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THE LOCATING CHROMATIC NUMBER OF THE JOIN OF GRAPHS

A. BEHTOEI* AND M. ANBARLOEI

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Abstract. Let f be a proper k-coloring of a connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple $c_{\Pi}(v) = (d(v, V_1), d(v, V_2), \ldots, d(v, V_k))$, where $d(v, V_i) = \min\{d(v, x) : x \in V_i\}, 1 \leq i \leq k$. If distinct vertices have distinct color codes, then f is called a locating coloring. The minimum number of colors needed in a locating coloring of G is the locating chromatic number of G, denoted by $\chi_{_L}(G)$. In this paper, we study the locating chromatic number of the join of graphs. We show that when G_1 and G_2 are two connected graphs with diameter at most two, then $\chi_{_L}(G_1 \vee G_2) = \chi_{_L}(G_1) + \chi_{_L}(G_2)$, where $G_1 \vee G_2$ is the join of G_1 and G_2. Also, we determine the locating chromatic number of the join of paths, cycles and complete multipartite graphs.

Keywords: Locating coloring, locating chromatic number, fan, wheel, join.
MSC(2010): Primary: 05C15; Secondary: 05C12.

1. Introduction

Let G be a graph, without loops and multiple edges, with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of G, $k \in \mathbb{N}$, is a function f defined from $V(G)$ onto a set of colors $[k] = \{1, 2, \ldots, k\}$ such that every two adjacent vertices have different colors. In fact, for every i, $1 \leq i \leq k$,

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the set $f^{-1}(i)$ is a nonempty independent set of vertices which is called the color class i. When $S \subseteq V(G)$, then $f(S) = \{f(u) : u \in S\}$. The minimum cardinality k for which G has a proper k-coloring is the chromatic number of G, denoted by $\chi(G)$. For a connected graph G, the distance $d(u, v)$ between two vertices u and v in G is the length of the shortest path between them, and for a subset S of $V(G)$, the distance between u and S is given by $d(u, S) = \min\{d(u, x) : x \in S\}$. The diameter of G is $\max\{d(u, v) : u, v \in V(G)\}$. When u is a vertex of G, then the neighbor of u in G is the set $N_G(u) = \{v : v \in V(G),\ d(u, v) = 1\}$.

Definition 1.1. [4] *Let f be a proper k-coloring of a connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. For a vertex v of G, the color code of v with respect to Π is defined to be the ordered k-tuple*

$$c_{\Pi}(v) = (d(v, V_1), d(v, V_2), \ldots, d(v, V_k)).$$

If distinct vertices of G have distinct color codes, then f is called a locating coloring of G. The locating chromatic number, denoted by $\chi_L(G)$, is the minimum number of colors in a locating coloring of G.

The concept of locating coloring was first introduced and studied by Chartrand et al. in [4]. They established some bounds for the locating chromatic number of a connected graph. They also proved that for a connected graph G with $n \geq 3$ vertices, we have $\chi_L(G) = n$ if and only if G is a complete multipartite graph. Hence, the locating chromatic number of the complete graph K_n is n. Also for paths and cycles of order $n \geq 3$ it is proved in [4] that $\chi_L(P_n) = 3$, $\chi_L(C_n) = 3$ when n is odd, and $\chi_L(C_n) = 4$ when n is even.
The locating chromatic number of trees, of Kneser graphs, and that of the Cartesian product of graphs were studied in [4] and [3], [2], and [1], respectively. For more results in the subject and related topics, see [5] to [10].

Obviously, $\chi(G) \leq \chi_L(G)$. Note that the i-th coordinate of the color code of each vertex in the color class V_i is zero and its other coordinates are non zero. Hence, a proper coloring is a locating coloring whenever the color codes of vertices in each color class are different.

Recall that the join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \bigcup V(G_2)$ and edge set $E(G_1) \bigcup E(G_2) \bigcup \{uv : u \in V(G_1),\ v \in V(G_2)\}$. For example $K_1 \vee P_n$ is the fan F_n, $K_1 \vee C_n$

is the wheel W_n, and the friendship graph Fr_n, $n = 2t+1$, is the graph obtained by joining K_1 to the t disjoint copies of K_2.

In this paper, we study the locating chromatic number of the join of graphs. Although we always have $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$, but it may happen that $\chi_{_L}(G_1 \vee G_2) \neq \chi_{_L}(G_1) + \chi_{_L}(G_2)$. For example we have $\chi_{_L}(P_{10}) = 3$ while, by Corollary 3.8 (see Section 3), $\chi_{_L}(P_{10} \vee P_{10}) = 8$.

The diameter of $G_1 \vee G_2$ is at most two. Hence, in each proper coloring of $G_1 \vee G_2$, the color code of no vertex of $G_1 \vee G_2$ has a coordinate greater than two. This fact motivated us to define a new parameter, the adjacency locating chromatic number, which is closely related to the locating chromatic number. Proposition 2.2 and Theorem 3.1 (see Section 3) show the relation of this parameter with the locating chromatic number. Using this new parameter we determine the exact value of the locating chromatic number of $P_m \vee P_n$, $K_m \vee P_n$, $P_m \vee C_n$, $K_m \vee C_n$, and $C_m \vee C_n$ in terms of m and n.

2. The adjacency locating chromatic number

The following parameter can be defined for disconnected graphs.

