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ON NONLINEAR PRESERVERS OF WEAK MATRIX MAJORIZATION

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ABSTRACT. For $X, Y \in M_{nm} (:= M_{nm}(\mathbb{R}))$, we say X is *weakly matrix majorized* or *matrix majorized from the left* by Y and write $X \prec_\ell Y$, if $X = RY$ for some row stochastic matrix R. Also we write $X \sim_\ell Y$ if $X \prec_\ell Y \prec_\ell X$. A mapping $T : M_{nm} \to M_{nm}$ is said to be a strong preserver of $\prec_\ell$, if $\{X \in M_{nm} : X \prec_\ell A\} = \{X \in M_{nm} : TX \prec_\ell TA\}$ for all $A \in M_{nm}$. Two such strong preservers T and τ are called equivalent if $TX \sim_\ell \tau X$ for all $X \in M_{nm}$. It is shown that if $m \geq 2$ and if $T : M_{nm} \to M_{nm}$ is a surjective (not necessarily linear) strong preserver of $\prec_\ell$, then $T - T0$ is equivalent to a linear strong preserver of $\prec_\ell$.

1. Introduction

Throughout this paper the following notations are fixed. The real vector space of all $1 \times m$ (row) vectors are denoted by $\mathbb{R}_m$ and the real linear space of all $n \times m$ matrices by M_{nm}, for any integers $n, m \geq 1$. For every $A \in M_{nm}$, $R(A) \subset \mathbb{R}_m$ will denote the set of all distinct rows of A. For every $x \in \mathbb{R}_m$, we let $x^{(n)}$ denote the $n \times m$ matrix such that $R(x^{(n)}) = \{x\}$. If $X, Y \in M_{nm}$, we say X is *matrix majorized from the left* or *weakly matrix majorized* by Y, and write $X \prec_\ell Y$, if the rows

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$X_1, \ldots, X_n$ of X and $Y_1, \ldots, Y_n$ of Y satisfy $X_i = \Sigma_{j=1}^{n} r_{ij} Y_j$ for some nonnegative scalars r_{ij} such that $\Sigma_{j=1}^{n} r_{ij} = 1$ $(i, j = 1, 2, \ldots, n)$. The matrix $R = [r_{ij}]$ is called a row stochastic matrix and the relation $X \prec_\ell Y$ can be illustrated as $X = RY$. We write $X \sim_\ell Y$ if $X \prec_\ell Y \prec_\ell X$. Also, we define $C(A) := \{X \in M_{nm} : X \prec_\ell A\}$ and $[A] := \{X \in M_{nm} : X \sim_\ell A\}$.

There is a right-sided type of matrix majorization $\prec_r$ on M_{nm} defined by $X \prec_r Y$ whenever $X = YR$ for some row stochastic matrix R depending on X and Y. In this paper, we deal only with the left-sided type and hence, for the remainder of the paper, we use the conventions $\prec$ and $\sim$ for $\prec_\ell$ and $\sim_\ell$, respectively. Throughout the paper, the letter T stands for a mapping satisfying the conditions set in the following Definition 1.1.

Definition 1.1. A (not necessarily linear) mapping $T : M_{nm} \to M_{nm}$ is said to be a strong preserver of $\prec$, if $\{X \in M_{nm} : X \prec A\} = \{X \in M_{nm} : TX \prec TA\}$ for all $A \in M_{nm}$.

Definition 1.2. Two strong preservers T and τ of $\prec$ on M_{nm} are said to be equivalent, if $TX \sim \tau X$ for all $X \in M_{nm}$.

The main result of the paper is to show that if $m \geq 2$ and if $T : M_{nm} \to M_{nm}$ is a surjective strong preserver of $\prec$, then the mapping $X \mapsto TX - T0$ is equivalent to a linear one. This extends results due to L.B. Beasley, S.-G. Lee and Y.-H. Lee [4] and A.M. Hasani and M. Radjabalipour [8]. Note that if T is a linear strong preserver, then it is injective and, hence, bijective [4,8,9]. Also, note that, if $T : \mathbb{R} \to \mathbb{R}$ is any function, then T is a strong preserver of $\prec$ on $M_1 = \mathbb{R}$ but $T - T0$ is not equivalent to a linear one. For more information on matrix majorization and the previous work on this subject we also refer to [1-3], [5-7] and [10-12]. In particular, the authors of [8] show that T is a linear strong preserver of $\prec_\ell$ if and only if there exist a permutation matrix P and an invertible matrix L in M_n such that $TX = PXL$ for all $X \in M_n$. We will obtain this result as a byproduct of our investigations in the present paper.

