COMMON FIXED POINTS OF TWO COMMUTING MAPPINGS IN COMPLETE METRIC SPACES

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ABSTRACT. We improve Park and Bae's common fixed point theorem which is a generalization of Meir and Keeler's fixed point theorem. We extend Kannan's fixed point theorem to a common fixed point theorem of two commuting maps. Also, using the notion of biased mappings, we prove another common fixed point theorem.

1. **Introduction**

In 1981, Park and Bae proved the following fixed point theorem.

Theorem 1.1 (Park and Bae [8])**.** *Let (X, d) be a complete metric space. Let S and T be mappings on X satisfying the following condition.*

(a) *$T(X) \subset S(X)$;*
(b) *S is continuous;*
(c) *S and T commute;*

(d) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(Sx, Sy) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \varepsilon.$$

Then S and T have a unique common fixed point.

When S is the identity mapping on X, Theorem 1.1 becomes Meir and Keeler's fixed point theorem [7], which is one of the generalizations of the Banach contraction principle [1] (see also Jungck [2]). On the other hand, in 1969, Kannan [5] proved the following interesting fixed point theorem, which is not an extension of the Banach contraction principle.

Theorem 1.2 (Kannan [5]). *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there exists $\alpha \in [0, 1/2)$ such that*

$$d(Tx, Ty) \le \alpha\, d(Tx, x) + \alpha\, d(Ty, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Some generalizations of Meir and Keeler's and Kannan's fixed point theorem are proved. (For example, see [9, 12, 14, 15]). Lim [6] characterized Meir-Keeler contractions.

The main purpose of this paper is to improve Theorems 1.1 and 1.2. Also, we will prove a fixed point theorem, using the notion of biased mappings introduced by Jungck and Pathak [3].

2. Preliminaries

Throughout this paper we denote the set of all positive integers by $\mathbb{N}$. In 2001, Suzuki introduced the notion of τ-distance in order to generalize results in Kada, Suzuki and Takahashi [4], Tataru [16], Zhong [17, 18] and others.

Definition 2.1 ([10]). Let (X, d) be a metric space. A function p from $X \times X$ into $[0, \infty)$ is called a τ-distance on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ such that the following are satisfied:

(τ1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \ge t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable;

(τ3) $\lim_n x_n = x$ and $\lim_n \sup \{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w, x) \le \liminf_n p(w, x_n)$ for all $w \in X$;

$(\tau 4)$ $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

$(\tau 5)$ $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and results on τ-distance are stated in [4, 10, 11, 12, 13]. The following is Lemma 2 in [10].

Lemma 2.2 ([10]). *Let X be a metric space with a τ-distance p. Then $p(z, x) = 0$ and $p(z, y) = 0$ implies $x = y$.*

Let (X, d) be a metric space with a τ-distance p. Then a sequence $\{x_n\}$ in X is called *p-Cauchy* if there exist a function η from $X \times [0, \infty)$ into $[0, \infty)$ satisfying $(\tau 2) - (\tau 5)$ and a sequence $\{z_n\}$ in X such that $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$. The following is known.

Lemma 2.3 ([10]). *Let X be a metric space with a τ-distance p. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence in the usual sense.*

Lemma 2.4 ([10]). *Let (X, d) be a metric space with a τ-distance p. If a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover if a sequence $\{y_n\}$ in X satisfies $\lim_n p(x_n, y_n) = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

Remark 2.5. We note that $x = y$ does not necessarily imply $p(x, y) = 0$, and $p(x, y) = 0$ does not necessarily imply $x = y$.

3. Main results

In this section, we prove our main results. We first improve Theorem 1.1. The following theorem is also a generalization of a result in [12].

Theorem 3.1. *Let (X, d) be a complete metric space with a τ-distance p. Let S and T be mappings on X satisfying the following conditions:*

(a) $T(X) \subset S(X)$;

(b) *if $\{x_n\}$ is p-Cauchy and converges to $z \in X$, then $\{Sx_n\}$ converges to Sz;*

(c) *S and T commute;*

(d) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$p(Sx, Sy) < \varepsilon + \delta \quad implies \quad p(Tx, Ty) < \varepsilon.$$

Then S and T have a unique common fixed point $z \in X$ such that $p(z, z) = 0$.

Remark 3.2. Condition (b) is weaker than the continuity of S.

