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ON δ-QUASI ARMENDARIZ MODULES

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ABSTRACT. Let δ be a derivation on R and $S = R[x;\delta]$ be the differential polynomial ring. A module M is called *Baer* (resp. *quasi-Baer*) if the annihilator of every subset (resp. submodule) of M is generated by an idempotent of R. In this note we impose δ-compatibility assumption on the module M and prove the following results. (1) The module M is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer). (2) If M_R is δ-Armendariz, then M_R is Baer (resp. p.p) if and only if $M[x]_S$ is Baer (resp. p.p). (3) A necessary and sufficient condition for the trivial extension $T(R,R)$ to be δ-quasi Armendariz is obtained.

1. Introduction

Throughout the paper R always denotes an associative ring with unity and M_R will stand for a right R-module. Recall from [16] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [16] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings. The class of Baer rings includes the von Neumann algebras. In [9] Clark defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. Another generalization of Baer rings are the p.p.-rings. A ring R is called *right (resp. left) p.p* if right (resp. left) annihilator of an element of R is

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generated by an idempotent. Birkenmeier et al. in [6] introduced the concept of principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal of R is generated by an idempotent.

In 1974, Armendariz considered the behavior of a polynomial ring over a Baer ring by obtaining the following result: Let R be a *reduced* ring (i.e. R has no nonzero nilpotent elements). Then $R[x]$ is a Baer ring if and only if R is a Baer ring ([4], Theorem B). Armendariz provided an example to show that the reduced condition is not superfluous. Recently, this result has been extended in several directions by Birkenmeier-Kim-Park [7], Han-Hirano-Kim [10], Hirano [12], Hong-Kim-Kwak [14], and Kim-Lee [18].

From now on, we always denote the differential polynomial ring by $S := R[x;\delta]$, where $\delta : R \to R$ is a derivation on R. Recall that a derivation δ is an additive operator on R with the property that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. The differential polynomial ring S is then the ring consisting of all (left) polynomials of the form $\sum a_i x^i$ $(a_i \in R)$, where the addition is defined as usual and the multiplication by $xb = bx + \delta(b)$ for any $b \in R$. From this rule, an inductive argument can be made to calculate an expression for $x^j a$, for all $j \in \mathbb{N}$ and $a \in R$.

One can show with a routine computation that

$$x^j a = \sum_{i=0}^{j} \binom{j}{i} \delta^{j-i}(a) x^i. \tag{1.1}$$

Given a right R-module M_R, we can make $M[x]$ into a right S-module by allowing polynomials from S to act on polynomials in $M[x]$ in the obvious way, and applying the above "twist" whenever necessary.

For a nonempty subset X of M, put $\mathrm{ann_R}(X) = \{a \in R \mid Xa = 0\}$.

In [22], Lee-Zhou introduced Baer, quasi-Baer and p.p.-modules as follows: (1) M_R is called *Baer* if, for any subset X of M, $\mathrm{ann_R}(X) = eR$ where $e^2 = e \in R$. (2) M_R is called *quasi-Baer* if, for any submodule $X \subseteq M$, $\mathrm{ann_R}(X) = eR$ where $e^2 = e \in R$. (3) M_R is called *p.p.* if, for any element $m \in M$, $\mathrm{ann_R}(m) = eR$ where $e^2 = e \in R$. Clearly, a ring R is Baer (resp. p.p. or quasi-Baer) if and only if R_R is Baer (resp. p.p. or quasi-Baer) module. If R is a Baer (resp. p.p. or quasi-Baer) ring, then for any right ideal I of R, I_R is Baer (resp. p.p. or quasi-Baer) module. Lee-Zhou have extended various results of reduced rings to reduced modules.

The module M_R is called *principally quasi-Baer* (or simply p.q.-Baer) if, for any $m \in M$, $\mathrm{ann_R}(mR) = eR$ where $e^2 = e \in R$. It is clear that R is a right p.q.-Baer ring if and only if R_R is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

In this note we impose δ-compatibility assumption on the module M_R and prove the following results which extend many results on rings to modules:

(1) The module M_R is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer), where $S = R[x;\delta]$. Also we give an example to show that δ-compatibility assumption on M_R is not superfluous.