Definition 2.1. *Let f be a proper k-coloring of a graph G. If for each two distinct vertices u and v with the same color $f(N_G(u)) \neq f(N_G(v))$, then we say f is an adjacency locating coloring of G. The adjacency locating chromatic number, $\chi_{_{L2}}(G)$, is the minimum number of colors in an adjacency locating coloring of G.*

Note that we always have $\chi(G) \leq \chi_{_{L2}}(G) \leq |V(G)|$. To see the relation between two parameters $\chi_{_L}$ and $\chi_{_{L2}}$, let f be a k-coloring of the connected graph G and $\Pi = (V_1, V_2, \ldots, V_k)$ be an ordered partition of $V(G)$ into the resulting color classes. Now for each $v \in V(G)$ determine the color code $c_{_\Pi}(v)$. Then, in the color code of each vertex replace by 2 all of the coordinates which are at least two. We call these new color codes modified color codes. Thus, in the modified color code of a vertex v exactly one coordinate is zero, $|N_G(v)|$ coordinates are 1, and the other coordinates are 2.

Now it is easy to see that f is an adjacency locating coloring if and only if different vertices of G have different modified color codes. Therefore, each adjacency locating coloring of G is a locating coloring. Hence, $\chi_{_L}(G) \leq \chi_{_{L2}}(G)$. Also, note that when the diameter of G is at most two, then each locating coloring of G is an adjacency locating coloring and hence, $\chi_{_{L2}}(G) \leq \chi_{_L}(G)$. Thus, we have the following proposition.

Proposition 2.2. *If* G *is a connected graph with diameter at most two, then* $\chi_{_L}(G) = \chi_{_{L2}}(G)$.

Specially, $\chi_{_{L2}}(K_n) = n$ and $\chi_{_{L2}}(K_{m,n}) = m + n$. It is well known that almost all graphs have diameter two. According to this fact, it is possible to compute $\chi_{_{L2}}$ instated of $\chi_{_L}$ almost for all graphs. This is interesting and useful, since computing the locating chromatic number of a graph directly may be infeasible if computing the distances in the graph is costly, while the adjacency locating chromatic number may be computed by only considering the equality, adjacency or non-adjacency of a vertex with each of the color classes. These two parameters can also be arbitrarily far apart. For example, by Theorem 3.6 (see Section 3), for each $n \geq 3$, $\chi_{_{L2}}(P_n) = \min\{k : \ k \in \mathbb{N}, \ n \leq \frac{1}{2}(k^3 - k^2)\}$ while $\chi_{_L}(P_n) = 3$. Note that the path P_9 is a graph whose diameter is greater than two but $\chi_{_{L2}}(P_9) = \chi_{_L}(P_9) = 3$. Theorem 3.6 also implies that for each positive integer m there exists a graph (a path) whose adjacency locating chromatic number is m.

3. The locating chromatic number and the join operation

In this section, we first study the locating chromatic number of the join of two arbitrary graphs. Then, we determine the locating chromatic number of the friendship graphs, and the join of paths, cycles and complete multipartite graphs. Specially, we determine $\chi_{_L}(F_n)$ and $\chi_{_L}(W_n)$.

Theorem 3.1. *For two arbitrary graphs* G_1 *and* G_2*, we have* $\chi_{_L}(G_1 \vee G_2) = \chi_{_{L2}}(G_1) + \chi_{_{L2}}(G_2)$.

Proof. The diameter of $G_1 \vee G_2$ is at most two and hence, by Proposition 2.2, $\chi_{_L}(G_1 \vee G_2) = \chi_{_{L2}}(G_1 \vee G_2)$. Let $k_1 = \chi_{_{L2}}(G_1)$, $k_2 = \chi_{_{L2}}(G_2)$, and $k = \chi_{_{L2}}(G_1 \vee G_2)$. Also, let f be an adjacency locating k-coloring of $G_1 \vee G_2$. Vertices of G_1 are adjacent to the vertices of G_2 and hence,

$$\{f(u) : \ u \in V(G_1)\} \bigcap \{f(v) : \ v \in V(G_2)\} = \emptyset.$$

Let $k'_1 = |\{f(u) : \ u \in V(G_1)\}|$ and $k'_2 = |\{f(v) : \ v \in V(G_2)\}|$. Thus, $k = k'_1 + k'_2$. Assume that u and u' are two vertices of G_1 with the same color. Since f is an adjacency locating coloring and $V(G_2) \subseteq N_{_{G_1 \vee G_2}}(u) \bigcap N_{_{G_1 \vee G_2}}(u')$, we have $f(N_{_{G_1}}(u)) \neq f(N_{_{G_1}}(u'))$. This means that the restriction of f to $V(G_1)$ is an adjacency locating k'_1-coloring of G_1. Hence, $k_1 \leq k'_1$. A similar argument holds for G_2. Thus, $k_1 + k_2 \leq k'_1 + k'_2 = k$.