Proof. We first show that

$$p(Tx, Ty) \leq p(Sx, Sy) \text{ holds for all } x, y \in X. \tag{3.1}$$

By (d), for every $\varepsilon > 0$ with $p(Sx, Sy) < \varepsilon$, there exists $\delta_0 > 0$ such that $p(Su, Sv) < \varepsilon + \delta_0$ implies $p(Tu, Tv) < \varepsilon$. Since $p(Sx, Sy) < \varepsilon + \delta_0$, we have $p(Tx, Ty) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (3.1). By (a), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$. We first show

$$\lim_{n \to \infty} p(SI^n x, SI^n y) = 0,$$

for all $x, y \in X$. Since

$$p(SI^{n+1}x, SI^{n+1}y) = p(TI^n x, TI^n y) \leq p(SI^n x, SI^n y),$$

for all $n \in \mathbb{N}$, $\{p(SI^n x, SI^n y)\}$ converges to some nonnegative number t_1. If $t_1 > 0$, then there exists $\delta_1 > 0$ such that $p(Su, Sv) < t_1 + \delta_1$ implies $p(Tu, Tv) < t_1$. We choose $\nu_1 \in \mathbb{N}$ with $p(SI^{\nu_1}x, SI^{\nu_1}y) < t_1 + \delta_1$. Then we have

$$p(SI^{\nu_1+1}x, SI^{\nu_1+1}y) < t_1 \leq p(SI^{\nu_1+1}x, SI^{\nu_1+1}y),$$

which is a contradiction. Therefore $t_1 = 0$.

We next fix $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = I^n x_0$ for $n \in \mathbb{N}$. We note that

$$Sx_{n+1} = SIx_n = Tx_n,$$

for all $n \in \mathbb{N}$. We will show

$$\lim_{n \to \infty} \sup_{m > n} p(Sx_n, Sx_m) = 0.$$

Fix $\varepsilon > 0$. There exists $\delta_2 \in (0, \varepsilon)$ such that $p(Su, Sv) < \varepsilon + \delta_2$ implies $p(Tu, Tv) < \varepsilon$. For such δ_2, there exists $\nu_2 \in \mathbb{N}$ such that $p(Sx_n, Sx_{n+1}) < \delta_2$ for all $n \geq \nu_2$.

We assume $\limsup_n \sup_{m>n} p(Sx_n, Sx_m) > 2\varepsilon$. Then there exist $k, \ell \in \mathbb{N}$ such that $\nu_2 \leq k < \ell$ and

$$p(Sx_k, Sx_\ell) \leq \varepsilon + \delta_2 < p(Sx_k, Sx_{\ell+1})$$

holds. We have

$$p(Sx_k, Sx_{\ell+1}) \leq p(Sx_k, Sx_{k+1}) + p(Sx_{k+1}, Sx_{\ell+1}) \leq \delta_2 + \varepsilon,$$

which leads to a contradiction. Hence $\limsup_n \sup_{m>n} p(Sx_n, Sx_m) \leq 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_n \sup_{m>n} p(Sx_n, Sx_m) = 0$. By Lemma 2.4, $\{Sx_n\}$ is p-Cauchy. So, by Lemma 2.3, $\{Sx_n\}$ is a Cauchy sequence in the usual sense. Since X is complete, $\{Sx_n\}$ converges to some $z \in X$. Hence $\{Tx_n\}$ also converges to z because $Sx_n = Tx_{n-1}$. By the assumptions, $Sz = \lim_n STx_n$ holds. Since

$$\begin{aligned} p(Sx_n, STx_n) = p(Sx_n, TSx_n) &= p(Tx_{n-1}, TSx_n) \\ &\leq p(Sx_{n-1}, SSx_n) = p(Sx_{n-1}, STx_{n-1}), \end{aligned}$$

for $n \in \mathbb{N}$, $\{p(Sx_n, STx_n)\}$ converges to some nonnegative number t_2. If $t_2 > 0$, then there exists $\delta_3 > 0$ such that $p(Su, Sv) < t_2 + \delta_3$ implies $p(Tu, Tv) < t_2$. We choose $\nu_3 \in \mathbb{N}$ with $p(Sx_{\nu_3}, STx_{\nu_3}) < t_2 + \delta_3$. Then we have

$$p(Sx_{\nu_3+1}, TSx_{\nu_3+1}) = p(Tx_{\nu_3}, TTx_{\nu_3}) < t_2 \leq p(Sx_{\nu_3+1}, TSx_{\nu_3+1}),$$

which leads to a contradiction. Therefore $t_2 = 0$. Hence $\lim_n p(Sx_n, STx_n) = 0$. Again by Lemma 2.4, $\lim_n d(Sx_n, STx_n) = 0$. This implies $Sz = z$. Using this, we have

$$\begin{aligned} \limsup_{n\to\infty} p(Sx_n, Tz) &= \limsup_{n\to\infty} p(Tx_{n-1}, Tz) \\ &\leq \limsup_{n\to\infty} p(Sx_{n-1}, Sz) = \limsup_{n\to\infty} p(Sx_{n-1}, z) \\ &\leq \limsup_{n\to\infty} \liminf_{m\to\infty} p(Sx_{n-1}, Sx_m) \\ &\leq \lim_{n\to\infty} \sup_{m\geq n} p(Sx_{n-1}, Sx_m) = 0, \end{aligned}$$

which implies $\lim_n d(Sx_n, Tz) = 0$ and hence $Tz = z$, Therefore z is a common fixed point of S and T. Let y be a common fixed point of S

and T, and define a mapping J on X by

$$Jx = \begin{cases} Ix, & \text{if } x \neq y \text{ and } x \neq z, \\ x, & \text{if } x = y \text{ or } x = z. \end{cases}$$