(2) If M_R is δ-Armendariz, then M_R is Baer (resp. p.p) if and only if $M[x]_S$ is Baer (resp. p.p).

(3) A necessary and sufficient condition for the trivial extension $T(R,R)$ to be δ-quasi Armendariz is obtained.

2. δ-quasi Armendariz modules and Ore extensions of quasi-Baer modules

Definition 2.1. (Annin, [3]) Given a module M_R and a derivation $\delta : R \to R$, we say that M_R is *δ-compatible* if for each $m \in M$, $r \in R$, we have $mr = 0 \Rightarrow m\delta(r) = 0$.

Remark 2.2. *If M_R is δ-compatible, then so is any submodule of M_R.*

Lemma 2.3. *Let M_R be a δ-compatible module. Let $m \in M$, $a, b \in R$. Then we have the following:*
(i) *if $ma = 0$, then $m\delta^j(a) = 0$ for any positive integer j;*
(ii) *if $mab = 0$, then $ma\delta^j(b) = 0 = m\delta^j(a)b$ for any positive integer j;*
(iii) $\mathrm{ann_R}(ma) \subseteq \mathrm{ann_R}(m\delta(a))$.

Proof. (i) This follows from Remark 2.2.

(ii) It is enough to show that $ma\delta(b) = 0 = m\delta(a)b$. Since M_R is δ-compatible, $mab = 0$ implies that $ma\delta(b) = 0$ and $m\delta(ab) = m\delta(a)b + ma\delta(b) = 0$. Hence $m\delta(a)b = 0$.

(iii) Let $mab = 0$ for some $b \in R$. Using δ-compatibility, we get $0 = m\delta(ab) = ma\delta(b) + m\delta(a)b = 0$ and hence $m\delta(a)b = 0$, as desired. $\square$

Lemma 2.4. *Let M_R be a δ-compatible module and $m(x) = m_0 + \cdots + m_k x^k \in M[x]$ and $r \in R$. Then $m(x)r = 0$ if and only if $m_i r = 0$ for all i.*

Proof. Assume $m(x)r = 0$. An easy calculation using (1.1) shows that $m(x)r = \sum_{i=0}^{k} \left[\sum_{j=i}^{k} \binom{j}{i} m_j \delta^{j-i}(r)\right] x^i$ and so

$$\sum_{j=i}^{k} \binom{j}{i} m_j \delta^{j-i}(r) = 0 \text{ for each } i \leq k. \tag{2.1}$$

Starting with $i = k$, Eq.(2.1) yields $m_k r = 0$. Now assume inductively that $m_j r = 0$ for each $j > i$. By δ-compatibility of M_R, for $j > i$, we have $m_j \delta^{j-i}(r) = 0$. Using (2.1) again, we deduce that $m_i r = 0$, as needed.

The converse follows from δ-compatibility assumption on M. $\square$

Following Anderson and Camillo [1], a module M_R is called *Armendariz* if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x]$, we have $m_i a_j = 0$ for all i, j.

Definition 2.5. Given a module M_R and a derivation $\delta : R \to R$, we say M_R is *δ-quasi Armendariz* (resp. δ-Armendariz), if whenever $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$, $f(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\delta]$ satisfy $m(x)R[x;\delta]f(x) = 0$ (resp. $m(x)f(x) = 0$), we have $m_i x^i R x^t b_j x^j = 0$ (resp. $m_i x^i a_j x^j = 0$) for $t \geq 0$, $i = 0, \cdots, k$ and $j = 0, \cdots, n$.

