]$  in  $G$ . Therefore,  $S' - S_1$  is a dominating set of  $G^m$  and  $\gamma(H^m) - 1 = |S' - S_1| \geq \gamma(G^m)$ . So  $\gamma(H^m) \geq \gamma(G^m) + 1$  and consequently,  $\gamma(H^m) = \gamma(G^m) + 1$ .

The proof of the second part is quite similar.  $\square$

**Corollary 2.4.** *Let  $G$  be a graph,  $m \in \mathbb{N}$ , and  $P_{2m+2}$  is an induced subgraph of  $G$  between vertices  $x$  and  $y$ . If we replace this subgraph with the edge  $xy$ , then for the resulting graph  $H$ , we have*

- (a)  $\gamma(H^m) = \gamma(G^m) - 1,$
- (b)  $i(H^m) = i(G^m) - 1.$

**Lemma 2.5.** *Let  $G$  be a graph and  $n, m \in \mathbb{N}$ . If  $n = (2m + 1)k + r$ ,  $1 \leq r \leq 2m + 1$ , and  $e \in E(G)$ , then*

- (a)  $\gamma((S_{e,n}(G))^m) = k + \gamma((S_{e,r}(G))^m),$
- (b)  $i((S_{e,n}(G))^m) = k + i((S_{e,r}(G))^m).$

*Proof.* (a) We proceed by induction on  $k$ . There is nothing to prove if  $k = 0$ . Suppose that  $n = (k + 1)(2m + 1) + r$  and  $n' = k(2m + 1) + r$ . We have  $\gamma((S_{e,n'}(G))^m) = k + \gamma((S_{e,r}(G))^m)$ . On the  $G$ -edge  $e = xy$  of  $S_{e,n'}(G)$  consider the edge  $e' = (xy)_0(xy)_1 \in S_{e,n'}(G)$ . It is easy to show that  $S_{e',2m+2}(S_{e,n'}(G)) \cong S_{e,n}(G)$ . By applying Lemma 2.3, we conclude that  $\gamma((S_{e,n}(G))^m) = \gamma((S_{e',2m+2}(S_{e,n'}(G)))^m) = 1 + \gamma((S_{e,n'}(G))^m) = 1 + k + \gamma((S_{e,r}(G))^m)$ . So (a) was proved for  $n = (k + 1)(2m + 1) + r$ . We can similarly prove the second part. □

Now we can prove Theorem 2.1.

**Proof of Theorem 2.1.** We only prove part (a). As seen before,

$$G^{\frac{1}{f}} = S_{e_1,f(e_1)}(S_{e_2,f(e_2)}(\cdots S_{e_q,f(e_q)}(G) \cdots)),$$

and

$$G^{\frac{1}{r}} = S_{e_1,r(e_1)}(S_{e_2,r(e_2)}(\cdots S_{e_q,r(e_q)}(G) \cdots)).$$

Now by applying Lemma 2.5,

$$\begin{aligned} \gamma(G^{\frac{m}{f}}) &= \gamma((S_{e_1,f(e_1)}(S_{e_2,f(e_2)}(\cdots S_{e_q,f(e_q)}(G))^m \cdots)) \\ &= k_1 + \gamma((S_{e_1,r(e_1)}(S_{e_2,f(e_2)}(\cdots S_{e_q,f(e_q)}(G))^m \cdots)) \\ &= k_1 + k_2 + \gamma((S_{e_1,r(e_1)}(S_{e_2,r(e_2)}(\cdots S_{e_q,f(e_q)}(G))^m \cdots)) \\ &= \dots = \sum_{i=1}^q k_i + \gamma((S_{e_1,r(e_1)}(S_{e_2,r(e_2)}(\cdots S_{e_q,r(e_q)}(G))^m \cdots)) \\ &= \sum_{i=1}^q k_i + \gamma(G^{\frac{m}{r}}). \end{aligned} \quad \square$$

**Corollary 2.6.** *If  $G$  is a graph,  $n, m \in \mathbb{N}$  and  $n = (2m + 1)k + r$  where  $1 \leq r \leq 2m + 1$ , then*

- (a)  $\gamma(G^{\frac{m}{n}}) = k.q(G) + \gamma(G^{\frac{m}{r}}),$
- (b)  $i(G^{\frac{m}{n}}) = k.q(G) + i(G^{\frac{m}{r}}).$

*Proof.* According to Theorem 2.1, we only need to consider the constant mappings  $F, R : E(G) \rightarrow \mathbb{N}$  defined by  $F(e) = n$  and  $R(e) = r$  for each edge  $e \in E(G)$ . □

In the next theorem, we prove that the set of  $t$ -vertices of the  $\frac{m}{2m+1}$ -power of any graph is an independent  $\gamma$ -set.