Now let f_1 be an adjacency locating k_1-colorings of G_1 with the color set $\{1, 2, \ldots, k_1\}$, and f_2 be an adjacency locating k_2-colorings of G_2 with the color set $\{k_1 + 1, k_1 + 2, \ldots, k_1 + k_2\}$. Define the $(k_1 + k_2)$-coloring f' of $G_1 \vee G_2$ as $f'(u) = f_1(u)$ when $u \in V(G_1)$, and $f'(v) = f_2(v)$ when $v \in V(G_2)$. If z_1 and z_2 are two vertices in $G_1 \vee G_2$ with $f'(z_1) = f'(z_2)$, then $\{z_1, z_2\} \subseteq V(G_1)$ or $\{z_1, z_2\} \subseteq V(G_2)$. Without loss of generality, assume that $\{z_1, z_2\} \subseteq V(G_1)$. Since f_1 is an adjacency locating k_1-coloring and $V(G_2) \subseteq N_{G_1 \vee G_2}(z_1) \bigcap N_{G_1 \vee G_2}(z_2)$, we have $f'(N_{G_1 \vee G_2}(z_1)) \neq f'(N_{G_1 \vee G_2}(z_2))$. This means that f' is an adjacency locating $(k_1 + k_2)$-coloring of $G_1 \vee G_2$ and hence, $k \leq k_1 + k_2$. Thus, $k = k_1 + k_2$, which completes the proof. □

Thus, $\chi_L(G_1 \vee G_2) \geq \chi_L(G_1) + \chi_L(G_2)$. Theorem 3.1 and Proposition 2.2 imply the following corollary.

Corollary 3.2. *If G_1 and G_2 are two connected graphs with diameter at most two, then $\chi_L(G_1 \vee G_2) = \chi_L(G_1) + \chi_L(G_2)$.*

Let m, t be two positive integers and $G = tK_m$ be the graph consisting of t disjoint copies of K_m. A coloring of G is an adjacency locating coloring if and only if no two different components of G have the same color set. For a positive integer k, the set $[k]$ has $\binom{k}{m}$ distinct subsets of size m. Thus, $\chi_{L2}(G) = \min\{k : \ t \leq \binom{k}{m}\}$. Now Theorem 3.1 implies the following result.

Proposition 3.3. *For a positive integer t, let $n = 2t + 1$. Then, the locating chromatic number of the friendship graph Fr_n is $1 + \min\{k : \ t \leq \binom{k}{2}\}$.*

Let $P_n = v_1 v_2 \cdots v_n$ be a path with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n\}$, and $C_n = v_1 v_2 \cdots v_n v_1$ be a cycle with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n, v_n v_1\}$. Let $G \in \{P_n, C_n\}$. Each coloring f of G can be represented by a sequence, say $[f(v_1), f(v_2), \ldots, f(v_n)]$. For convenience, we identify each coloring with its sequence and work with the colors instated of vertices. For $1 \leq n_1 \leq n$, let $f_{|_{[n_1]}} = [f(v_1), f(v_2), \ldots, f(v_{n_1})]$ be the restriction of f to the subgraph induced by the vertices $\{v_1, v_2, \ldots, v_{n_1}\}$.

If there exists a vertex $v_l \in V(G)$ such that $f(v_l) = s$ and $f(N_G(v_l)) = \{r, t\}$, then we say that the segment $[[r, s, t]]$ occurs in (the corresponding sequence of) f. This notation indicates that in G there exists a vertex with color s between two vertices with colors r and t. Note that

$[[r,s,t]] = [[t,s,r]]$. Also, if $f(v_l) = s$ and $f(N_G(v_l)) = \{r\}$, then we say that the segment $[[r,s,r]]$ occurs in f. This indicates that there exists a vertex with color s between two vertices with color r, or there exists a vertex of degree one (a leaf) with color s whose neighbor has color r. When r,s,t are elements of $[k]$ with $r \neq s$ and $t \neq s$, then we say that $[[r,s,t]]$ is a proper segment over the set $[k]$. Using these notations we have the following observation.

Observation 3.4. *Let f be a k-coloring of P_n or C_n. Then, f is an adjacency locating k-coloring if and only if each proper segment over the set $[k]$ occurs at most once in f.*

Now assume that f is an adjacency locating k-coloring of G, $G \in \{P_n, C_n\}$, for some $k \in \mathbb{N}$. If u and v are two vertices in G with the same color i, $1 \leq i \leq k$, then $f(N_G(u)) \neq f(N_G(v))$. Note that for each $u \in V(G)$, $|f(N_G(u))| \leq 2$. Hence, we have

$$|\{u :\ u \in V(G),\ f(u) = i,\ |f(N_G(u))| = 1\}| \leq k-1$$

and,

$$|\{u :\ u \in V(G),\ f(u) = i,\ |f(N_G(u))| = 2\}| \leq \binom{k-1}{2}.$$

This means that there are at most $(k-1) + \binom{k-1}{2} = \frac{1}{2}(k^2-k)$ vertices in G with color i. Hence, $n \leq k(\frac{k^2-k}{2})$. Therefore, we have the following proposition.

Proposition 3.5. *Let n,k be two positive integers. If there exists an adjacency locating k-coloring f of P_n or C_n, then $n \leq \frac{1}{2}(k^3-k^2)$. The equality holds if and only if each proper segment over the set $[k]$ occurs exactly once in f.*

When $f = [f(v_1), f(v_2), \ldots, f(v_t)]$ is a coloring of $P_t = v_1v_2\cdots v_t$, $f' = [f'(v'_1), f'(v'_2), \ldots, f'(v'_{t'})]$ is a coloring of $P_{t'} = v'_1v'_2\cdots v'_{t'}$, $f(v_t) \neq f'(v'_1)$, and $\{v_1, v_2, \ldots, v_t\} \bigcap \{v'_1, v'_2, \ldots, v'_{t'}\} = \emptyset$, then by $f \oplus f'$ we mean

$$[f(v_1), f(v_2), \ldots, f(v_t), f'(v'_1), f'(v'_2), \ldots, f'(v'_{t'})],$$

which is a coloring of the path $P_{t+t'} = v_1v_2\cdots v_tv'_1v'_2\cdots v'_{t'}$. In fact, we stick the colorings of two small paths in order to get a coloring of a larger path. Note that the segment corresponding to v_t in f is $[[f(v_{t-1}), f(v_t), f(v_{t-1})]]$, while the segment corresponding to v_t in $f \oplus f'$ is $[[f(v_{t-1}), f(v_t), f'(v'_1)]]$. In this case we say that the segment $[[f(v_{t-1}), f(v_t), f'(v'_1)]]$ occurs between f an f'. A similar argument holds for v_1'. For convenience, for the empty sequence $\emptyset$ we define $f \oplus \emptyset = \emptyset \oplus f = f$.

