Bulletin of the Iranian Mathematical Society Vol. 33 *No.* 2 (2007), *pp* 49-59.

THE NEHARI MANIFOLD FOR A CLASS OF INDEFINITE WEIGHT SEMILINEAR ELLIPTIC EQUATIONS

G. A. AFROUZI* AND S. KHADEMLOO

Communicated by Henrik Shahgholian

Abstract. Using the Nehari manifold, we prove the existence of positive solutions of the problem $-\Delta u = \lambda a(x)u + b(x)|u|^{\gamma-2}u$ for $x \in \Omega$, together with the boundary condition $\alpha u+(1-\alpha)(\partial u/\partial n) = 0$. Exploiting the relationship between the Nehari manifold and fibrering maps (i.e., maps of the form $t \to J_\lambda(u)$, where J_λ is the Euler functional associated with the equation) and a condition on $b(x)$, we discuss how the Nehari manifold changes as λ changes, and show how existence results for positive solutions of the equation are linked to properties of the Nehari manifold.

1. Introduction

In this paper we study the existence of positive solutions for the problem

$$\begin{cases} -\Delta u = \lambda a(x)u + b(x)|u|^{\gamma-2}u, & x \in \Omega, \\ \alpha u + (1-\alpha)(\partial u/\partial n) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in R^N, $N \geq 2$, $\lambda > 0$ is a real parameter and $a, b : \Omega \to R$ are smooth functions which may change sign. Here we say a function f changes sign if the lebesgue measure of

MSC(2000): Primary: 35J20; Secondary: 35J25
Keywords: Nehari manifold, Fibrering maps, Critical exponent, Indefinite weight function
Received: 14 July 2007, Revised: 26 August 2007, Accepted: 13 September 2007
∗Corresponding author

$\{x \in \Omega; f(x) > 0\}$ and $\{x \in \Omega; f(x) < 0\}$ are both positive.
We shall regard α, λ and γ as real parameters assuming throughout that $2 < \gamma < 2^* = \frac{2N}{N-2}$. Thus we shall study a superlinear perturbation of a linear problem. In this paper we suppose $0 \leq \alpha \leq 1$. Thus $\alpha = 0$ corresponds to the Neumann problem, $\alpha = 1$ to the Dirichlet problem and $0 < \alpha < 1$ to the usual Robin problem.
Similar problems in the case $\alpha = 1$ have been studied by Binding et al. [3, 4] by using variational methods and by Amann and Lopez-Gomez [2] by using global bifurcation theory and by Brown and Zhang [5] by using the Nehari manifold. Furthermore this problem in the case $b(x) = \lambda a(x)$ and $\gamma = 3$ that arises in population genetics has been studied by Fleming [7]. In this setting, (1.1) is a reaction-diffusion equation where the real parameter $\lambda > 0$ corresponds to the reciprocal of the diffusion coefficient and the unknown function u represents a relative frequency so that the interest is only in solutions satisfying $0 \leq u \leq 1$.
The purpose of this paper is to discuss the problem (1.1) from the variational viewpoint and, in particular, from the viewpoint of the Nehari manifold ([9]).
The existence results can be summarized as follows. Let $\lambda_1(\alpha)$ denotes the positive principal eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda a(x)u, & x \in \Omega, \\ \alpha u + (1-\alpha)(\partial u/\partial n) = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with corresponding positive principal eigenfunction u_1. We shall prove in this paper that if $\int_\Omega b(x)u_1^\gamma dx > 0$, then there exists a positive solution of (1.2) whenever $0 < \lambda < \lambda_1(\alpha)$. This result can be understood in terms of global bifurcation theory as the sign of $\int_\Omega b(x)u_1^\gamma dx$ determines the direction of bifurcation from the branch of zero solutions of the bifurcation point at $\lambda = \lambda_1(\alpha)$ so that bifurcation is to the left when $\int_\Omega b(x)u_1^\gamma dx > 0$.
We will demonstrate how the structure of the Nehari manifold is determined by the sign of $\int_\Omega b(x)u_1^\gamma dx$.
The plane of the paper is as follows. In section 2 we first recall the facts that we shall require about the Nehari manifold and examine carefully the connection between the Nehari manifold and the fibrering maps. In section 3 we discuss the solutions of equation (1.1) when $\lambda < \lambda_1(\alpha)$ and show how the behaviour of the manifold as $\lambda \to (\lambda_1(\alpha))^-$ depends on the sign of $\int_\Omega b(x)u_1^\gamma dx$, where $\alpha \in (0, 1]$.