**Theorem 2.7.** *Let  $G$  be a graph and  $m \in \mathbb{N}$ ; and let  $f : E(G) \rightarrow \mathbb{N}$  be a mapping such that for each  $e_i \in E(G) = \{e_1, e_2, \dots, e_q\}$ ,  $f(e_i) = (2m + 1)k_i$  and  $k_i \in \mathbb{N}(1 \leq i \leq q)$ . Then*

- (a)  $i(G^{\frac{m}{2m+1}}) = \gamma(G^{\frac{m}{2m+1}}) = p(G)$ .  
 (b)  $i(G^{\frac{m}{f}}) = \gamma(G^{\frac{m}{f}}) = \sum_{i=1}^q k_i - q(G) + p(G)$ .

*Proof.* (a) Consider the closed  $m$ -neighborhoods of the  $t$ -vertices of  $G^{\frac{m}{2m+1}}$ . The intersection of any two of them is empty. Additionally, any dominating set has a common vertex with each of them. Therefore,  $\gamma(G^{\frac{m}{2m+1}}) \geq p(G)$ . Furthermore, the set of  $t$ -vertices of  $G^{\frac{m}{2m+1}}$  is an independent dominating set. Hence,  $i(G^{\frac{m}{2m+1}}) = \gamma(G^{\frac{m}{2m+1}}) = p(G)$ .

(b) This part can be deduced from Theorem 2.1 and part (a).  $\square$

Next theorems and corollaries give us some inequalities between domination numbers of fractional powers of a graph.

**Theorem 2.8.** *Let  $G$  be a graph and  $m, n \in \mathbb{N}$ .*

- (a)  $\gamma(G^{\frac{m}{n}}) \geq \gamma(G^{\frac{m+1}{n}})$ ,  
 (b)  $\gamma(G^{\frac{m}{n}}) \leq \gamma(G^{\frac{m}{n+1}})$ ,  
 (c) if  $\frac{m}{n} < 1$  then  $\gamma(G^{\frac{m}{n}}) \geq \gamma(G^{\frac{m+1}{n+1}})$ , and  
 (d) if  $\frac{m}{n} > 1$  then  $\gamma(G^{\frac{m}{n}}) \leq \gamma(G^{\frac{m+1}{n+1}})$ .

*Proof.* (a) Because  $G^{\frac{m}{n}}$  is a spanning subgraph of  $G^{\frac{m+1}{n}}$ , any  $\gamma$ -set of  $G^{\frac{m}{n}}$  is a dominating set for  $G^{\frac{m+1}{n}}$  and so  $\gamma(G^{\frac{m}{n}}) \geq \gamma(G^{\frac{m+1}{n}})$ .

(b) Suppose that  $S$  is a  $\gamma$ -set of  $G^{\frac{m}{n+1}}$ . We select one  $i$ -vertex from each  $G$ -edge of  $G^{\frac{1}{n+1}}$  and collect them in  $S_1$ . Assume that  $x \in S \cap S_1$  and  $y$  and  $z$  are two neighbours of  $x$  in  $G^{\frac{1}{n+1}}$ . It is easy to see that either  $y \notin S$  or  $z \notin S$ . To see this, suppose that  $\{x, y, z\} \subseteq S$ . Any vertex that is dominated by  $x$ , is also dominated by  $y$  or  $z$ . In addition,  $x$  is dominated by  $y$  and  $z$ . So  $S \setminus \{x\}$  is a dominating set of  $G^{\frac{m}{n+1}}$ , a contradiction. For each  $v \in S_1$  we do the following operation on  $G^{\frac{1}{n+1}}$ : If  $v \notin S$  we connect two neighbours of  $v$  in  $G^{\frac{1}{n+1}}$ , and if  $v \in S$  we connect two neighbours  $v'$  and  $v''$  of  $v$  in  $G^{\frac{1}{n+1}}$  and change  $S$  to  $(S \setminus \{v\}) \cup \{v'\}$  where  $v' \notin S$ . If we denote this graph by  $H$  then we can easily show that  $G^{\frac{1}{n}}$  is a subgraph of  $H$  and (the new)  $S$  is a dominating set for  $H^m$ . Therefore,  $S$  is a dominating set of  $G^{\frac{m}{n}}$  and so  $\gamma(G^{\frac{m}{n}}) \leq \gamma(H^m) \leq |S| = \gamma(G^{\frac{m}{n+1}})$ .