The following lemma enables us to assume, without loss of generality, that $T0 = 0$.

Lemma 1.3. *Let $T : M_{nm} \to M_{nm}$ be a strong preserver of $\prec$. Then the following assertions are true.*

(a) *Assume T is surjective. Then $C(TA) = TC(A)$ for all $A \in M_{nm}$.*

(b) *Assume $C(TA) = TC(A)$ for all $A \in M_{nm}$. Then $R(X)$ is a singleton if and only if $R(TX)$ is a singleton.*

(c) *Assume $C(TA) = TC(A)$ for all $A \in M_{nm}$. The mapping $T' : M_{nm} \to M_{nm}$ defined by $T'X = TX - T0$ for all $X \in M_{nm}$ is a strong preserver of $\prec$ satisfying $C(T'A) = T'C(A)$ for all $A \in M_{nm}$ and $T'0 = 0$.*

Proof. (a) Let $A \in M_{nm}$. By definition, $TC(A) \subset C(TA)$. Now, let $Y \in C(TA)$. Then there exists $X \in M_{nm}$ such that $Y = TX$. Since $TX \prec TA$, $X \prec A$ and hence, $Y \in TC(A)$.

(b) The set $R(A)$ is a singleton if and only if $C(A) = \{A\}$ if and only if $C(TA) = TC(A) = \{TA\}$ if and only if $R(TA)$ is a singleton.

(c) By part (b), $R(T0) = \{a\}$ for some ($1 \times m$ row) vector $a \in \mathbb{R}_m$. Now, let $A, X \in M_{nm}$ and let $B = TA$ and $Y = TX$. Let B_i, Y_i be the i^{th} rows of B and Y, respectively ($i = 1, 2, \ldots, n$). Then $X \in C(A)$ if and only if $Y = TX \in C(B)$ if and only if $Y_i = \mu_{i1}B_1 + \ldots + \mu_{in}B_n$ or, equivalently, $Y_i - a = \mu_{i1}(B_1 - a) + \ldots + \mu_{in}(B_n - a)$ for some nonnegative scalars μ_{ij} satisfying $\Sigma_{j=1}^{n}\mu_{ij} = 1$ ($i, j = 1, 2, \ldots, n$). The latter shows that $T'X = TX - T0 \prec TA - T0 = T'A$ and hence, T' is a strong preserver of matrix majorization which satisfies $T'0 = 0$. Now, if $Y \prec T'A$ for some $A \in M_{nm}$, then $Y \prec TA - T0$ or, equivalently, $Y + T0 \prec TA$. Then $Y + T0 = TX$ for some $X \in M_{nm}$ and hence, $Y = TX - T0 = T'X$. Obviously, $C(T'A) = T'C(A)$ for all $A \in M_{nm}$. $\square$

The converse of part (a) of Lemma 1.3 will be proven in Theorem 2.2. Throughout the remainder of the paper we impose the following assumption on T unless otherwise stated.

Assumption 1.4. $T0 = 0$.

We conclude this section by a technical lemma needed in the sequel. If W is a subset of a real vector space V, co W will denote the convex hull of W, and, if W is convex, ext W will denote the set of extreme points of W.

Lemma 1.5. *Let $A \in M_{nm}$. Then the following assertions are true.*

(a) *Up to the natural identification of the vector spaces M_{nm} and $(\mathbb{R}_m)^n$, $C(A) = (\text{co } R(A))^n$. In particular, $C(A)$ is a convex subset of M_{nm}.*

(b) ext $C(A) = (\text{ext co } R(A))^n$.

Proof. Part (a) is easy, and part (b) follows from the fact that $(\text{ext } W_1)\times \ldots \times (\text{ext } W_k) = \text{ext } (W_1 \times \ldots \times W_k)$, whenever $W_1, \ldots , W_k$ are convex subsets of the real vector spaces $V_1, \ldots , V_k$, respectively. □

2. Nonlinear preservers

In this section we study the structure of the surjective strong preservers of matrix majorization $\prec$ which are not necessarily linear. We will show that if $m \geq 2$, such mappings are equivalent to linear ones.

We begin with a lemma which strengthens Lemma 1.3.

