Title:

A generalization of Villarreal's result for unmixed tripartite graphs

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A GENERALIZATION OF VILLARREAL'S RESULT FOR UNMIXED TRIPARTITE GRAPHS

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(Communicated by Ebadollah S. Mahmoodian)

ABSTRACT. In this paper we give a characterization of unmixed
tripartite graphs under certain conditions which is a generalization
of a result of Villarreal on bipartite graphs. For bipartite graphs two
different characterizations were given by Ravindra and Villarreal.
We show that these two characterizations imply each other.
Keywords: Well-covered graph, unmixed graph, perfect matching.
MSC(2010): Primary: 5E40; Secondary: 5C69, 5C75.

1. **Introduction**

Let $G = (V(G), E(G))$ be a finite, undirected, and simple graph, i.e.,
without multiple edges or loops. A graph G is said to be connected
if any two vertices of G can be connected by a sequence of edges of
G. A subset C of $V(G)$ is called a vertex cover of G if for every edge
$\{x, y\} \in E(G)$, $C \cap \{x, y\} \neq \emptyset$. A vertex cover C is called minimal, if
for every $A \subsetneq C$, A is not a vertex cover of G. A graph G is called
unmixed, if every minimal vertex cover of G has the same cardinality.
A subset H of $V(G)$ is called an independent (stable) set of G if no
two element subset of H is an edge of H. A maximal independent set
of G is an independent set M of G such for every $H \supsetneq M$, H is not
an independent set of G. Note that a set H is a maximal independent
set of G if and only if $V(G) \setminus H$ is a minimal vertex cover of G. The
maximum cardinality of all maximal independent set of G is called the

independence number of G and is denoted by $\alpha(G)$. A graph G is called well-covered, if every maximal independent set of G has the same cardinality, i.e., the cardinality of every maximal independent set of G is $\alpha(G)$. Therefore a graph is unmixed if and only if it is well-covered.

The problem of determining the maximum cardinality of maximal independent sets of a graph is an old and well known problem in graph theory. It is shown in [5], that for an arbitrary graph G, $\alpha(G)$ would be determined in a nondeterministic polynomial time and hence it is an NP-complete problem. Chvátal and Slater, [1], independently proved that the problem of failing to be well-covered is an NP-complete problem. In other words, there is no deterministic algorithm to decide whether a graph is well-covered. Nevertheless, there are interesting classes of graphs for which theoretical criteria are available to check whether they are well-covered. Well-covered graphs were first introduced and studied by Plummer. The reference [7] contains an interesting survey on well-covered graphs and their properties.

Recall that for an integer $r \geq 2$, a graph G is called r-partite if its vertices can be partitioned into r disjoint parts such that for every edge $\{x, y\} \in E(G)$, the vertices x and y lie in different parts. An r-partite graph for $r = 2, 3$, is called bipartite and tripartite, respectively. The class of bipartite graphs is among the classes of graphs for which there exist graph-theoretical criteria to check that a given graph is well-covered (or, equivalently, unmixed), (see Section 2).

An immediate natural question is that what would be the criteria which characterize unmixed tripartite graphs. In this paper we provide a combinatorial criterion which characterizes a special class of unmixed tripartite graphs (see Theorem 3.2). Moreover, we provide an example of an unmixed tripartite graph which does not belong to the class of graphs characterized by our result. Furthermore, we show that in the absence of either of the conditions (1) and (2) of Theorem 3.2, a tripartite graph may fail to be unmixed.

2. **Unmixed bipartite graphs**

Unmixed bipartite graphs have already been characterized. In fact there are combinatorial [8, 9], as well as, algebraic characterizations for unmixed bipartite graphs [10]. In this section we show that each of these two characterizations; [8] and [9] follows from the other one.