Let R be a ring. The trivial extension of R is given by:
$T(R,R) = \left\{ \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} | a, r \in R \right\}$. Clearly, $T(R,R)$ is a subring of the ring of 2×2 matrices over R. The derivation δ on R is extended to $\overline{\delta} : T(R,R) \to T(R,R)$ by $\overline{\delta}\left(\begin{pmatrix} a & r \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \delta(a) & \delta(r) \\ 0 & \delta(a) \end{pmatrix}$. One can show that $\overline{\delta}$ is a derivation on $T(R,R)$ and $T(R,R)[x;\overline{\delta}] \cong T(R[x;\delta], R[x;\delta])$.

Proposition 2.6. *Let R be a δ-compatible ring. If the trivial extension $T(R,R)$ is $\overline{\delta}$-quasi Armendariz, then R is δ-quasi Armendariz.*

Proof. Let $f(x)=a_0+\cdots+a_nx^n, g(x)=b_0+\cdots+b_mx^m\in R[x;\delta]$ and $f(x)R[x;\delta]g(x)=0$. For each $a,r\in R$ and $t\geq 0$, we have the following equation:

$$\left[\sum_{i=0}^{n}\begin{pmatrix} a_i & 0\\ 0 & a_i\end{pmatrix}x^i\right]\begin{pmatrix} a & r\\ 0 & a\end{pmatrix}x^t\left[\sum_{j=0}^{m}\begin{pmatrix} 0 & b_j\\ 0 & 0\end{pmatrix}x^j\right]=$$

$$\begin{pmatrix} f(x) & 0\\ 0 & f(x)\end{pmatrix}\begin{pmatrix} ax^t & rx^t\\ 0 & ax^t\end{pmatrix}\begin{pmatrix} 0 & g(x)\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & f(x)ax^tg(x)\\ 0 & 0\end{pmatrix}=0.$$

Since $T(R,R)$ is $\overline{\delta}$-quasi Armendariz,
$\begin{pmatrix} a_ix^i & 0\\ 0 & a_ix^i\end{pmatrix}\begin{pmatrix} ax^t & rx^t\\ 0 & ax^t\end{pmatrix}\begin{pmatrix} 0 & b_jx^j\\ 0 & 0\end{pmatrix}=0$ and so $a_ix^iax^tb_jx^j=0$ for all i,j,t. Therefore R is δ-quasi rmendariz. $\square$

When the trivial extension $T(R,R)$ is δ-quasi Armendariz?

Theorem 2.7. *Let R be a δ-compatible ring such that*
(i) *R is δ-quasi Armendariz;*
(ii) *if $f(x)R[x;\delta]g(x)=0$, then $f(x)R[x;\delta]\cap R[x;\delta]g(x)=0$.*
Then the trivial extension $T(R,R)$ is $\overline{\delta}$-quasi Armendariz.

Proof. Suppose that $\alpha(x)T(R,R)\beta(x)=0$, where
$\alpha(x)=\begin{pmatrix} a_0 & r_0\\ 0 & a_0\end{pmatrix}+\begin{pmatrix} a_1 & r_1\\ 0 & a_1\end{pmatrix}x+\cdots+\begin{pmatrix} a_n & r_n\\ 0 & a_n\end{pmatrix}x^n$ and
$\beta(x)=\begin{pmatrix} b_0 & s_0\\ 0 & b_0\end{pmatrix}+\begin{pmatrix} b_1 & s_1\\ 0 & b_1\end{pmatrix}x+\cdots+\begin{pmatrix} b_m & s_m\\ 0 & b_m\end{pmatrix}x^m\in T(R,R)[x;\overline{\delta}]$.
Let $f(x)=a_0+a_1x+\cdots+a_nx^n, r(x)=r_0+r_1x+\cdots+r_nx^n$,
$g(x)=b_0+b_1x+\cdots+b_mx^m$ and $s(x)=s_0+s_1x+\cdots+s_mx^m\in R[x;\delta]$.
For each $\begin{pmatrix} a & r\\ 0 & a\end{pmatrix}x^t\in T(R,R)[x;\overline{\delta}]$, it follows that
$0=\begin{pmatrix} f(x) & r(x)\\ 0 & f(x)\end{pmatrix}\begin{pmatrix} ax^t & rx^t\\ 0 & ax^t\end{pmatrix}\begin{pmatrix} g(x) & s(x)\\ 0 & g(x)\end{pmatrix}$
$=\begin{pmatrix} f(x)ax^tg(x) & f(x)ax^ts(x)+f(x)rx^tg(x)+r(x)ax^tg(x)\\ 0 & f(x)ax^tg(x)\end{pmatrix}$. Hence
$f(x)ax^tg(x)=0$ and $f(x)ax^ts(x)+f(x)rx^tg(x)+r(x)ax^tg(x)=0$. Since ax^t is an arbitrary element of $R[x;\delta]$, $f(x)R[x;\delta]g(x)=0$. But R is δ-quasi Armendariz and hence $a_ix^iRx^tb_jx^j=0$ for all i,j,t. Since $f(x)[ax^ts(x)+r(x)x^tg(x)]+[r(x)ax^t]g(x)=0$, $f(x)[ax^ts(x)+r(x)x^tg(x)]=-[r(x)ax^t]g(x)\in f(x)R[x;\delta]\cap R[x;\delta]g(x)$, so