(c) Assume that  $S$  is a  $\gamma$ -set of  $G^{\frac{m}{n}}$ . To construct a dominating set from  $S$  for  $G^{\frac{m+1}{n+1}}$ , select a central edge (or one of the two central edges) of each  $G$ -edges of  $G^{\frac{1}{n}}$  and subdivide each of them to the two edges. Let  $v_{xy}$  be a vertex which is added to  $G$ -edge  $xy$ . The resulting graph is isomorphic to  $G^{\frac{1}{n+1}}$  and we denote it by  $H$ . We prove that  $S$  is a dominating set for  $(m+1)$ -power of  $H$ , that is isomorphic to  $G^{\frac{m+1}{n+1}}$ . Suppose that  $x$  is a vertex of  $G^{\frac{1}{n}}$ . Since  $S$  is a  $\gamma$ -set for  $G^{\frac{m}{n}}$ , we deduce that  $d_{G^{\frac{1}{n}}}(x, S) \leq m$ . Therefore, there is  $v \in S$  such that  $d_{G^{\frac{1}{n}}}(x, v) \leq m$ . Call the shortest  $xv$ -path  $P$ . Because the distance of any two new vertices  $v_{xy}$  and  $v_{x'y'}$  in  $H$  is at least  $n$ , we deduce that  $P$  contains at most one of these new vertices and hence  $d_H(x, v) \leq m + 1$ . So  $x$  is dominated by  $S$  in  $H^{m+1}$ . In addition, for any vertex  $v_{xy}$  of  $H$ ,  $d_H(v_{xy}, S) \leq d_H((xy)_{[\frac{n}{2}]}, S) + 1$  where  $(xy)_{[\frac{n}{2}]}$  is a neighbour of  $v_{xy}$  in  $H$ . Therefore,  $d_H(v_{xy}, S) \leq m + 1$  and  $v_{xy}$  is also dominated by  $S$  in  $H^{m+1}$ . So,  $S$  is a dominating set in  $G^{\frac{m+1}{n+1}}$  and hence  $\gamma(G^{\frac{m+1}{n+1}}) \leq |S| = \gamma(G^{\frac{m}{n}})$ .

(d) Suppose that  $S$  is a  $\gamma$ -set of  $G^{\frac{m+1}{n+1}}$ . To construct a dominating set from  $S$  for  $G^{\frac{m}{n}}$ , select a central  $i$ -vertex (or one of the two central  $i$ -vertices) from each  $G$ -edge of  $G^{\frac{1}{n+1}}$  and collect them in  $S_1$ . Assume that  $x \in S \cap S_1$  and  $y$  and  $z$  are two neighbours of  $x$  in  $G^{\frac{1}{n+1}}$ . As seen in part (b), either  $y \notin S$  or  $z \notin S$ . For each  $v \in S_1$ , we do the following operation on  $G^{\frac{1}{n+1}}$ : If  $v \notin S$  we connect two neighbours of  $v$  in  $G^{\frac{1}{n+1}}$  and remove  $v$ , and if  $v \in S$  we connect two neighbours  $v'$  and  $v''$  of  $v$  in  $G^{\frac{1}{n+1}}$  and change  $S$  to  $(S \setminus \{v\}) \cup \{v'\}$  where  $v' \notin S$  and then remove  $v$ . If we denote this graph by  $H$ , then we can easily show that  $H \cong G^{\frac{1}{n}}$  and  $H^m \cong G^{\frac{m}{n}}$ . We prove that the resulting set  $S$  is also a dominating set of  $H^m$  and conclude that  $\gamma(G^{\frac{m}{n}}) = \gamma(H^m) \leq |S| = \gamma(G^{\frac{m+1}{n+1}})$ . Suppose that  $x$  is a vertex of  $G^{\frac{1}{n+1}}$  which is not in  $S_1$ . Because  $S$  is a  $\gamma$ -set for  $G^{\frac{m+1}{n+1}}$ , so  $d_{G^{\frac{1}{n+1}}}(x, S) \leq m + 1$ . Therefore, there is  $v \in S$  such that  $d_{G^{\frac{1}{n+1}}}(x, v) \leq m + 1$ . Since the distance of any two vertices of  $S_1$  in  $G^{\frac{1}{n+1}}$  is at least  $n$  and  $m > n$ , so  $d_H(x, v) \leq m$ . Therefore,  $x$  is dominated by  $S$  in  $H^m$ , and hence,  $S$  is a dominating set of  $H^m$ .  $\square$

We know that  $G^{\frac{m}{n}}$  is an induced subgraph of  $G^{\frac{mk}{nk}}$  which is not spanning subgraph when  $k \geq 2$ . The next theorem and corollary show the

relations between domination numbers of  $\frac{mk}{nk}$ -powers of a graph when  $k \in \mathbb{N}$ .