Now we are ready to determine the adjacency locating chromatic number of paths.

Theorem 3.6. *For a positive integer $n \geq 2$, $\chi_{_{L2}}(P_n) = m$, where $m = \min\{k : \ k \in \mathbb{N},\ n \leq \frac{1}{2}(k^3 - k^2)\}$. More precisely, there exist an adjacency locating m-coloring f_n of the path $P_n = v_1 v_2 \cdots v_n$ with the color set $\{1, 2, ..., m\}$, and two specified colors (say "1" and "2") such that f_n satisfies the following properties.*

(a) *$f_n(v_{n-1}) = 2$ and $f_n(v_n) = 1$.*
(b) *If $n \geq 9$, then $f_n(v_{n-2}) = m$.*
(c) *If $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$, then $f_n(v_1) = 2$ and $f_n(v_2) = 1$.*

Proof. Since
$$\frac{1}{2}((m-1)^3 - (m-1)^2) < n \leq \frac{1}{2}(m^3 - m^2),$$
if we give an adjacency locating m-coloring of P_n, then Proposition 3.5 implies that $\chi_{_{L2}}(P_n) = m$.

For $2 \leq n \leq 50$, consider the colorings which are listed in Table 1. It is not hard to check that each proper segment over the set [5] occurs at most once in the f_i, $2 \leq i \leq 50$. Hence, each f_i is an adjacency locating coloring. Note that $\frac{3^3-3^2}{2} = 9$, $\frac{4^3-4^2}{2} = 24$, and $\frac{5^3-5^2}{2} = 50$. Also, note that all of the proper segments over the sets [3], [4], and [5] occur in f_9, f_{24}, and f_{50}, respectively.

Here after let $n \geq 51$. Thus, $m \geq 6$. Now in an inductive way we prove the theorem. Let $n' = \frac{1}{2}((m-1)^3 - (m-1)^2)$, and assume that $f_{n'}$ is an adjacency locating $(m-1)$-coloring of $P_{n'}$ with the mentioned properties in the theorem (let us denote this by writing $f_{n'} = [2, 1, \ldots, m-1, 2, 1]$). Specially, by Proposition 3.5, all of the proper segments over the set $[m-1]$ occur in $f_{n'}$. Note that $\frac{1}{2}(m^3 - m^2) = n' + (2(m-1) + 3\binom{m-1}{2})$. Using the new color "$m$", we will add $2(m-1) + 3\binom{m-1}{2}$ new entries to $f_{n'}$. These new entries are $(m-1)$ pairs of the form $[m, i]$, and $\binom{m-1}{2}$ triples of the form $[m, i, j]$, $\{i, j\} \subseteq [m-1]$. Step by step, we provide an adjacency locating m-coloring f_i for each i, $n' < i \leq n' + (2(m-1) + 3\binom{m-1}{2})$. In each step we modify the coloring for a path with one more vertex. Equivalently, we add a new entry to somewhere in the coloring sequence and probably, we change some other entries.

Let $T = [m, 1, 3, m, 3, 2, m, 2, 1]$ and $A = [m, m-4, m, m-5, \ldots, m, 2, m, 1]$.

TABLE 1. Optimal adjacency locating colorings of the small paths.

$f_2=[2,1]$	$f_{19}=f_9\oplus[4,1,4,3,4,3,2,4,2,1]$
$f_3=[3,2,1]$	$f_{20}=f_9\oplus[4,3,1,4,2,4,3,2,4,2,1]$
$f_4=[1,3,2,1]$	$f_{21}=f_9\oplus[4,1,4,2,4,3,4,3,2,4,2,1]$
$f_5=[2,1,3,2,1]$	$f_{22}=f_7\oplus[4,3,4,2,4,1,4,1,3,4,3,2,4,2,1]$
$f_6=[3,2,3,1,2,1]$	$f_{23}=f_8\oplus[4,3,4,2,4,1,4,1,3,4,3,2,4,2,1]$
$f_7=[2,1,3,2,3,2,1]$	$f_{24}=f_9\oplus[4,3,4,2,4,1,4,1,3,4,3,2,4,2,1]$
$f_8=[3,2,3,1,3,1,2,1]$	$f_{25}=f_{24\|_{[22]}}\oplus[5,2,1]$
$f_9=[2,1,3,1,3,2,3,2,1]$	$f_{26}=f_{24\|_{[22]}}\oplus[2,5,2,1]$
$f_{10}=[2,1,3,1,3,2,3,4,2,1]$	$f_{27}=f_{24\|_{[22]}}\oplus[2,1,5,2,1]$
$f_{11}=[2,1,3,1,3,2,3,2,4,2,1]$	$f_{28}=f_{24\|_{[22]}}\oplus[5,3,1,5,2,1]$
$f_{12}=[2,1,3,1,3,2,3,2,1,4,2,1]$	$f_{29}=f_{24\|_{[22]}}\oplus[5,3,5,1,5,2,1]$
$f_{13}=[2,1,3,1,3,2,3,4,3,1,4,2,1]$	$f_{30}=f_{24\|_{[22]}}\oplus[5,3,5,1,3,5,2,1]$
$f_{14}=[2,1,3,1,3,2,3,4,3,4,1,4,2,1]$	$f_{31}=f_{24\|_{[22]}}\oplus[5,3,5,1,5,2,5,2,1]$
$f_{15}=[2,1,3,1,3,2,3,4,3,4,1,3,4,2,1]$	$f_{32}=f_{24\|_{[22]}}\oplus[5,3,5,1,3,5,2,5,2,1]$
$f_{16}=[2,1,3,1,3,2,3,4,3,4,1,4,2,4,2,1]$	$f_{33}=f_{24}\oplus[5,1,3,5,3,2,5,2,1]$
$f_{17}=[2,1,3,1,3,2,3,4,3,4,1,3,4,2,4,2,1]$	$f_{34}=f_{24}\oplus[5,1,5,3,5,3,2,5,2,1]$
$f_{18}=F_9\oplus[4,1,3,4,3,2,4,2,1]$	
$f_{26+i}=f_i\oplus[5,4,5,3,5,2,5,1,5,3,4,5,4,2,5,4,1,5,1,3,5,3,2,5,2,1],\ 9\le i\le 24$	