2. **Notation and preliminaries**

Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$, $N \geq 2$. We will work in the Sobolev space $W := W^{1,2}(\Omega)$ equipped with the norm

$$||u|| = \{\int_\Omega (|\nabla u(x)|^2 + u(x)^2)dx\}^{1/2}.$$

We assume that $a(x), b(x) \in L^\infty(\Omega)$.
Consider the map $\lambda \to \mu(\alpha, \lambda)$, where $\mu(\alpha, \lambda)$ denotes the principal eigenvalue of the linear problem

$$\begin{cases} -\Delta u - \lambda a(x)u = \mu u, & x \in \Omega, \\ \alpha u + (1-\alpha)\frac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Our method works provided the linearized problem for equation (1.1), viz, (2.1) has principal eigenvalues. It is shown in [1] that this occurs on an interval $[\alpha_0, 1]$ where $\alpha_0 \leq 0$. Thus we are able to obtain existence results for equation (1.1) even in the case of nonstandard Robin boundary conditions.
Clearly λ is a principal eigenvalue of the equation (2.1) if and only if $\mu(\alpha, \lambda) = 0$. It can be shown that $\mu(\alpha, \lambda)$ has the variational characterization

$$\mu(\alpha, \lambda) = \inf\{\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma; u \in W, \int_\Omega u^2 dx = 1\}$$

whence it follows that
(i) $\alpha \to \mu(\alpha, \lambda)$ is an increasing function;
(ii) $\lambda \to \mu(\alpha, \lambda)$ is a concave function with a unique maximum such that $\mu(\alpha, \lambda) \to \infty$ as $\lambda \to \pm\infty$;
(iii) if $\alpha \in (0, 1]$, then $\mu(\alpha, 0) > 0$ and so it follows that $\mu(\alpha, \lambda)$ as a function of λ has exactly one negative zero $\lambda_{-1}(\alpha)$ and one positive zero $\lambda_1(\alpha)$. Thus $\lambda_{-1}(\alpha)$ and $\lambda_1(\alpha)$ are principal eigenvalues for (2.1).

Proposition 2.1. [8] *Assume that $0 < \alpha < 1$. Then for any λ strictly between $\lambda_{-1}(\alpha)$ and $\lambda_1(\alpha)$, the relation*

$$||u||_{\alpha,\lambda} := (\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma)^{1/2}$$

defines an equivalent norm on W.

We can now introduce the structure of the Nehari manifold. It is well known that steady state solutions of (1.1) correspond to critical points of the functional

$$J_\lambda(u)=\int_\Omega(\frac{1}{2}|\nabla u|^2-\frac{1}{2}\lambda a(x)u^2-\frac{1}{\gamma}b(x)|u|^\gamma)dx+\frac{\alpha}{2(1-\alpha)}\int_{\partial\Omega}u^2d\sigma$$

that is bounded neither above nor below. In order to obtain existence results in this case we introduce the Nehari manifold

$$S=\{u\in W;<J'_\lambda(u),u>=0\},$$

where $<,>$ denotes the usual duality. Thus $u\in S$ if and only if

$$\int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2d\sigma=\int_\Omega b(x)|u|^\gamma dx.$$

Clearly S is a much smaller set than W and so it is easier to study J_λ on S.

For $u\in S$ we have that

$$\begin{aligned}J_\lambda(u) &= (\frac{1}{2}-\frac{1}{\gamma})\left(\int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2d\sigma\right)\\ &= (\frac{1}{2}-\frac{1}{\gamma})\int_\Omega b(x)|u|^\gamma dx.\end{aligned}$$

The Nehari manifold is closely linked to the behavior of functions of the form $\phi_u:t\rightarrow J_\lambda(tu)$ $(t>0)$. Such maps are known as fibrering maps and were introduced by Drabek and Pohozaev in [6]. If $u\in W$, we have

$$\begin{aligned}\phi_u(t) &= \frac{t^2}{2}\left(\int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2d\sigma\right)\\ &\quad-\frac{t^\gamma}{\gamma}\int_\Omega b(x)|u|^\gamma dx,\\ \phi'_u(t) &= t\left(\int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2d\sigma\right)\\ &\quad-t^{\gamma-1}\int_\Omega b(x)|u|^\gamma dx,\\ \phi''_u(t) &= \int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2d\sigma\\ &\quad-(\gamma-1)t^{\gamma-2}\int_\Omega b(x)|u|^\gamma dx.\end{aligned}$$