Let G be a graph. For a vertex $v \in V(G)$, let $N(v)$ be the set of a vertices $u \in V(G)$ where $\{u, v\} \in E(G)$. A set $F \subset E(G)$ is called a matching of G if for any two edges $\{x, y\}, \{a, b\} \in F$, $\{x, y\} \cap \{a, b\} = \emptyset$. A matching F of G is called perfect if for every $v \in V(G)$, there exists an edge $\{x, y\} \in F$ such that $v \in \{x, y\}$.

Let G be a bipartite graph and let $e = \{x, y\}$ be an edge of G. Let G_e be the subgraph induced on $N(x) \cup N(y)$, i.e., $V(G_e) = N(x) \cup N(y)$ and $E(G_e) = \{\{u, v\} \in E(G) | u, v \in V(G_e)\}$.
Ravindra in 1977 gave the following necessary and sufficient condition for G to be unmixed.

Theorem 2.1. [8][A] *Let G be a connected bipartite graph. Then G is unmixed if and only if G contains a perfect matching F such that for every edge $e = \{x, y\} \in F$ the induced subgraph G_e is a complete bipartite graph.*

In 2007, Villarreal, inspired by [4], and in an attempt to provide a criterion for unmixedness property of hypergraphs, gave the following characterization of unmixed bipartite graphs.

Theorem 2.2. [9, Theorem 1.1][B] *Let G be a bipartite connected graph. Then G is unmixed if and only if there is a bipartition $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ of G such that:*

  (a) *$\{x_i, y_i\} \in E(G)$ for all i, (i.e., a perfect matching),*
  (b) *if $\{x_i, y_j\} \in E(G)$ and $\{x_j, y_k\}$ are in $E(G)$ and i, k, j are distinct, then $\{x_i, y_j\} \in E(G)$.*

We now show that each of these results imply the other one.

Proposition 2.3. *Each of the above two characterizations imply the other one.*

Proof. $A \Rightarrow B$): Let G be a bipartite graph which satisfies the conditions of Theorem A. Then $E(G)$ contains a perfect matching $F = \{\{x_i, y_i\} \mid 1 \leq i \leq n\}$. Hence $X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_n\}$ gives a bipartition of $V(G)$. Let $\{x_i, y_j\}$ and $\{x_j, y_k\}$ be edges of G. Then by Ravindra's criterion, G induces a complete bipartite graph on $N(x_j) \cup N(y_j)$. Since $y_k \in N(x_j)$ and $x_i \in N(y_j)$ and the induced graph on $N(x_j) \cup N(y_j)$ is complete bipartite, $\{x_i, y_k\} \in N(x_j) \cup N(y_j)$, hence this edge is in $E(G)$.

$B \Rightarrow A$). First note that condition (a), gives a perfect matching $F = \{\{x_i, y_i\} \mid 1 \leq i \leq n\}$ for G. Let $e = \{x_j, y_j\} \in F$. If $N(x_j) \cup N(y_j) \subset X \cup \{y_j\}$ or $N(x_j) \cup N(y_j) \subset Y \cup \{x_j\}$, then the induced graph on $N(x_j) \cup N(y_j)$ is $K_{1,m}$ for some positive integer m and the Theorem A holds. So let $x_i, y_k \in N(x_j) \cup N(y_j)$ where i, k are different from j. Since the induced graph on $N(x_j) \cup N(y_j)$ is a bipartite graph, hence $x_i \in N(y_j)$ and $y_k \in N(x_j)$. So $\{x_i, y_j\} \in E(G)$ and $\{x_j, y_k\} \in E(G)$. Therefore, by condition (b), $\{x_i, y_k\} \in E(G)$ and consequently $\{x_i, y_k\} \in E(N(x_j) \cup N(y_j))$, which means that the induced graph on $N(x_j) \cup N(y_j)$ is a complete bipartite graph. $\qquad\square$