For each i,j, $4\le i\le m-1$, and $1\le j\le i-2$, let $D_{i,j}=[m,i,j,m,i,j-1,\dots,m,i,1]$. Also, let $D_i=[m,i-1,i]\oplus D_{i,i-2}$ and $D_{[i]}=D_i\oplus D_{i-1}\oplus\cdots\oplus D_4$. For example we have $D_5=[m,4,5,m,5,3,m,5,2,m,5,1]$. For convenience, define $D_{i,0}=D_3=D_{[3]}=\emptyset$. Now consider the following coloring which is an m-coloring of a path with $n'+2(m-1)+3\binom{m-1}{2}$ vertices.

$$f_{n'+2(m-1)+3\binom{m-1}{2}}=f_{n'}\oplus[m,m-1,m,m-2,m,m-3]\oplus A\oplus D_{[m-1]}\oplus T.$$

This is our final "complete model". Using this complete model we want to build the smaller colorings $\{f_{n'+i}\ :\ 1\le i<2(m-1)+3\binom{m-1}{2}\}$. Note that all of the proper segments over the set $[m]$ occur in $f_{n'+2(m-1)+3\binom{m-1}{2}}$, each of them just once. More precisely,

- All of the proper segments over the set $[m-1]$ occur in $f_{n'}$, except the segment $[[2,1,2]]$ which occurs at the end of T.

- The segments of the form $[[m,i,j]]$, where $i,j \in [m-1]$ and $i \neq j$, occur in $D_{[m-1]} \oplus T$, except the segment $[[m,1,2]] = [[2,1,m]]$ which occurs between $f_{n'}$ and $[m,m-1,m,m-2,m,m-3]$.
- The segments of the form $[[m,i,m]]$, $2 \leq i \leq m-1$, occur in $[m,m-1,m,m-2,m,m-3] \oplus A$. The segment $[[m,1,m]]$ occurs between A and $D_{[m-1]}$.
- The segments of the form $[[i+1,m,i]]$, $1 \leq i \leq m-2$, occur in $[m,m-1,m,m-2,m,m-3] \oplus A$.
- The segments of the forms $[[j,m,i]]$ and $[[i,m,i]]$, where $4 \leq i \leq m-1$ and $2 \leq j \leq i-2$, occur in D_i, inside $D_{[m-1]}$.
- The segments of the form $[[1,m,j]]$, $3 \leq j \leq m-3$, occur between D_{j+2} and D_{j+1}, inside $D_{[m-1]}$. The segment $[[1,m,1]]$ occurs between $D_{[m-1]}$ and T. The segment $[[1,m,m-2]]$ occurs between A and $D_{[m-1]}$, and the segment $[[1,m,m-1]]$ occurs between $f_{n'}$ and $[m,m-1,m,m-2,m,m-3]$.
- The segments of the form $[[2,m,j]]$, $4 \leq j \leq m-1$, occur in D_j. The segment $[[2,m,2]]$ occurs in T.
- The segments of the form $[[3,m,j]]$, $5 \leq j \leq m-1$, occur in D_j. The segment $[[3,m,3]]$ occurs in T.

Note that f_{24} and f_{50} are given using this complete model. Now we proceed to build the other smaller colorings. Note that by the hypothesis, we have $f_{n'} = [2,1,\ldots,m-1,2,1]$. Let