It is easy to see that $u\in S$ if and only if $\phi'_u(1)=0$ and more generally that $tu\in S$ if and only if $\phi'_u(t)=0$, i.e., elements in S correspond to stationary points of fibrering maps. Thus it is natural to subdivide

S into sets corresponding to local minima, local maxima and points of inflection. Thus we define

$$\begin{aligned} S^+ &= \{u \in W; \phi_u'(1) = 0, \phi_u''(1) > 0\} \\ &= \left\{u \in S; \int_\Omega b(x)|u|^\gamma dx < 0\right\}, \\ S^- &= \left\{u \in S; \int_\Omega b(x)|u|^\gamma dx > 0\right\}, \\ S^0 &= \left\{u \in S; \int_\Omega b(x)|u|^\gamma dx = 0\right\}. \end{aligned}$$

Let $u \in W$. Then
(i) if $\int_\Omega(|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma$ and $\int_\Omega b(x)|u|^\gamma dx$ have the same sign, ϕ_u has a unique turning point at

$$t(u) = \left[\frac{\int_\Omega(|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma}{\int b(x)|u|^\gamma dx}\right]^{\frac{1}{\gamma-2}},$$

this turning point is a local minimum (maximum) so that $t(u)u \in S^+(S^-)$ if and only if $\int_\Omega b(x)|u|^\gamma dx > 0(< 0)$;
(ii) if $\int_\Omega(|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma$ and $\int_\Omega b(x)|u|^\gamma dx$ have opposite signs, ϕ_u has no turning points and so no multiples of u lie in S.

We define

$$L^+ = \left\{u \in W; ||u|| = 1, \int_\Omega(|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma > 0\right\},$$

and L^-, L^0 similarly by replacing $>$ by $<$ and $=$, respectively. We also define

$$B^+ = \left\{u \in W; ||u|| = 1, \int_\Omega b(x)|u|^\gamma dx > 0\right\},$$

and B^- and B^0 analogously. So we have
(i) if $u \in L^+ \cap B^+$, then $t \to \phi_u(t)$ has a local maximum at $t = t(u)$ and $t(u)u \in S^-$;
(ii) if $u \in L^- \cap B^-$, then $t \to \phi_u(t)$ has a local minimum at $t = t(u)$ and $t(u)u \in S^+$;
(iii) if $u \in L^+ \cap B^-$, then $t \to \phi_u(t)$ is strictly increasing and no multiple of u lies in S;
(iii) if $u \in L^- \cap B^+$, then $t \to \phi_u(t)$ is strictly decreasing and no multiple of u lies in S.

We shall prove the existence of solutions of (1.1) by investigating the existence of minimizers on S. Although S is only a small subset of W, it turns out that minimizers of J_λ on S are generically also critical points of J_λ on W. In fact, as proved in [5], we have:

Proposition 2.2. *Suppose that* u_0 *is a local minimizer for* J_λ *on* S *and* $u_0 \notin S^0$, *then* $J'_\lambda(u_0) = 0$.

3. Main results

The following lemmas and theorems are central in proving that bifurcation occurs and in determining the direction of bifurcation.

Lemma 3.1. *If* $0 < \lambda < \lambda_1(\alpha)$, *then* L^-, L^0 *and* S^+ *are empty and* $S^0 = \{0\}$.

Proof. It is easy to deduce, by the equivalence property mentioned in Proposition 2.1, that there exists $M > 0$ such that

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 d\sigma \geq M||u||^2$$

for all $u \in W$. Thus L^- and L^0 are empty and so S^+ is empty and $S^0 = \{0\}$. Moreover $S^- = \{t(u)u; u \in B^+\}$ and $S = S^- \cup \{0\}$. □

In view of the preceding lemma we just investigate the behavior of J_λ on S^-.

Theorem 3.2. *If* $0 < \lambda < \lambda_1(\alpha)$, *then* $\inf_{u\in S^-} J_\lambda(u) > 0$.