$f(x)[ax^ts(x)+r(x)x^tg(x)] = [r(x)ax^t]g(x) = 0$. Since ax^t is an arbitrary element of $R[x;\delta]$, $r(x)R[x;\delta]g(x) = 0$. Then $r_ix^iRx^tb_jx^j = 0$ for all i,j,t, since R is δ-quasi Armendariz. Thus $f(x)[ax^ts(x)] = -[f(x)r(x)x^t]g(x) \in f(x)R[x;\delta] \cap R[x;\delta]g(x) = 0$. So $f(x)R[x;\delta]s(x) = 0$ and $a_ix^iRx^ts_jx^j = 0$ for all i,j,t, since R is δ-quasi Armendariz. Therefore

$$\begin{pmatrix} a_i & r_i \\ 0 & a_i \end{pmatrix} x^i \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} x^t \begin{pmatrix} b_j & s_j \\ 0 & b_j \end{pmatrix} x^j =$$
$$\begin{pmatrix} a_ix^iax^tb_jx^j & a_ix^irx^tb_jx^j + a_ix^irx^tb_jx^j + r_ix^iax^tb_jx^j \\ 0 & a_ix^iax^tb_jx^j \end{pmatrix} = 0 \text{ for all } i,j$$

and each $\begin{pmatrix} a & r \\ 0 & a \end{pmatrix} x^t \in T(R,R)$. Therefore the trivial extension $T(R,R)$ is $\overline{\delta}$-quasi Armendariz. □

Theorem 2.8. *Let M_R be a δ-compatible and δ-quasi Armendariz module. Then M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[x]_S$.*

Proof. Assume that M_R satisfies the ascending chain condition on annihilator of submodules. Let $I_1 \subseteq I_2 \subseteq I_3 \cdots$ be a chain of annihilator of submodules of $M[x]_S$. Then there exist submodules K_i of $M[x]_S$ such that $\mathrm{ann_S}(K_i) = I_i$ and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ for all $i \geq 1$. Let $M_i = \{$all coefficients of elements of $K_i\}$. Since M is δ-quasi Armendariz, M_i is submodule of M for all $i \geq 1$. Clearly $M_i \supseteq M_{i+1}$ for all $i \geq 1$. Thus $\mathrm{ann_R}(M_1) \subseteq \mathrm{ann_R}(M_2) \subseteq \mathrm{ann_R}(M_3) \subseteq \cdots$. Since M_R satisfies the ascending chain condition on annihilator of submodules, there exists $n \geq 1$ such that $\mathrm{ann_R}(M_i) = \mathrm{ann_R}(M_n)$ for all $i \geq n$. We show that $\mathrm{ann_S}(K_i) = \mathrm{ann_S}(K_n)$ for all $i \geq n$. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in \mathrm{ann_S}(K_i)$. Then $M_ia_j = 0$ for $j = 0, \cdots, m$, since M is δ-quasi Armendariz. Thus $M_na_j = 0$ for $j = 0, \cdots, m$ and so $K_nf(x) = 0$ by Lemma 2.4. Therefore $\mathrm{ann_S}(K_i) = \mathrm{ann_S}(K_n)$ for all $i \geq n$ and $M[x]_S$ satisfies the ascending chain condition on annihilator of submodules.