$$\begin{aligned}
f_{n'+1} &= f_{n'}|_{[n'-2]} \oplus [m,2,1],\\
f_{n'+2} &= f_{n'}|_{[n'-2]} \oplus [2,m,2,1],\\
f_{n'+3} &= f_{n'}|_{[n'-2]} \oplus [2,1,m,2,1],\\
f_{n'+4} &= f_{n'}|_{[n'-2]} \oplus [m,3,1,m,2,1],\\
f_{n'+5} &= f_{n'}|_{[n'-2]} \oplus [m,3,m,1,m,2,1],\\
f_{n'+6} &= f_{n'}|_{[n'-2]} \oplus [m,3,m,1,3,m,2,1],\\
f_{n'+7} &= f_{n'}|_{[n'-2]} \oplus [m,3,m,1,m,2,m,2,1],\\
f_{n'+8} &= f_{n'}|_{[n'-2]} \oplus [m,3,m,1,3,m,2,m,2,1],\\
f_{n'+9} &= f_{n'}|_{[n'-2]} \oplus [2,1,m,1,3,m,3,2,m,2,1],\\
f_{n'+10} &= f_{n'}|_{[n'-2]} \oplus [2,1,m,1,m,3,m,3,2,m,2,1],\\
f_{n'+11} &= f_{n'}|_{[n'-2]} \oplus [2,1,m,m-1,1,m,3,m,3,2,m,2,1],\\
f_{n'+12} &= f_{n'}|_{[n'-2]} \oplus [2,1,m,m-1,m-2,m,1,3,m,3,2,m,2,1].
\end{aligned}$$

Let $1 \leq i \leq 12$. The coloring $f_{n'+i}$ has two parts. The first part is $f_{n'}|_{[n'-2]}$ which the color m does not appear in it, and the second part which m appears in it. Since $f_{n'}$ is an adjacency locating $(m-1)$-coloring, each proper segment over the set $[m-1]$ occurs at most once in $f_{n'}|_{[n'-2]}$. Note that the segment $[[2,1,2]]$ occurs at the end of the second pat of $f_{n'+i}$ and not in the first part. Also, it is easy to see that each segment in $f_{n'+i}$ which contains m occurs just once. Hence, $f_{n'+i}$ is an adjacency locating m-coloring. Note that $f_{n'+12} = f_{n'} \oplus [m, m-1, m-2] \oplus T$. Now step by step we add the part A. Let

$$f_{n'+12+1} = f_{n'} \oplus [m, m-1, m, m-2] \oplus T,$$

and

$$f_{n'+12+2} = f_{n'} \oplus [m, m-1, m-2] \oplus [m, 1] \oplus T.$$

Also, for each i, $2 \leq i \leq m-4$, let

$$f_{n'+12+2i-1} = f_{n'} \oplus [m, m-1, m, m-2] \oplus [m, i-1, m, i-2, \ldots, m, 1] \oplus T,$$

and

$$f_{n'+12+2i} = f_{n'} \oplus [m, m-1, m-2] \oplus [m, i, m, i-1, \ldots, m, 1] \oplus T.$$

Specially, $f_{n'+12+2(m-4)} = f_{n'} \oplus [m, m-1, m-2] \oplus A \oplus T$. Let $1 \leq j \leq 2(m-4)$. In the coloring $f_{n'+12+j}$ the segment $[[2,1,2]]$ occurs at the end of part T instead of part $f_{n'}$. Note that in $f_{n'+12+j}$, the segment corresponding to the final entry of $f_{n'}$ is $[[2,1,m]]$, not $[[2,1,2]]$. Each proper segment over the set $[m-1]$ occurs at most once and, except $[[2,1,2]]$, each one occurs just in the part $f_{n'}$. Also, by the case by case investigation, we can see that each proper segment containing m occurs at most once. Hence, $f_{n'+12+j}$ is an adjacency locating m-coloring.

For adding the parts D_4, D_5, …, D_{m-3} we proceed as follows. Let $4 \leq i \leq m-3$ and assume that D_{i-1} is added (note that $D_3 = \emptyset$). For adding D_i, alternately, we add a new entry m, then we remove it in order to add the portion $[m, m-3]$ to the beginning of A, and then we remove this portion in order to add a portion of the form $[m, i, j]$. More precisely, assume that $D_{i,j-1}$ is completed, where $1 \leq j \leq i-1$. We want to add the portion $[m, i, j]$ or $[m, j, i]$ of D_i. Let $n_i = n' + 12 + 2(m-4) + 3(3 + 4 + \cdots + (i-1-1))$. Note that $n_4 = n + 12 + 2(m-4)$ and $D_{i,0} = D_{[3]} = \emptyset$. Let
$f_{n_i+3j-2} = f_{n'} \oplus [m, m-1, m, m-2] \oplus A \oplus D_{i,j-1} \oplus D_{[i-1]} \oplus T$,
$f_{n_i+3j-1} = f_{n'} \oplus [m, m-1, m-2] \oplus [m, m-3] \oplus A \oplus D_{i,j-1} \oplus D_{[i-1]} \oplus T$,
and $f_{n_i+3j} =$

$$\begin{cases} f_{n'} \oplus [m, m-1, m-2] \oplus A \oplus [m, i, j] \oplus D_{i,j-1} \oplus D_{[i-1]} \oplus T & j < i-1 \\ f_{n'} \oplus [m, m-1, m-2] \oplus A \oplus [m, j, i] \oplus D_{i,j-1} \oplus D_{[i-1]} \oplus T & j = i-1. \end{cases}$$

Except the segment $[[2, 1, 2]]$ which occurs at the end of T, all of the other proper segments over the set $[m-1]$ occur just in $f_{n'}$. Also, by considering the structures of A, $D_{i,j-1}$, $D_{[i-1]}$ and T (and similar to what we said about the complete model) it is not hard to see that each segment containing m occurs at most once in this colorings. Hence, these colorings are adjacency locating m-colorings.