Lemma 2.1. *Assume $T : M_{nm} \to M_{nm}$ is a strong preserver of $\prec$ satisfying $T0 = 0$ and $TC(A) = C(TA)$ for all $A \in M_{nm}$. Let $A \in M_{nm}$, let* ext co $R(A) = \{x_1, x_2, \ldots , x_k\}$, *and let $Tx_i^{(n)} = y_i^{(n)}$, $i = 1, 2, \ldots , k$, where, as before, $u^{(n)}$ denotes an $n \times m$ matrix whose rows are all equal to some $u \in \mathbb{R}_m$. Then* ext co $R(TA) = \{y_1, y_2, \ldots , y_k\}$.

Proof. Define $S : \mathbb{R}_m \to \mathbb{R}_m$ by $Sx = y$, where $y^{(n)} = Tx^{(n)}$. Since $Tx^{(n)} = Ty^{(n)}$ if and only if $x^{(n)} = y^{(n)}$, it follows that S is an injective mapping whose range contains co $R(TX)$ for all $X \in M_{nm}$. Assume $x \in \mathbb{R}_m$ is an extreme point of co $R(A)$ and, if possible, $y = Sx$ is not an extreme point of co $R(TA)$. Then, there exists $u \neq v \in$ co $R(TA)$ such that $y = (u+v)/2$. Let $B \in M_{nm}$ be any matrix such that $R(B) = \{u, v\}$. Since $B \prec TA$, there exists $D \prec A$ such that $B = TD$. Let $r = S^{-1}u$ and $s = S^{-1}v$. Then $\{x, r, s\} \subset R(D)$. Let $E, F, G \in C(D)$ be such that $R(E) = \{x, r\}$, $R(F) = \{x, s\}$ and $R(G) = \{r, s\}$. Then co $R(TE) \supset [y, u]$, co $R(TF) \supset [y, v]$, and co $R(TG) \supset [u, v]$, where $[a, b]$ denotes the closed line segment joining the vectors $a, b \in \mathbb{R}_m$. Replacing TE, TF, and TG by minor matrices having, respectively, $\{y, u\}$, $\{y, v\}$, and $\{u, v\}$ as their exact collection of rows, one can easily see that E, F and G have still the same collections $\{x, r\}$, $\{x, s\}$ and $\{r, s\}$ as their exact collections of rows, respectively. Thus, we can assume without loss of generality that $R(TE) = \{x, y\}$, $R(TF) = \{y, v\}$ and $R(TG) = \{u, v\}$. This implies that $S[x, r] = [y, u]$, $S[x, s] = [y, v]$ and $S[r, s] = [u, v]$. If x, r, s form a nontrivial triangle, choose a point w in the interior of the triangle and observe that $w^{(n)} \prec D$ and hence $Tw^{(n)} \prec TD$ or, equivalently, $Sw \in [u, v]$, which implies that S is not injective; a contradiction. If x, r, s are collinear, then $[x, r] \cap [x, s] \backslash \{x\} \neq \emptyset$ which implies that S is multivalued; again a contradiction. Summing up, we have shown that $S(\text{ext co } R(A)) \subset \text{ext co } R(TA)$.

To prove the converse, assume y is an extreme point of co $R(TA)$. Since $y^{(n)} \prec TA$, there exists $x \in$ co $R(A)$ such that $Tx^{(n)} = y^{(n)}$. We claim x is an extreme point of co $R(A)$. If not, there exists $r, s \in$ co $R(A)$ such that $x = (r+s)/2$. Assume without loss of generality that $R(A) = \{r, s\}$. Let $Tr^{(n)} = u^{(n)}$ and $Ts^{(n)} = v^{(n)}$. By the previous paragraph, u, v are extreme points of co $R(TA)$ and, hence, y, u, v are noncollinear extreme points of co $R(TA)$. By an argument similar to the one given for the first part, there exist matrices E, F and G such that co $R(E) = [x, r]$, co $R(F) = [x, s]$, co $R(G) = [r, s]$, co $R(TE) = [y, u]$, co $R(TF) = [y, v]$, and co $R(TG) = [u, v]$. Choosing $t \in [x, r] \backslash \{x, r\}$, it follows that $t^{(n)} \prec E$, $t^{(n)} \prec G$ and hence, $Tt^{(n)} \in [y, u] \cap [u, v]$. Thus $St = y, u$, or v. Equivalently, $t = x, r$, or s; a contradiction. $\square$

Corollary 2.2. *Let $S : \mathbb{R}_m \to \mathbb{R}_m$ be as in the proof of Lemma 2.1; i.e., $Tx^{(n)} = (Sx)^{(n)}$ for all $x \in \mathbb{R}_m$. Then S is injective and* ext co $R(TA) = S(\text{ext co } R(A))$. *In particular, if $m \geq 2$ and if x and y are distinct vectors in $\mathbb{R}_m$, then $S((1-t)x + ty) = (1 - f(t))Sx + f(t)Sy$ for some strictly increasing function f from $[0,1]$ onto $[0,1]$.*

Definition 2.3. The operator $S : \mathbb{R}_m \to \mathbb{R}_m$ defined in Corollary 2.2 will be called the *border operator* corresponding to T.