**Theorem 2.9.** *Let  $G$  be a graph and  $m, k \in \mathbb{N}$ . Then  $\gamma(G^{\frac{mk}{k}}) \leq \gamma(G^{\frac{m(k+1)}{k+1}})$ .*

*Proof.* Assume that  $S$  is a  $\gamma$ -set of  $G^{\frac{m(k+1)}{k+1}}$ . To construct a dominating set from  $S$  for  $G^{\frac{mk}{k}}$ , select a central  $i$ -vertex (or one of the two central  $i$ -vertices) from each  $G$ -edge of  $G^{\frac{1}{k+1}}$  and collect them in  $S_1$ . Assume that  $x \in S \cap S_1$  and  $y$  and  $z$  are two neighbours of  $x$  in  $G^{\frac{1}{k+1}}$ . As you see in part (b) of Theorem 2.8, either  $y \notin S$  or  $z \notin S$ . For each  $v \in S_1$  we do the following operation on  $G^{\frac{1}{k+1}}$ : If  $v \notin S$  we connect two neighbours of  $v$  in  $G^{\frac{1}{k+1}}$  and remove  $v$ , and if  $v \in S$  we connect two neighbours  $v'$  and  $v''$  of  $v$  in  $G^{\frac{1}{k+1}}$  and change  $S$  to  $(S - \{v\}) \cup \{v'\}$  where  $v' \notin S$  and then remove  $v$ . If we denote this graph by  $H$  then we can easily show that  $H \cong G^{\frac{1}{k}}$  and  $H^{mk} \cong G^{\frac{mk}{k}}$ . We prove that the resulting set  $S$  is also a dominating set for  $H^{mk}$  and we conclude that  $\gamma(G^{\frac{mk}{k}}) = \gamma(H^{mk}) \leq |S| = \gamma(G^{\frac{m(k+1)}{k+1}})$ . Suppose that  $x$  is a vertex of  $G^{\frac{1}{k+1}}$  which is not in  $S_1$ . Since  $S$  is a  $\gamma$ -set for  $G^{\frac{m(k+1)}{k+1}}$ , so  $d_{G^{\frac{1}{k+1}}}(x, S) \leq mk + k$  and therefore, there is  $v \in S$  such that  $d_{G^{\frac{1}{k+1}}}(x, v) \leq mk + k$ . We consider two cases:

**Case1.**  $d_{G^{\frac{1}{k+1}}}(x, v) \leq mk$

In this case,  $d_H(x, v) \leq d_{G^{\frac{1}{k+1}}}(x, v) \leq mk$  and hence  $x$  is dominated by  $S$  in  $H^{mk}$ .

**Case2.**  $mk + 1 \leq d_{G^{\frac{1}{k+1}}}(x, v) \leq mk + k$

Call the shortest  $xv$ -path  $P$  (in  $G^{\frac{1}{k+1}}$ ) and assume that the length of  $P$  is  $mk + i$  where  $1 \leq i \leq m$ .  $P$  contains at least  $n_1 = \lfloor \frac{mk+i}{k+1} \rfloor$  vertices of  $S_1$ . Therefore,  $d_H(x, v) \leq mk + i - n_1$ . Because  $i \leq m$ ,  $i \leq n_1$  and so  $mk + i - n_1 \leq mk$ . Thus,  $d_H(x, v) \leq mk$  and  $x$  is dominated by  $S$  in  $H^{mk}$ . Therefore,  $S$  is a dominating set of  $H^{mk}$ .  $\square$

**Corollary 2.10.** *Let  $G$  be a graph and  $m, n, k, l \in \mathbb{N}$ .*

(a)  $\gamma(G^{\frac{mk}{k}}) \leq \gamma(G^{\frac{ml}{l}})$  when  $k < l$ ,

(b)  $\gamma(G^{\frac{mk}{nk}}) \leq \gamma(G^{\frac{ml}{nl}})$  when  $k < l$ ,

(c)  $\gamma(G^m) \leq \gamma(G^{\frac{mk}{k}})$ , and

(d)  $\gamma(G^{\frac{m}{n}}) \leq \gamma(G^{\frac{mk}{nk}})$ .