Let $n'' = n_{m-3} + 3(m-4)$. Since $(n' + 2(m-1) + 3\binom{m-1}{2}) - n'' = 6m-12$, we need $6m-12$ steps to complete the proof. Adding D_{m-2} and D_{m-1} in this way is complicated and requires more details. Instead, we use the completed model, just we replace $f_{n'}$ with the smaller colorings. Let

$$f_{n''+j} = f_{n'-6m+12+j} \oplus [m, m-1, m, m-2, m, m-3] \oplus A \oplus D_{[m-1]} \oplus T,$$

where $1 \leq j \leq 6m-12$. Note that since $m \geq 6$, $n' - (6m-12) \geq 9$. $\square$

Note that the proof of Theorem 3.6 provides an algorithm which runs in polynomial time and explicitly produces an adjacency locating coloring of each path. In fact, using the proof of Theorem 3.9, this algorithm also produces an adjacency locating coloring of each cycle. Hence, it explicitly provides optimal locating coloring of the fan graph F_n and the wheel W_n in polynomial time. Theorems 3.1 and 3.6 imply the following two corollaries.

Corollary 3.7. *For $m \geq 1$ and $n \geq 2$, we have $\chi_L(K_m \vee P_n) = m + \min\{k : k \in \mathbb{N}, n \leq \frac{1}{2}(k^3 - k^2)\}$. Specially, the locating chromatic number of the fan F_n is $\chi_L(K_1 \vee P_n)$.*

Corollary 3.8. *For two positive integers $m \geq 2$ and $n \geq 2$, let $m_0 = \min\{k : k \in \mathbb{N}, m \leq \frac{1}{2}(k^3 - k^2)\}$ and $n_0 = \min\{k : k \in \mathbb{N}, n \leq \frac{1}{2}(k^3 - k^2)\}$. Then, $\chi_L(P_m \vee P_n) = m_0 + n_0$.*

Now we determine the adjacency locating chromatic number of the cycles. Then using it we determine the exact values of $\chi_L(P_m \vee C_n)$, $\chi_L(K_m \vee C_n)$, and $\chi_L(C_m \vee C_n)$.

For each n, $3 \leq n < 9$, consider the following coloring (sequence) h_n of the cycle C_n.

$$h_3 = [1, 2, 3],\ h_4 = [1, 2, 3, 4],\ h_5 = [1, 2, 1, 2, 3],\ h_6 = [1, 2, 1, 3, 2, 4],$$

$$h_7 = [2, 1, 3, 2, 3, 2, 1],\ h_8 = [3, 2, 3, 1, 3, 1, 2, 1, 4].$$

It is easy to check that each coloring h_n is an adjacency locating coloring. Note that $\chi_{_L}(C_n)$ is three or four depending on the parity of n, and $\chi_{_L}(C_n) \leq \chi_{_{L2}}(C_n)$. Therefore, $\chi_{_L}(C_n) = \chi_{_{L2}}(C_n)$ for $3 \leq n < 9$. For the general case $n \geq 9$, we have the following theorem.

Theorem 3.9. *For a positive integer $n \geq 9$, let $n_{_0} = \min\{k : \ k \in \mathbb{N},\ n \leq \frac{1}{2}(k^3 - k^2)\}$. Then,*

$$\chi_{_{L2}}(C_n) = \begin{cases} n_{_0} & n \neq \frac{1}{2}(n_{_0}^3 - n_{_0}^2) - 1 \\ n_{_0} + 1 & n = \frac{1}{2}(n_{_0}^3 - n_{_0}^2) - 1. \end{cases}$$

Proof. Suppose that $C_n = v_1 v_2 \cdots v_n v_1$. By Proposition 3.5, we have $\chi_{_{L2}}(C_n) \geq n_{_0}$. First assume that $n \neq \frac{1}{2}(n_{_0}^3 - n_{_0}^2) - 1$. By Theorem 3.6, there exists an adjacency locating $n_{_0}$-coloring f_n of the path $P_n = v_1 v_2 \cdots v_n$ such that $f_n(v_1) = 2$, $f_n(v_2) = 1$, $f_n(v_{n-1}) = 2$, and $f_n(v_n) = 1$. Consider f_n as a coloring of the vertices of C_n. Since $f_n(v_1) \neq f_n(v_n)$, this is a proper coloring of C_n. Note that $E(C_n) = E(P_n) \bigcup \{v_n v_1\}$. Hence, for each i, $1 \leq i \leq n$, we have $f_n(N_{C_n}(v_i)) = f_n(N_{P_n}(v_i))$. Therefore, f_n is also an adjacency locating $n_{_0}$-coloring of C_n. This implies that $\chi_{_{L2}}(C_n) = n_{_0}$.

Now assume that $n = \frac{1}{2}(n_{_0}^3 - n_{_0}^2) - 1$. By Theorem 3.6, there exists an adjacency locating $n_{_0}$-coloring f_{n-1} of the path $P_{n-1} = v_1 v_2 \cdots v_{n-1}$ such that $f_{n-1}(v_1) = 2$ and $f_{n-1}(v_{n-1}) = 1$. Define the coloring f'_n of C_n as $f'_n(v_n) = n_{_0} + 1$ and $f'_n(v_i) = f_{n-1}(v_i)$ for $1 \leq i \leq n-1$. Note that

$$n_{_0} + 1 \in f'_n(N_{C_n}(v_1)) \bigcap f'_n(N_{C_n}(v_{n-1})), f'_n(v_1) \neq f'_n(v_{n-1}),$$

and $f'_n(N_{C_n}(v_i)) = f_{n-1}(N_{P_{n-1}}(v_i))$ for each i, $2 \leq i \leq n-2$. Thus, f'_n is an adjacency locating $(n_{_0}+1)$-coloring of C_n. Hence, $\chi_{_{L2}}(C_n) \leq n_{_0} + 1$.