In the proof of the next theorem, we will make use of the following version of a fact due to Zs. Pales [14] as interpreted by L. Molnar [13] : If K is a noncollinear convex set in $\mathbb{R}_m$ and if $S : K \to K$ is a bijective mapping such that for any $x, y \in K$ and any $\lambda \in [0,1]$, there exists $\mu \in [0,1]$ satisfying $S(\lambda x + (1-\lambda)y) = \mu S(x) + (1-\mu)S(y)$, then there exist a linear operator $\Psi : \mathbb{R}_m \to \mathbb{R}_m$, a constant vector $a \in \mathbb{R}_m$, a linear functional f on $\mathbb{R}_m$, and a constant $b \in \mathbb{R}$ such that

$$S(x) = \frac{\Psi(x) + a}{f(x) + b} \quad \text{for all } x \in K, \tag{2.1}$$

and $f(x) + b$ is always positive on K. In particular, if $K = \mathbb{R}_m$, then f has to be zero.

Theorem 2.4. *Let $m \geq 2$ and assume $C(TA) = TC(A)$ for all $A \in M_{nm}$. The border operator $S : \mathbb{R}_m \to \mathbb{R}_m$, is a bijective linear operator. Also, T is bijective.*

Proof. We break the proof into various steps.

Step 1. *Claim: For any* $x \in \mathbb{R}_m$*, the ray* $[0, Sx \rightarrow) := \{tSx : t \geq 0\}$ *is a subset of* $S\mathbb{R}_m$. Fix $x \in \mathbb{R}_m$ and assume, if possible, the ray $[0, Sx \rightarrow)$ contains some point z not lying in the range of S. Choose $y \in \mathbb{R}_m$ such that $0, x, y$ are noncollinear. By Lemma 2.1, the points $0, Sx, Sy$ are noncollinear too. Let $\lambda \in (0, 1)$ be such that $2^{-1}Sy = S(\lambda y)$, and join the point $w = S(\lambda y)$ to the point z. Choose $u \in [w, z] \cap [Sx, Sy]$. Then $u = Sv$ for some $v \in [y, x]$. (See Corollary 2.2.) Replacing z by $2z$, if necessary, we can assume without loss of generality that the extension of $[\lambda y, v]$ intersects the extension of $[0, x]$ at some point r. In view of Corollary 2.2, this means that Sr is equal to z; a contradiction.

Step 2. *Claim:* $S\mathbb{R}_m = \mathbb{R}_m$.

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}_m$, and choose $j_1, \ldots, j_k$ such that $\{Se_{j_1}, \ldots, Se_{j_k}\}$ forms a basis for $< Se_1, Se_2, \ldots, Se_m >$. Assume $k < m$ and choose $j \in \{1, 2, \ldots, m\} \backslash \{j_1, \ldots, j_k\}$. Then there exist real numbers $c_1, \ldots, c_k$ such that $Se_j = c_1 Se_{j_1} + \ldots + c_k Se_{j_k}$. In view of Step 1 and Corollary 2.2, if $0 \neq x \in \mathbb{R}_m$ and if $\eta > 0$, then $-Sx = S(-\mu x)$ and $\eta Sx = S(\gamma x)$ for some positive numbers μ and γ. Hence, there exist real numbers $d, d_1, d_2, \ldots, d_k$ with $d \neq 0$ such that