Now assume $M[x]_S$ satisfies the ascending chain condition on annihilator of submodules. Let $J_1 \subseteq J_2 \subseteq J_3 \cdots$ be a chain of annihilator of submodules of M_R. Then there exist submodules M_i of M such that $\mathrm{ann_R}(M_i) = J_i$ and
$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ for all $i \geq 1$. Hence $M_i[x]$ is a submodule of $M[x]$ and $M_i[x] \supseteq M_{i+1}[x]$ and $\mathrm{ann_S}(M_i[x]) \subseteq \mathrm{ann_S}(M_{i+1}[x])$ for all $i \geq 1$.

Since $M_S[x]$ satisfies the ascending chain condition on annihilator of submodules, there exists $n \geq 1$ such that $\mathrm{ann_S}(M_i[x]) = \mathrm{ann_S}(M_n[x])$ for all $i \geq n$. Since M is δ-quasi Armendariz, by a similar argument as used in the previous paragraph, one can show that $\mathrm{ann_R}(M_i) = \mathrm{ann_R}(M_n)$ for all $i \geq n$. □

Theorem 2.9. *Let M_R be a δ-compatible module. Then M_R is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer). In this case M_R is δ-quasi Armendariz.*

Proof. Assume M_R is quasi-Baer. First we shall prove that M_R is δ-quasi Armendariz. Suppose that $(m_0 + m_1x + ... + m_kx^k)S(b_0 + b_1x + ... + b_nx^n) = 0$, with $m_i \in M$, $b_j \in R$. Then

$$(m_0 + m_1x + ... + m_kx^k)R(b_0 + b_1x + ... + b_nx^n) = 0. \tag{2.2}$$

Thus $m_kRb_n = 0$ and $b_n \in \mathrm{ann_R}(m_kR)$. Then $m_kx^kRx^tb_nx^n = 0$, by Lemma 2.3. Since M_R is qusi-Baer, there exists $e_k^2 = e_k \in R$ such that $\mathrm{ann_R}(m_kR) = e_kR$ and so $b_n = e_kb_n$. Replacing R by Re_k in Eq.(2.2) and using Lemma 2.3, we obtain $(m_0 + m_1x + ... + m_{k-1}x^{k-1})Re_k(b_0 + b_1x + ... + b_nx^n) = 0$. Hence $m_{k-1}Rb_n = 0$ and $b_n \in \mathrm{ann_R}(m_{k-1}R)$. Then $m_{k-1}x^{k-1}Rx^tb_nx^n = 0$, by Lemma 2.3. Hence $b_n \in \mathrm{ann_R}(m_kR) \cap \mathrm{ann_R}(m_{k-1}R)$. Since M_R is 0qusi-Baer, there exists $f^2 = f \in R$ such that $\mathrm{ann_R}(m_kR) = fR$ and so $b_n = fb_n$. If we put $e_{k-1} = e_mf$, then $e_{k-1}b_n = b_n$ and $e_{k-1} \in \mathrm{ann_R}(m_kR) \cap \mathrm{ann_R}(m_{k-1}R)$. Next, replacing R by Re_{k-1} in Eq.(2.2), and using Lemma 2.3, we obtain $(m_0 + m_1x + ... + m_{k-2}x^{k-2})Re_{k-1}(b_0 + b_1x + ... + b_nx^n) = 0$. Hence we have $b_n \in \mathrm{ann_R}(m_{k-2}R)$ and so $m_{k-2}x^{k-2}Rx^tb_nx^n = 0$, by Lemma 2.3. Continuing this process, we get $m_ix^iRx^tb_nx^n = 0$ for $i = 0, \cdots, k$. Using induction on $k + n$, we obtain $m_ix^iRx^tb_jx^j = 0$ for all i, j, t. Therefore M_R is δ-quasi Armendariz. Let J be a S-submodule of $M[x]$. Let $N = \{m \in M | m$ is a leding coefficient of some non-zero element of $J\} \cup \{0\}$. Clearly, N is a submodule of M. Since M_R is quasi-Baer, there exists $e^2 = e \in R$ such that $\mathrm{ann_R}(N) = eR$. Hence $eS \subseteq \mathrm{ann_S}(J)$ by Lemma 2.3. Let $f(x) = b_0 + b_1x + ... + b_nx^n \in \mathrm{ann_S}(J)$. Then $Nb_j = 0$ for each $j = 0, \cdots, n$ since M_R is δ-quasi Armendariz. Hence $b_j = eb_j$ for each $j = 0, \cdots, n$ and $f(x) = ef(x) \in eS$. Thus $\mathrm{ann_S}(J) = eS$ and $M[x]_S$ is quasi-Baer.