Proof. It is an immediate consequence of the structure of S^- that $J_\lambda(u) \geq 0$ whenever $u \in S^-$ and so J_λ is bounded below by 0 on S^-. Let $u \in S^-$. Then $v = \frac{u}{||u||} \in L^+ \cap B^+$ and $u = t(v)v$, where $t(v) = \left[\frac{\int_\Omega(|\nabla v|^2 - \lambda a(x)v^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} v^2 d\sigma}{\int_\Omega b(x)|v|^\gamma dx}\right]^{\frac{1}{\gamma-2}}$. Now we have

$$\int_\Omega b(x)|v|^\gamma dx \leq \bar{b}\int_\Omega |v|^\gamma dx \leq \bar{b}K(\int_\Omega(|\nabla v|^2 + v^2)dx)^{\frac{\gamma}{2}} = \bar{b}K||v||^{\frac{\gamma}{2}} = \bar{b}K,$$

where $\bar{b} = \sup_{x\in\overline{\Omega}} b(x)$ and K is a Sobolev embedding constant. Hence

$$\begin{aligned}
J_\lambda(u) &= J_\lambda(t(v)v) \\
&= (\frac{1}{2}-\frac{1}{\gamma})t(v)^2\left[\int_\Omega(|\nabla v|^2-\lambda a(x)v^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}v^2d\sigma\right] \\
&= (\frac{1}{2}-\frac{1}{\gamma})\frac{\left[\int_\Omega(|\nabla v|^2-\lambda a(x)v^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}v^2d\sigma\right]^{\frac{\gamma}{\gamma-2}}}{[\int_\Omega b(x)|v|^\gamma dx]^{\frac{2}{\gamma-2}}} \\
&\geq (\frac{1}{2}-\frac{1}{\gamma})\frac{M^{\frac{\gamma}{\gamma-2}}}{(\bar{b}K)^{\frac{2}{\gamma-2}}} := c,
\end{aligned}$$

and so $J_\lambda(u) > c > 0$ for all $u \in S^-$, i.e., $\inf_{u\in S^-} J_\lambda(u) \geq c > 0$. □

We conclude this section by proving a technical theorem which gives us a non-trivial solution of the problem (1.1).

Theorem 3.3. *If $0 < \lambda < \lambda_1(\alpha)$, then there exists a minimizer on S^- which is a critical point of $J_\lambda(u)$.*

Proof. Let $\{u_n\} \subset S^-$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^-} J_\lambda(u).$$

By Theorem 2.1, we have

$$\begin{aligned}
J_\lambda(u_n) = (\frac{1}{2}-\frac{1}{\gamma})&\left[\int_\Omega(|\nabla v|^2-\lambda a(x)v^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}v^2d\sigma\right] \\
&\geq (\frac{1}{2}-\frac{1}{\gamma})M||u_n||^2.
\end{aligned}$$

Hence $\{u_n\}$ is bounded in W, and so we may assume that $u_n \to u_0$ weakly in W for some $u_0 \in W$ and strongly in L^2 and L^γ. Now

$$0 < \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty}(\frac{1}{2}-\frac{1}{\gamma})\int_\Omega b(x)|u_n|^\gamma dx = (\frac{1}{2}-\frac{1}{\gamma})\int_\Omega b(x)|u_0|^\gamma dx,$$

and so $u_0 \not\equiv 0$. Since $\lambda < \lambda_1(\alpha)$, we have $\int_\Omega(|\nabla u_0|^2 - \lambda a(x)u_0^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u_0^2 d\sigma > 0$ and a multiple of u_0 lies in $L^+ \cap B^+$. We claim that $u_n \to u_0$ strongly in W. Suppose that this is false. Then $||u_0|| <$

$\liminf_{n\to\infty}||u_n||$ and so

$$\begin{aligned}\int_\Omega(|\nabla u_0|^2-\lambda a(x)u_0^2-b(x)|u_0|^\gamma)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u_0^2d\sigma &<\\ \liminf_{n\to\infty}\int_\Omega(|\nabla u_n|^2-\lambda a(x)u_n^2-b(x)|u_n|^\gamma)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u_n^2d\sigma &= 0.\end{aligned}$$