Remark 2.4. *As the above theorems exhibit, in an unmixed bipartite graph, the cardinality of a minimal vertex cover is equal to the cardinality of a maximal perfect matching, and both are equal to $\mid V(G) \mid /2$. An unmixed graph which its independence number is equal to the half of the number of its vertices is called very well-covered. This class of graphs contains unmixed bipartite graphs. Moreover, a characterization of very well-covered graph is given in (see $[2$, Theorem 1.2]$)$, which is very close to the one in Theorem 2.2.*

Remark 2.5. *Since the cardinality of a maximum perfect matching of a bipartite graph is polynomially computable $[6$, Theorem 9.1.8]$,$ the independence number, as well as, the cardinality of a minimal vertex cover of these graphs are also polynomially computable.*

3. The Main Result

In the above two characterizations of unmixed bipartite graphs, the existence of a perfect matching is essential to prove the unmixedness property of bipartite graphs, and conversely, unmixedness of these graphs implies the existence of a perfect matching. But for unmixed tripartite graphs, there seems no natural perfect matching attached to a three-partition. To consider the unmixedness of tripartite graphs, we impose the following condition.
Let G be a tripartite graph with partitions

$$U = \{u_1, \ldots, u_n\}, V = \{v_1, \ldots, v_n\}, W = \{w_1, \ldots, w_n\}.$$

We will consider those tripartite graphs for which the following condition holds:

$(*)$ $\qquad \{u_i, v_i\}, \{u_i, w_i\}, \{v_i, w_i\} \in E(G),$ for all $i = 1, \ldots, n.$

Lemma 3.1. *Let G be a tripartite graph which satisfies the condition $(*)$. Let G be unmixed. Then every minimal vertex cover of G contains $2n$ vertices. In particular the independence number of G is $n = |V(G)| / 3$.*

Proof. Let C be a minimal vertex cover of G. Since for each i; $1 \le i \le n$, u_i is adjacent to v_i and w_i, C must contain at least two vertices in $\{u_i, v_i, w_i\}$ hence it contains at least $2n$ vertices of G. By hypothesis, $U \cup V$ is a vertex-cover of G with $2n$ vertices and G is unmixed, hence C must have exactly $2n$ elements. Since the complement of a minimal vertex-cover is a maximal independent set the last claim of the lemma is immediate. $\square$

To simplify the notations in the following theorem, we use $\{x_i, y_i, z_i\}$ and $\{r_i, s_i, t_i\}$ as two permutations of $\{u_i, v_i, w_i\}$.

Theorem 3.2. *Let G be a tripartite graph which satisfies the condition $(*)$. Then the graph G is unmixed if and only if the following conditions hold:*

(1) *If $\{u_i, x_q\}, \{v_j, y_q\}, \{w_k, z_q\} \in E(G)$, where no two vertices of $\{x_q, y_q, z_q\}$ lie in one of the three parts of $V(G)$ and i, j, k, q are distinct, then the set $\{u_i, v_j, w_k\}$ contains an edge of G.*

(2) *If $\{r, x_q\}, \{s, y_q\}$ and $\{t, z_q\}$ are edges of G, where r and s belong to one of the three parts of $V(G)$ and t belongs to another part, then the set $\{r, s, t\}$ contains an edge of G (here r and s may be equal).*

Proof. Let G be an unmixed graph, we prove that both conditions (1) and (2) hold. First we prove the condition (1). Assume the contrary. Let $\{u_i, x_q\}, \{v_j, y_q\}$ and $\{w_k, z_q\}$ be edges of G such that the set $\{u_i, v_j, w_k\}$ does not contain any edge of G. Therefore, $\{x_i, y_j, z_k\}$ is an independent set of G. Hence there exists a maximal independent set M of G which contains $\{x_i, y_j, z_k\}$. Since M is maximal, the set $C = V(G) \setminus M$ is a minimal vertex cover of G. Since $\{u_i, v_j, w_k\} \subset M$, the vertices x_i, y_j, z_k are not in C.