We want to show that $\chi_{_{L2}}(C_n) \neq n_{_0}$. Suppose on the contrary there exists an adjacency locating $n_{_0}$-coloring f of C_n. For each i, $1 \leq i \leq n_{_0}$, let $V_i = \{x : \ x \in V(C_n),\ f(x) = i\}$. Since f is an adjacency locating $n_{_0}$-coloring, each color class contains at most $\frac{1}{2}(n_{_0}^2 - n_{_0})$ vertices (see the argument before Proposition 3.5). Now since $n = \frac{1}{2}(n_{_0}^3 - n_{_0}^2) - 1$, exactly one of the color classes, say V_1, has size $\frac{1}{2}(n_{_0}^2 - n_{_0}) - 1$ and the others have size $\frac{1}{2}(n_{_0}^2 - n_{_0})$. For each i, $2 \leq i \leq n_{_0}$, let $X_i = \{(x, y) : \ x \in N_{C_n}(y),\ f(x) = 1,\ f(y) = i\}$. Let $2 \leq i \leq n_{_0}$. Since $|V_i| = \frac{1}{2}(n_{_0}^2 - n_{_0})$, all of the proper segments of the form $[[r, i, j]]$, where $r \in [n_{_0}]$ and $j \in [n_{_0}]$, occur in f. Thus, for each j with $j \notin \{1, i\}$, there exists $y \in V_i$ such that $f(N_{C_n}(y)) = \{1, j\}$. Also, there exists $z \in V_i$ such that $f(N_{C_n}(z)) = \{1\}$. This implies that $|X_i| = (n_{_0} - 2) + 2 = n_{_0}$. Hence, $|X| = (n_{_0} - 1) n_{_0}$, where $X = X_2 \bigcup X_3 \bigcup \cdots \bigcup X_{n_0}$. Each vertex x with color 1 has two neighbors and hence, $|X| = 2\ |\{x:\ x \in V(C_n),\ f(x) = 1\}|$. This means that there are $\frac{|X|}{2} = \frac{(n_0-1)n_0}{2}$ vertices with color 1, which is a contradiction. □

Theorems 3.1 and 3.9 imply the following corollaries.

Corollary 3.10. *For two positive integers $m \geq 2$ and $n \geq 3$, let $m_0 = \min\{k:\ k \in \mathbb{N},\ m \leq \frac{1}{2}(k^3 - k^2)\}$ and $n_0 = \min\{k:\ k \in \mathbb{N},\ n \leq \frac{1}{2}(k^3 - k^2)\}$. Then,*

$$\chi_L(P_m \vee C_n) = \begin{cases} m_0 + \chi_L(C_n) & 3 \leq n < 9 \\ m_0 + n_0 & n \geq 9,\ n \neq \frac{1}{2}(n_0^3 - n_0^2) - 1 \\ m_0 + n_0 + 1 & n \geq 9,\ n = \frac{1}{2}(n_0^3 - n_0^2) - 1. \end{cases}$$

Corollary 3.11. *For two positive integers $m \geq 1$ and $n \geq 3$, let $n_0 = \min\{k:\ k \in \mathbb{N},\ n \leq \frac{1}{2}(k^3 - k^2)\}$. Then,*

$$\chi_L(K_m \vee C_n) = \begin{cases} m + \chi_L(C_n) & 3 \leq n < 9 \\ m + n_0 & n \geq 9,\ n \neq \frac{1}{2}(n_0^3 - n_0^2) - 1 \\ m + n_0 + 1 & n \geq 9,\ n = \frac{1}{2}(n_0^3 - n_0^2) - 1. \end{cases}$$

Specially, the locating chromatic number of the wheel W_n is $\chi_L(K_1 \vee C_n)$.

Corollary 3.12. *For positive integers m and n, $3 \leq m \leq n$, let $m_0 = \min\{k:\ k \in \mathbb{N},\ m \leq \frac{1}{2}(k^3 - k^2)\}$ and $n_0 = \min\{k:\ k \in \mathbb{N},\ n \leq \frac{1}{2}(k^3 - k^2)\}$. Also, let $m_1 = \frac{1}{2}(m_0^3 - m_0^2) - 1$ and $n_1 = \frac{1}{2}(n_0^3 - n_0^2) - 1$. Then,*

$$\chi_L(C_m \vee C_n) = \begin{cases} \chi_L(C_m) + \chi_L(C_n) & n < 9 \\ \chi_L(C_m) + n_0 & m < 9 \leq n,\ n \neq n_1 \\ \chi_L(C_m) + n_0 + 1 & m < 9 \leq n,\ n = n_1 \\ m_0 + n_0 & m \geq 9,\ m \neq m_1,\ n \neq n_1 \\ m_0 + n_0 + 1 & m \geq 9,\ m = m_1,\ n \neq n_1 \\ m_0 + n_0 + 1 & m \geq 9,\ m \neq m_1,\ n = n_1 \\ m_0 + n_0 + 2 & m \geq 9,\ m = m_1,\ n = n_1. \end{cases}$$

Remark 3.13. *Note that the diameter of a complete multipartite graph is two and its locating chromatic number is equal to the number of its vertices. Hence Corollaries 3.7 and 3.11 hold also for complete multipartite graphs (such as stars) instead of complete graphs.*

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