Assume that $M[x]_S$ is quasi-Baer and I is a submodule of M. Then $I[x]$ is a submodule of $M[x]$. Since $M[x]$ is quasi-Baer, there exists an idempotent $e(x) = e_0 + \cdots + e_n x^n \in S$ such that $\mathrm{ann_S}(I[x]) = e(x)S$. Hence $Ie_0 = 0$ and $e_0R \subseteq \mathrm{ann_R}(I)$. Let $t \in \mathrm{ann_R}(I)$. Then $I[x]t = 0$ by Lemma 2.4. Hence $t = e(x)t$ and so $t = e_0t \in e_0R$. Thus $\mathrm{ann_R}(I) = e_0R$ and M_R is quasi-Baer. □

Corollary 2.10. *Let R be a δ-compatible ring. Then R is quasi-Baer (resp. right p.q.-Baer) if and only if $R[x;\delta]$ is quasi-Baer (resp. right p.q.-Baer).*

The following example shows that δ-compatibility condition on R_R in Corollary 2.10 is not superfluous.

Example 2.11. [4, Example 11] There is a ring R and a derivation δ of R such that $R[x;\delta]$ is a Baer (hence quasi-Baer) ring, but R is not quasi-Baer. In fact let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$ where $\bar{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Consider the Ore extension $R[x;\delta]$. If we set $e_{11} = \bar{t}x, e_{12} = \bar{t}, e_{21} = \bar{t}x^2 + x$, and $e_{22} = 1 + \bar{t}x$ in $R[x;\delta]$, then they form a system of matrix units in $R[x;\delta]$. Now the centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2])$ $\cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So the ring $R[x;\delta]$ is a Baer ring, but R is not quasi-Baer.

Corollary 2.12. [7, Corollary 2.8] *Let R be a ring. Then R is quasi-Baer (resp. right p.q.-Baer) if and only if $R[x]$ is quasi-Baer (resp. right p.q.-Baer).*

According to Lee-Zhou [22], a module M_R is called *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is clear that R is a reduced ring if an only if R_R is reduced. If M_R is reduced, then M_R is p.p. if and only if M_R is p.q.-Baer.

Lemma 2.13. *The following are equivalent for a module M_R.*
(i) *M_R is reduced and δ-compatible;*
(ii) *The following conditions hold. For any $m \in M$ and $a \in R$,*
  (a) *$ma = 0$ implies $mRa = 0$,*
  (b) *$ma = 0$ implies $m\delta(a) = 0$,*
  (c) *$ma^2 = 0$ implies $ma = 0$.*

Proof. The proof is straightforward. $\square$

McCoy [23,Theorem 2] proved that if R is a commutative ring, then whenever $g(x)$ is a zero-divisor in $R[x]$ there exists a nonzero $c \in R$ such that $cg(x) = 0$. We shall extend this result as follows.