The assumptions $\{u_i, x_q\} \in E(G)$ and $u_i \notin C$ imply that $x_q \in C$. Similarly, since $\{v_j, y_q\} \in E(G)$ and $v_j \notin C$, we have $y_q \in C$. And, while $\{w_k, z_q\} \in E(G)$ and $w_k \notin C$, it follows that $z_q \in C$. Since for each $m \in \{1, \ldots, n\} \setminus \{q\}$, the edges $\{u_m, v_m\}, \{u_m, w_m\}$ and $\{w_m, v_m\}$ are among the edges of G, C must contain at least $2n - 2$ vertices. Furthermore, it contains $\{x_q, y_q, z_q\}$, hence the cardinality of C is at least

$2n + 1$ which contradicts Lemma 3.1.

Now we prove condition (2). Let $\{r, x_q\}, \{s, y_q\}$ and $\{t, z_q\}$ be edges of G, and assume that $\{r, s, t\}$ does not contain any edge of G. Similar to the above argument, let M be a maximal independent set which contains $\{r, s, q\}$ and put $C = V(G) \setminus M$.

The assumptions $\{r, x_q\} \in E(G)$ and $r \notin C$ imply that $x_q \in C$. Similarly, since $\{s, y_q\} \in E(G)$ and $s \notin C$ it follows that $y_q \in C$. Furthermore, while $\{t, z_q\} \in E(G)$ and $t \notin C$ we have $z_q \in C$. Hence similar to the above case, $\mid C \mid \geq 2n + 1$ which contradicts Lemma 3.1.

Conversely let the conditions (1), (2) hold, we have to prove that G is unmixed.

Let C be a minimal vertex cover of G. We have to show that for each $i = 1, \ldots, n, C$ intersects the set $\{x_i, y_i, z_i\}$ in exactly two elements. Since C is a vertex cover of G, it intersects this set in at least two elements. In contrary, assume that for some q, the cardinality of $C \cap \{x_q, y_q, z_q\}$ is 3. Since G is connected, the minimality of C implies that

(a) $N(z_q)$ contains another element different from x_q, y_q; say x_i or y_j, such that:

$x_i \notin C$ and $\{x_i, z_q\} \in E(G)$, or, $y_j \notin C$ and $\{y_j, z_q\} \in E(G)$.

(b) $N(y_q)$ contains another element different from x_q, z_q; say x_m or z_n, such that:

$x_m \notin C$ and $\{x_m, y_q\} \in E(G)$, or, $z_n \notin C$ and $\{z_n, y_q\} \in E(G)$.

(c) $N(x_q)$ contains another element different from x_q, y_q; say y_ℓ or z_k, such that:

$y_\ell \notin C$ and $\{y_\ell, x_q\} \in E(G)$, or, $z_k \notin C$ and $\{z_k, x_q\} \in E(G)$.

Depending on the choices of the vertices $x_i, x_m, y_j, y_\ell, z_k, z_n$, we encounter with the following cases, where each case, gives rise to a contradiction. Cases 1 and 2, follow from condition (1) and cases 3, 4, ..., 8, follow from condition (2).

Case 1: If $\{y_j, z_q\}, \{z_k, x_q\}, \{x_m, y_q\} \in E(G) \Rightarrow C \cap \{y_j, z_k, x_m\} \neq \emptyset$ while $\{y_j, z_k, x_m\} \cap C = \emptyset$.

Case 2: If $\{x_i, z_q\}, \{y_\ell, x_q\}, \{z_n, y_q\} \in E(G) \Rightarrow C \cap \{x_i, y_\ell, z_n\} \neq \emptyset$ while $\{x_i, y_\ell, z_n\} \cap C = \emptyset$.

Case 3: If $\{x_i, z_q\}, \{z_k, x_q\}, \{x_m, y_q\} \in E(G) \Rightarrow C \cap \{x_i, z_k, x_m\} \neq \emptyset$ while $\{x_i, z_k, x_m\} \cap C = \emptyset$.