*Proof.* We only prove (b). Since  $G^{\frac{1}{mn}} = (G^{\frac{1}{m}})^{\frac{1}{n}}$ , we have  $\gamma(G^{\frac{mk}{nk}}) = \gamma((G^{\frac{1}{nk}})^{mk}) = \gamma(((G^{\frac{1}{n}})^{\frac{1}{k}})^{mk}) = \gamma((G^{\frac{1}{n}})^{\frac{mk}{k}}) \leq \gamma((G^{\frac{1}{n}})^{\frac{ml}{l}}) = \gamma(G^{\frac{ml}{nl}})$ .  $\square$

In the next theorem and its corollary, we find an upper bound for the increasing sequence  $\{\gamma(G^{\frac{mk}{nk}})\}$  when  $m, n, k \in \mathbb{N}$ .

**Theorem 2.11.** *Let  $G$  be a graph and  $m, k \in \mathbb{N}$ . Then  $\gamma(G^{\frac{mk}{k}}) \leq \gamma(G^{m-1})$  when  $m > 1$ , and  $\gamma(G^{\frac{k}{k}}) \leq p(G)$ .*

*Proof.* Suppose that  $m > 1$  and  $S$  is a  $\gamma$ -set of  $G^{m-1}$ . Note that  $S$  is also a subset of  $t$ -vertices of  $G^{\frac{mk}{k}}$ . We prove that  $S$  is a dominating set of  $G^{\frac{mk}{k}}$ . Consider the vertex  $(xy)_i$  of  $G^{\frac{mk}{k}}$  where  $0 \leq i \leq k$ . Now we have  $d_{G^{\frac{1}{k}}}(xy)_i, S \leq d_{G^{\frac{1}{k}}}(xy)_i, x + d_{G^{\frac{1}{k}}}(x, S) \leq k + k \cdot d_G(x, S) \leq k + k(m - 1) = km$ . Thus,  $\gamma(G^{\frac{mk}{k}}) \leq |S| = \gamma(G^{m-1})$ .

Obviously, we can prove that the set of terminal vertices of  $G^{\frac{k}{k}}$  is a dominating set for this graph.  $\square$

**Corollary 2.12.** *Let  $G$  be a graph and  $m, n, k \in \mathbb{N}$ . Then  $\gamma(G^{\frac{mk}{nk}}) \leq \gamma(G^{\frac{m-1}{n}})$  when  $m > 1$ , and  $\gamma(G^{\frac{k}{nk}}) \leq p(G^{\frac{1}{n}})$ .*

*Proof.* Assume that  $m > 1$ . Because  $G^{\frac{1}{mn}} = (G^{\frac{1}{m}})^{\frac{1}{n}}$ , we have  $\gamma(G^{\frac{mk}{nk}}) = \gamma((G^{\frac{1}{nk}})^{mk}) = \gamma(((G^{\frac{1}{n}})^{\frac{1}{k}})^{mk}) = \gamma((G^{\frac{1}{n}})^{\frac{mk}{k}}) \leq \gamma((G^{\frac{1}{n}})^{m-1}) = \gamma(G^{\frac{m-1}{n}})$ .  $\square$

### 3. Concluding remarks

Let  $G$  be a graph. When  $\frac{m}{n} < 1$ , the sequence  $x_k = \frac{m+k}{n+k}$  is an increasing sequence that converges to 1, and when  $\frac{m}{n} > 1$ , the sequence  $y_k = \frac{m+k}{n+k}$  is a decreasing sequence that converges to 1. By using Theorem 2.8,  $\{a_k\} = \{\gamma(G^{x_k})\}$  is a decreasing sequence and  $\{b_k\} = \{\gamma(G^{y_k})\}$  is an increasing sequence. In addition,  $\gamma(G^{x_k}) \geq 1$  and  $\gamma(G^{y_k}) \leq p(G)$  (because in this case the set of terminal vertices of  $G^{\frac{m+k}{n+k}}$  is a dominating set). Thus, the sequences  $\{a_k\}$  and  $\{b_k\}$  are convergent.

**Problem 3.1.** *What are the limits of the sequences  $\{a_k\}$  and  $\{b_k\}$ ?*

Similarly, consider the constant sequence  $z_k = \frac{mk}{nk}$ . According to Theorem 2.9 and Corollary 2.12, the sequence  $\{c_k\} = \{\gamma(G^{z_k})\}$  is increasing and bounded. Therefore, the sequence  $\{c_k\}$  is convergent.

**Problem 3.2.** *What is the limit of the sequence  $\{c_k\}$ ?*

We know that  $\gamma(G^k) \geq \gamma(G^{k+1})$  for any  $k \in \mathbb{N}$  and we proved the similar inequality for the fractional powers in some special cases.

**Problem 3.3.** *Let  $r, r' \in \mathbb{Q}^+$  and  $r < r'$ . Is it correct that  $\gamma(G^r) \geq \gamma(G^{r'})$ ?*

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