Proposition 2.14. *Let M be a reduced and δ-compatible module. If $m(x)$ is a torsion element in $M[x]$ (i.e. $m(x)h(x) = 0$ for some $0 \neq h(x) \in R[x;\delta]$), then there exists a non-zero element c of R such that $m(x)c = 0$.*

Proof. Let $m(x) = \sum_{i=0}^{n} m_i x^i$ and $h(x) = \sum_{j=0}^{s} h_j x^j$ and $m(x)h(x) = 0$. Then

$$m_n h_s = 0. \tag{2.4}$$

Note that for a reduced module M, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$ and $ma^2 = 0$ implies $ma = 0$ by Lemma 2.13. By (2.4) $m_n R h_s = 0$ and $m_n \delta^j(h_s) = 0$ for each $j \geq 0$. Hence the coefficient of x^{n+s-1} in $m(x)h(x) = 0$ is

$$m_n h_{s-1} + m_{n-1} h_s = 0. \tag{2.5}$$

Multiplying Eq. (2.5) by h_s from the right-hand side and using the hypothesis we obtain $m_{n-1}h_s = 0$. Hence $m_{n-1}Rh_s = 0$ and $m_{n-1}\delta^j(h_s) = 0$ for each $j \geq 0$. Thus the coefficient of x^{n+s-2} in $m(x)h(x) = 0$ is

$$m_n h_{s-2} + m_{n-1} h_{s-1} + m_{n-2} h_s = 0. \tag{2.6}$$

Multiplying Eq. (2.6) by h_s from the right-hand side and using the hypothesis, we obtain $m_{n-2}h_s = 0$. Continuing this process, we may prove $m_j h_s = 0$ for each j. Since $h(x) \neq 0$ we may assume $h_s \neq 0$. Then $m(x)h_s = 0$ by Lemma 2.4. $\square$

Proposition 2.15. *Let M_R be a reduced and δ-compatible module. Then M_R is δ-Armendariz.*

Proof. Let $m(x) = m_0 + \cdots + m_k x^k \in M[x]$, $f(x) = a_0 + \cdots + a_n x^n \in R[x;\delta]$ and $m(x)f(x) = 0$. Hence $m_k R a_n = 0$. Thus the coefficient of x^{k+n-1} in equation $m(x)f(x) = 0$ is $m_k a_{n-1} + m_{k-1} a_n = 0$. Multiplying this equation from the right-hand side by a_n, we obtain $m_{k-1}a_n^2 = 0$. Hence $m_{k-1}a_n = 0$ by Lemma 2.13. Therefore $m_k a_{n-1} = 0$, and so

$m_k x^k a_{n-1} x^{n-1} = m_{k-1} x^{k-1} a_n x^n = 0$ by Lemma 2.3. Continuing this process, we can prove $m_i x^i a_j x^j = 0$ for each i, j. □

Theorem 2.16. *Let M_R be a δ-compatible module and $S = R[x;\delta]$. If M_R is δ-Armendariz, then M_R is Baer (resp. p.p) if and only if $M[x]_S$ is Baer (resp. p.p).*

Proof. The proof is similar to that of Theorem 2.9. □

Corollary 2.17. *Let M_R be a reduced and δ-compatible module and $S = R[x;\delta]$. Then M_R is Baer (resp. p.p) if and only if $M[x]_S$ is Baer (resp. p.p).*

Proof. This follows from Proposition 2.14 and Theorem 2.16. □

Corollary 2.18. *Let R be a reduced ring and $S = R[x;\delta]$. Then R is Baer (resp. p.p) if and only if S is Baer (resp. p.p).*

Proof. By using Corollary 2.17, it remains to show that R is δ-compatible. Let $ab = 0$. Then $\delta(ab) = \delta(a)b + a\delta(b) = 0$. Multiplying this equation by b from the right-hand side, we obtain $\delta(a)b^2 = 0$ and so $\delta(a)b = 0 = a\delta(b)$, since R is reduced. □

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