This means $\phi'_{u_0}(1)=\int_\Omega(|\nabla u_0|^2-\lambda a(x)u_0^2-b(x)|u_0|^\gamma)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u_0^2d\sigma<0$. Now by using the mean value theorem for ϕ'_{u_0} we can see that there exists $\theta<1$ such that $\phi'_{u_0}(\theta)=0$, i.e., $\theta u_0\in S^-$ and $\theta u_0\not\equiv 0$. Therefore $\theta u_0\notin S^0$, and as $\{u_n\}\in S^-$, we conclude that ϕ_{u_n} attains its maximum at $t=1$. Hence

$$\begin{aligned}J_\lambda(\theta u_0) &= (\frac{1}{2}-\frac{1}{\gamma})\theta^2\left[\int_\Omega(|\nabla u_0|^2-\lambda a(x)u_0^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u_0^2d\sigma\right]\\ &< (\frac{1}{2}-\frac{1}{\gamma})\theta^2\liminf_{n\to\infty}\left[\int_\Omega(|\nabla u_n|^2-\lambda a(x)u_n^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u_n^2d\sigma\right]\\ &= \liminf_{n\to\infty}J_\lambda(\theta u_n).\end{aligned}$$

Using the auxiliary function $f(t)=\frac{1}{2}t^2-\frac{1}{\gamma}t^\gamma$, we have

$$\liminf_{n\to\infty}J_\lambda(\theta u_n)\le\liminf_{n\to\infty}J_\lambda(u_n)=\lim_{n\to\infty}J_\lambda(u_n)=\inf_{u\in S^-}J_\lambda(u),$$

which is a contradiction. Therefore $u_n\to u_0$ in W. Hence $\int_\Omega(|\nabla u_0|^2-\lambda a(x)u_0^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u_0^2d\sigma=\int_\Omega b(x)|u_0|^\gamma dx$ and so $u_0\in S^-$. Also $J_\lambda(u_0)=\lim_{n\to\infty}J_\lambda(u_n)=\inf_{u\in S^-}J_\lambda(u)$ and so u_0 is a minimizer on S^-, and since $\int_\Omega b(x)|u_0|^\gamma dx>0$ and $0\neq u_0\notin S^0$, u_0 is a critical point of J_λ. Note that J_λ is an even functional, i.e., $J_\lambda(u)=J_\lambda(|u|)$. Therefore we may assume u_0 is nonnegative. $\square$

We can also prove some properties of the branch of solutions bifurcated from $\lambda_1(a)$ whenever the condition $\int_\Omega b(x)u_1^\gamma dx>0$ is satisfied.

Theorem 3.4. *Let $\int_\Omega b(x)u_1^\gamma dx>0$. Then*
(i) $\lim_{\lambda\to(\lambda_1(\alpha))^-}\inf_{u\in S^-}J_\lambda(u)=0$,
(ii) *if $\lambda_n\to(\lambda_1(\alpha))^-$ and $\{u_n\}$ is a minimizer of J_{λ_n} on S^- then* $\lim_{n\to\infty}u_n=0$.

Proof. (i) We may assume, without loss of generality, that $||u_1||=1$ and so $u_1\in B^+$. By Lemma 3.1, we get $u_1\in B^+\cap L^+$ and so $t(u_1)u_1\in S^-$, where

$$\begin{aligned} t(u_1) &= \left[\frac{\int_\Omega(|\nabla u_1|^2 - \lambda a(x)u_1^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u_1^2 d\sigma}{\int_\Omega b(x)u_1^\gamma dx}\right]^{\frac{1}{\gamma-2}} \\ &= \left[\frac{\lambda_1(\alpha)\int_\Omega a(x)u_1^2dx - \lambda\int_\Omega a(x)u_1^2dx}{\int_\Omega b(x)u_1^\gamma dx}\right]^{\frac{1}{\gamma-2}} \\ &= \left[\frac{(\lambda_1(\alpha)-\lambda)\int_\Omega a(x)u_1^2dx}{\int_\Omega b(x)u_1^\gamma dx}\right]^{\frac{1}{\gamma-2}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} J_\lambda(t(u_1)u_1) &= (\frac{1}{2}-\frac{1}{\gamma})\int_\Omega b(x)(t(u_1)u_1)^\gamma dx \\ &= (\frac{1}{2}-\frac{1}{\gamma})(\lambda_1(\alpha)-\lambda)^{\frac{\gamma}{\gamma-2}}\left[\frac{\int_\Omega a(x)u_1^2dx}{\int_\Omega b(x)u_1^\gamma dx}\right]^{\frac{\gamma}{\gamma-2}}\int_\Omega b(x)u_1^\gamma dx \\ &= (\frac{1}{2}-\frac{1}{\gamma})(\lambda_1(\alpha)-\lambda)^{\frac{\gamma}{\gamma-2}}\frac{(\int_\Omega a(x)u_1^2dx)^{\frac{\gamma}{\gamma-2}}}{(\int_\Omega b(x)u_1^\gamma dx)^{\frac{2}{\gamma-2}}} \\ &\longrightarrow 0 \text{ as } \lambda\to(\lambda_1(\alpha))^-. \end{aligned}$$