Case 4: If $\{x_i, z_q\}, \{z_k, x_q\}, \{z_n, y_q\} \in E(G) \Rightarrow C \cap \{x_i, z_k, z_n\} \neq \emptyset$ while $\{x_i, z_k, z_n\} \cap C = \emptyset$.

Case 5: If $\{y_j, z_q\}, \{y_\ell, x_q\}, \{x_m, y_q\} \in E(G) \Rightarrow C \cap \{y_j, y_\ell, x_m\} \neq \emptyset$

while $\{y_j, y_\ell, x_m\} \cap C = \emptyset$.

Case 6: If $\{y_j, z_q\}, \{y_\ell, x_q\}, \{z_n, \ y_q\} \in E(G) \ \Rightarrow C \cap \{y_j, y_\ell, z_n \ \} \neq \emptyset$ while $\{y_j, y_\ell, z_n \ \} \cap C = \emptyset$.

Case 7: If $\{x_i, z_q\}, \{y_\ell, x_q\}, \{x_m, y_q\} \in E(G) \ \Rightarrow C \cap \{x_i, y_\ell, x_m\} \neq \emptyset$ while $\{x_i, y_\ell, x_m\} \cap C = \emptyset$.

Case 8: If $\{y_j, z_q\}, \{z_k, x_q\}, \{z_n, \ y_q\} \in E(G) \ \Rightarrow C \cap \{y_j, z_k, z_n \ \} \neq \emptyset$ while $\{y_j, z_k, z_n\} \cap C = \emptyset$.

$\square$

4. **Some examples and counterexamples**

Theorem 3.2 does not characterize all unmixed tripartite graphs. In fact, the condition $(*)$ is not valid for all unmixed tripartite graphs. Hence our result characterizes just a sub-class of unmixed tripartite graphs. We first provide an example of an unmixed tripartite graph which does not satisfy the condition $(*)$.

Example 4.1. *The following graph is an unmixed tripartite graph with partitions $\{y_1, y_4\}$, $\{y_2, y_5\}$, $\{y_3, y_6\}$, where the cardinality of each minimal vertex cover is 4, and does not satisfy the condition $(*)$. Indeed, if it satisfies the condition $(*)$, it must have two disjoint triangles, while even by any relabeling of its vertices it is not possible to find two disjoint triangles in it. Therefore Theorem 3.2 is not applicable to it.*
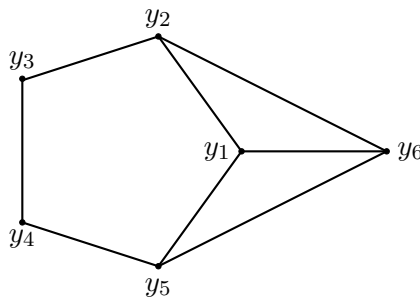

Figure 1

Example 4.2. *The tripartite graphs depicted in Figure 2 (which is a complete tripartite graph, $K_{2,2,2}$) and in Figure 3 satisfy the condition $(*)$ and the assumptions of Theorem 3.2 hold, hence they are unmixed.*
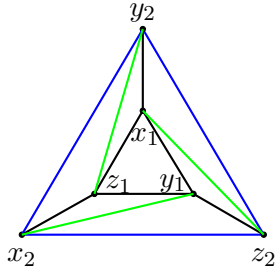
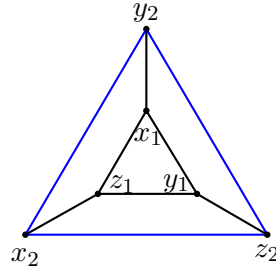Figure 2   Figure 3

As the second part of the proof of Theorem 3.2 shows, in this theorem neither condition (1) nor condition (2) alone, does imply the unmixedness of the tripartite graph. This fact is illustrated in the following examples.