$$S(de_j) = n^{-1}Se_j = n^{-1}S(d_1 e_{j_1}) + \ldots + n^{-1}S(d_k e_{j_k}).$$

Let $A, B \in M_{nm}$ be such that $R(A) = \{0, d_1 e_{j_1}, \ldots, d_k e_{j_k}\}$ and $R(B) = \{0, S(d_1 e_{j_1}), \ldots, S(d_k e_{j_k})\}$. Fix $i = 1, 2, \ldots, m$. Since $(d_i e_{j_i})^{(n)} \prec A$, it follows that $T(d_1 e_{j_1})^{(n)} \prec TA$ and hence, in view of Corollary 2.2, the unique distinct row $S(d_i e_{j_i})$ of $T(d_i e_{j_i})^{(n)}$ is a convex combination of the rows of TA. Since $0 \prec A$ and $T0 = 0$, it follows that $0_{1\times m} \in \mathrm{co}R(TA)$ and hence, $B \prec TA$. Thus, $T(de_j)^{(n)} \prec B \prec TA$, and hence, $(de_j)^{n)} \prec A$. Therefore, de_j is a convex combination of $0, d_1 e_{j_1}, \ldots, d_k e_{j_k}$; a contradiction. Thus $k = m$ and $S\mathbb{R}_m$ contains m (full) lines with linearly independent directions. (See Step 1 and Corollary 2.2.) Since $S\mathbb{R}_m$ is convex, $S\mathbb{R}_m = \mathbb{R}_m$.

Step 3. *Claim:* S *is linear.*

The proof follows from (2.1). Note that $K = \mathbb{R}_m$ and hence $f = 0$ and $b > 0$. Also, $b^{-1}a = b^{-1}(\Psi(0) + a) = S0 = 0$. Thus $a = 0$ and $S = b^{-1}\Psi$.

Step 4. *Claim: There exists an invertible matrix* K *such that the mapping* $T_1 : M_{nm} \rightarrow M_{nm}$ *defined by* $T_1 X = (TX)K$ *satisfies the properties of* T *set in Definition 1.1 and Assumption 1.4. Moreover, if* S_1 *is the border operator corresponding to* T_1*, then* $S_1 = I$.

Let $\varphi_i = Se_i$, where $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{R}_m$. Since S is linear and injective, it follows that $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ is also a basis. Choose $K \in M_m$ such that $\varphi_i K = e_i$, for $i = 1, 2, \dots, m$. It is easy to see that the mapping $T_1 X = (TX)K$ satisfies the properties of T set in Definition 1.1 and Assumption 1.4. Moreover, $C(T_1 A) = T_1(C(A))$ for all $A \in M_{nm}$. Also, $T_1 e_i^{(n)} = (Te_i^{(n)})K = \varphi_i^{(n)} K = e_i^{(n)}$ for $i = 1, 2, \dots, m$. Now, if S_1 is the (linear) border operator corresponding to T_1, then $S_1 e_i = e_i$ for $i = 1, 2, \dots, n$ and hence, $S_1 = I$.

Step 5. *Claim: T is surjective.*

Let T_1 and S_1 be as in the previous step. Let $Y \in M_{nm}$ be arbitrary and choose $Z \in M_{nm}$ such that $R(Z) = \text{ext co } R(Y) = \{z_1, z_2, \dots, z_k\}$. Since $T_1 z_i^{(n)} = z_i^{(n)} \prec Z$, it follows that $z_i^{(n)} \prec T_1 Z$ and hence $z_i \in \text{co } R(T_1 Z)$, for $i = 1, 2, \dots, k$. Then $Y \prec T_1 Z$ and therefore, $Y = T_1 U$ for some $U \in M_{nm}$. This proves that T_1 and hence, T is surjective. $\square$

Corollary 2.5. *If $T : M_{nm} \to M_{nm}$ is a surjective strong preserver of $\prec$ for some $n \geq 2$, then S is linear.*

Proof. In view of Lemma 1.3, T satisfies the conditions of the above theorem. $\square$

Example 2.6. Let $T : M_{11} = \mathbb{R}_m \to M_{1m} = \mathbb{R}_m$ be any function. In this case $C(TA) = T(C(A)) = \{TA\}$ for all $A \in M_{1m}$. Also T defines a strong preserver of $\prec$ on M_{1m} if and only if it is injective. So we assume T is injective. (Now, T may or may not be surjective.) Thus T and the corresponding border operator S are the same and hence, they need not be linear. Thus, when T is not linear, it cannot be equivalent to a linear one.

Proposition 2.7. *Let $T : M_{nm} \to M_{nm}$ be a strong preserver of $\prec$ such that $T0 = 0$. Assume $K \in M_m$ is invertible. Define $T_1 : M_{nm} \to M_{nm}$ by $T_1 X = (TX)K$ for all $X \in M_{nm}$. Then T is linear if and only if T_1 is linear. Moreover, T is equivalent to a strong preserver $\tau : M_{nm} \to M_{nm}$ if and only if T_1 is equivalent to τ_1 defined by $\tau_1 X = [\tau(X)]K$ for all $X \in M_{nm}$.*

The simple proof of Proposition 2.7 is omitted.