On the other hand, we have $t(u_1)u_1 \in S^-$ and so

$$0 < \inf_{u\in S^-} J_\lambda(u) \le J_\lambda(t(u_1)u_1) \to 0 \text{ as } \lambda\to(\lambda_1(\alpha))^-.$$

Hence $\lim_{\lambda\to(\lambda_1(\alpha))^-}\inf_{u\in S^-} J_\lambda(u) = 0$.

(ii) First we show that every minimizer sequence of J_λ on S^- is bounded. Otherwise we may assume, without loss of generality, that $||u_n|| \to \infty$ as $n\to\infty$. Let $v_n = \frac{u_n}{||u_n||}$, $\{v_n\}$ is a bounded sequence and so $v_n \to v_0$ weakly in W for some $v_0 \in W$ and strongly in L^2 and L^γ. Hence $\lim_{n\to\infty}\int_\Omega a(x)v_n^2dx = \int_\Omega a(x)v_0^2dx$ and $\lim_{n\to\infty}\int_\Omega b(x)|v_n|^\gamma dx = \int_\Omega b(x)|v_0|^\gamma dx$, and

$$0 \le \lim_{n\to\infty} J_{\lambda_n}(u_n) = \inf_{u_n\in S^-} J_{\lambda_n}(u_n) \le \sup_{\lambda_n\to(\lambda_1(\alpha))^-}\inf_{u_n\in S^-} J_{\lambda_n}(u_n) = 0.$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} J_{\lambda_n}(u_n) &= \lim_{n\to\infty}(\frac{1}{2}-\frac{1}{\gamma})\left[\int_\Omega(|\nabla u_n|^2 - \lambda_n a(x)u_n^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u_n^2 d\sigma\right] \\ &= \lim_{n\to\infty}(\frac{1}{2}-\frac{1}{\gamma})\int_\Omega b(x)|u_n|^\gamma dx = 0. \end{aligned}$$

By dividing by $||u_n||^\gamma$ we have

$$\int_\Omega (|\nabla v_n|^2 - \lambda_n a(x) v_n^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} v_n^2 d\sigma = ||u_n||^{\gamma-2}\int_\Omega b(x)|v_n|^\gamma dx \to 0$$

as $n \to \infty$. Hence $\lim_{n\to\infty}\int_\Omega b(x)|v_n|^\gamma dx = \int_\Omega b(x)|v_0|^\gamma dx = 0$.
Now we show that $v_n \to v_0$ in W. Suppose this is false. From the weak convergence of v_n to v_0 in W and $\int_\Omega a(x)v_n^2 dx \to \int_\Omega a(x)v_0^2 dx$, we conclude that

$$\begin{aligned} 0 &\leq \int_\Omega (|\nabla v_0|^2 - \lambda_1(\alpha) a(x) v_0^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} v_0^2 d\sigma \\ &\leq \lim_{n\to\infty}\left[\int_\Omega (|\nabla v_n|^2 - \lambda_1(\alpha) a(x) v_n^2)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} v_n^2 d\sigma\right] = 0. \end{aligned}$$

If $v_0 = 0$ we then derive that $v_n \rightharpoonup 0$ in W, which contradicts the fact that $||v_n|| = 1$. It then follows from the uniqueness of u_1 that $v_0 = ku_1$ for some positive constant k. On the other hand, we have $k^\gamma \int_\Omega b(x)u_1^\gamma dx = \int_\Omega b(x)v_0^\gamma dx = 0$, which is impossible. Hence $\{u_n\}$ is bounded.
Thus we may assume, without loss of generality, that $u_n \to u_0$ weakly in W. Then using essentially the same argument on $\{u_n\}$ as has just been used above on $\{v_n\}$, it follows that $u_n \to u_0$ and $u_0 = 0$ and so the proof is complete. □

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G. A. Afrouzi
Department of Mathematics
Faculty of Basic Sciences
University of Mazandaran
Babolsar, Iran
e-mail: afrouzi@umz.ac.ir

S. Khademloo
Faculty of Basic Sciences
Noushirvani Institute of Technology
University of Mazandaran
Babol, Iran
e-mail: s.khademloo@umz.ac.ir