Example 4.3. *The following graph shows that condition (1) of the Theorem 3.2 alone does not imply unmixedness (see Figure 4). In this graph, the edge set is:*

$$E(G) = \{\{u_1, v_1\}, \{u_2, v_2\}, \{u_3, v_3\}, \{u_4, v_4\}, \{w_1, v_1\}, \{w_2, v_2\},$$

(4.1)
$$\{w_3, v_3\}, \{w_4, v_4\}, \{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_3\}, \{u_4, w_4\},$$

$$\{w_1, u_4\}, \{v_2, w_4\}, \{u_3, v_4\}, \{w_1, u_2\}\}.$$

Observe that the following two sets are minimal vertex covers of G:

$C_1 = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$, $C_2 = \{u_1, u_2, u_4, v_1, v_3, v_4, w_2, w_3, w_4\}$,

where $\mid C_1 \mid = 8$, while $\mid C_2 \mid = 9$, which shows that this graph is not well-covered.

Example 4.4. *This example shows that the condition (2) of the Theorem 3.2 alone does not imply unmixedness of a tripartite graph (see Figure 5). In this graph, the edge set is:*

$$E(G) = \{\{u_1, v_1\}, \{u_2, v_2\}, \{u_3, v_3\}, \{u_4, v_4\}, \{v_1, w_1\}, \{v_2, w_2\},$$

$$\{v_3, w_3\}, \{v_4, w_4\}, \{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_3\}, \{u_4, w_4\},$$

(4.2)
$$\{u_3, w_4\}, \{v_3, w_2\}, \{v_1, w_3\}, \{v_1, w_2\}, \{u_1, v_2\}, \{v_1, w_3\},$$

$$\{v_4, w_1\}, \{v_4, w_3\}, \{u_4, v_2\}, \{v_1, w_2\}, \{u_2, v_1\}, \{u_4, v_1\}\}.$$

The following two sets are among its minimal vertex covers of G:

$C_1 = \{u_1, u_3, u_4, v_1, v_2, v_4, w_2, w_3\}$, $C_2 = \{u_1, u_2, u_4, v_1, v_3, v_4, w_2, w_3, w_4\}$,

where $\mid C_1 \mid = 8$, while $\mid C_2 \mid = 9$, which shows that this graph is not well-covered.

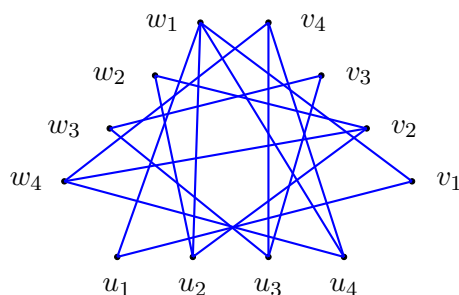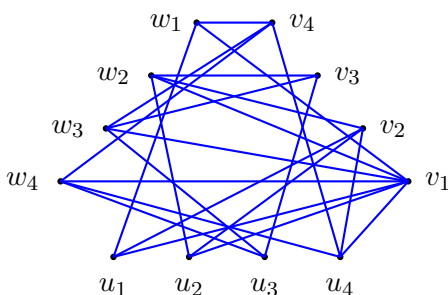Figure 4   Figure 5

We end this section with a relevant remark. We refer to [3] for concepts which are not defined in this paper.

Remark 4.5. *In* [3, Theorem 1.3] *an algebraic characterization of unmixed bipartite graphs was given. In this result it has been shown that for bipartite graphs unmixedness property is equivalent to Cohen-Macaulay property of the edge ideal associated to the graph. For unmixed tripartite graphs a similar characterization is more subtle. For example, the graph given in 4.1, does not satisfy the condition* (∗) *but it is Cohen-Macaulay, while the graph given in Figure 2, satisfies the condition* (∗) *but it is not Cohen-Macaulay.*

Acknowledgments

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