Example 2.8. Let $m \geq 2$ and choose $A \in M_{nm}$ such that $|\text{ext co } R(A)| \geq 2$. Let $f : [A] \to [A]$ be an arbitrary bijective function satisfying

$f(A) \neq A$. Define $T : M_{nm} \to M_{nm}$ by $TX = f(X)$ for all $X \in [A]$, and $TX = X$, otherwise. Then T is a strong preserver of $\prec$, $T0 = 0$, and $Tx^{(n)} = x^{(n)}$ for all $x \in \mathbb{R}_m$. However, $T(2A) = 2A \neq 2TA$. That is, a strong preserver T of $\prec$ satisfying $T0 = 0$ need not be linear. The next theorem shows that any such T is equivalent to a linear one.

Theorem 2.9. *Let $T : M_{nm} \to M_{nm}$ be a strong preserver of $\prec$. Then $T - T0$ is equivalent to a linear strong preserver of $\prec$. In fact, there exists an invertible $K \in M_m$ such that $TX - T0 \sim XK^{-1}$ for all $X \in M_{nm}$.*

Proof. In view of Lemma 1.3(c), we assume without loss of generality that $T0 = 0$. Letting K and T_1 be as in Step 4 of the proof of Theorem 2.4, we have $T_1x^{(n)} = x^{(n)}$ for all $x \in \mathbb{R}_m$. By Lemma 2.1 and Proposition 2.7, $T_1X = X$ for all $X \in M_{nm}$ and hence, $TX \sim XK^{-1}$ for all $X \in M_{nm}$. □

The following corollary is due to Hasani-Radjabalipour [8]. So is the alternative proof given below which is based on the results of the present paper.

Corollary 2.10. *Let $m \geq 1$. For a linear strong preserver $T : M_{nm} \to M_{nm}$ there exist a permutation matrix $P \in \mathcal{P}(n)$ and an invertible matrix $L \in M_m$ such that $TX = PXL$ for all $X \in M_{nm}$.*

Proof. Assume without loss of generality that $m \geq 2$. Since T is assumed to be a linear mapping, it is clear that $T0 = 0$. Also, by letting K and T_1 to be as in Theorem 2.9, and replacing T by T_1, we can assume without loss of generality that $Tx^{(n)} = x^{(n)}$ for all $x \in \mathbb{R}_m$ or, equivalently, $S = I$.

Fix $j = 1, 2, \ldots, m$. For $i = 1, 2, \ldots, n$, let E_{ij} be the $n \times m$ matrix whose (r, s) entry is $\delta_{ir}\delta_{js}$, for $r = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n$. By Theorem 2.8, $R(TE_{ij}) = \{0, e_j\}$. Since $\Sigma_{i=1}^{n} E_{ij} = e_j^{(n)}$, it follows that, for each i, there exists an integer $\sigma(i)$ (depending on j too) such that $TE_{ij} = E_{\sigma(i)j}$ and, if $i \neq k$, then $\sigma(i) \neq \sigma(k)$. Thus, there exists an $n \times n$ permutation matrix P_j such that $TE_{ij} = P_jE_{ij}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

We claim $P_1 = P_2 = \ldots = P_m$. If not, then $P_r \neq P_s$ for some $r \neq s$. Let $\{e_1, \ldots, e_m\}$ and $\{\varphi_1, \ldots, \varphi_n\}$ be the standard bases for $\mathbb{R}_m$ and $\mathbb{R}_n$, respectively. Hence, there exists i such that $P_r\varphi_i \neq P_s\varphi_i$.

Let $A = E_{ir} + E_{is}$ and observe that $R(A) = \{0, e_r + e_s\}$. But $TA = TE_{ir} + TE_{is} = P_rE_{ir} + P_sE_{is}$ and hence, $R(TA) \cup \{0\} = \{0, e_r, e_s\}$. Thus $\{0, e_r + e_s\} = \{0, e_r, e_s\}$; a contradiction. Hence, $TE_{ij} = PE_{ij}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where $P = P_1 = \ldots = P_m$. Now, if $X = [x_{ij}] \in M_{nm}$, then

$$TX = T(\Sigma_{i,j}x_{ij}E_{ij}) = \Sigma_{i,j}x_{ij}TE_{ij} =$$
$$\Sigma_{i,j}x_{ij}PE_{ij} = P\Sigma_{i,j}x_{ij}E_{ij} = PX.$$

□

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