Then we note that $SJx = Tx$ for all $x \in X$. So we obtain

$$p(z,y) = \lim_{n \to \infty} p(SJ^n z, SJ^n y) = 0.$$

Similarly we can prove $p(z,z) = 0$. So, by Lemma 2.2, we obtain $y = z$. This means that z is the unique common fixed point of S and T. This completes the proof. □

We next improve Theorem 1.2 as follows.

Theorem 3.3. *Let (X, d) be a complete metric space with a τ-distance p. Let S and T be mappings on X satisfying the following conditions:*

(a) *$T(X) \subset S(X)$;*
(b) *if $\{x_n\}$ is p-Cauchy and converges to $z \in X$, then $\{Sx_n\}$ converges to Sz;*
(c) *S and T commute;*
(d) *there exists $\alpha \in [0, 1/2)$ satisfying either of the following:*
    (A) *$p(Tx, Ty) \leq \alpha\, p(Tx, Sx) + \alpha\, p(Ty, Sy)$ for all $x, y \in X$;*
    (B) *$p(Tx, Ty) \leq \alpha\, p(Tx, Sx) + \alpha\, p(Sy, Ty)$ for all $x, y \in X$.*

Then S and T have a unique common fixed point $z \in X$ such that $p(z,z) = 0$.

Proof. By (a), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$. Fix $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = I^n x_0$ for $n \in \mathbb{N}$. We note that $Sx_{n+1} = SIx_n = Tx_n$ for all $n \in \mathbb{N}$. We first assume (A). Since

$$\begin{aligned} p(Sx_{n+2}, Sx_{n+1}) &= p(Tx_{n+1}, Tx_n) \\ &\leq \alpha\, p(Tx_{n+1}, Sx_{n+1}) + \alpha\, p(Tx_n, Sx_n) \\ &= \alpha\, p(Sx_{n+2}, Sx_{n+1}) + \alpha\, p(Sx_{n+1}, Sx_n), \end{aligned}$$

we have

$$p(Sx_{n+2}, Sx_{n+1}) \leq r\, p(Sx_{n+1}, Sx_n) \leq r^{n+1}\, p(Sx_1, Sx_0),$$

for $n \in \mathbb{N}$, where $r := \alpha/(1-\alpha) \in [0,1)$. Hence

$$
\begin{aligned}
\lim_{m,n\to\infty} p(Sx_n, Sx_m) &= \lim_{m,n\to\infty} p(Tx_{n-1}, Tx_{m-1}) \\
&\leq \lim_{m,n\to\infty} \left(\alpha\, p(Tx_{n-1}, Sx_{n-1}) + \alpha\, p(Tx_{m-1}, Sx_{m-1}) \right) \\
&\leq \lim_{m,n\to\infty} \alpha \left(r^{n-1} + r^{m-1} \right) p(Sx_1, Sx_0) \\
&= 0.
\end{aligned}
$$

By Lemma 2.4, $\{Sx_n\}$ is p-Cauchy. Hence, by Lemma 2.3, $\{Sx_n\}$ is a Cauchy sequence in the usual sense. Since X is complete, $\{Sx_n\}$ converges to some $z \in X$. By (b), $\{SSx_n\}$ converges to Sz. Also we have

$$
\begin{aligned}
p(SSx_{n+2}, SSx_{n+1}) &= p(STx_{n+1}, STx_n) = p(TSx_{n+1}, TSx_n) \\
&\leq \alpha\, p(TSx_{n+1}, SSx_{n+1}) + \alpha\, p(TSx_n, SSx_n) \\
&= \alpha\, p(STx_{n+1}, SSx_{n+1}) + \alpha\, p(STx_n, SSx_n) \\
&= \alpha\, p(SSx_{n+2}, SSx_{n+1}) + \alpha\, p(SSx_{n+1}, SSx_n).
\end{aligned}
$$

Thus,

$$
p(SSx_{n+1}, SSx_n) \leq r^n\, p(SSx_1, SSx_0),
$$

for every $n \in \mathbb{N}$. This implies $\lim_{m,n\to\infty} p(SSx_n, SSx_m) = 0$ and so $\{SSx_n\}$ is p-Cauchy. Since

$$
\begin{aligned}
\lim_{n\to\infty} p(Sx_n, z) &\leq \lim_{n\to\infty} \liminf_{m\to\infty} p(Sx_n, Sx_m) \\
&\leq \lim_{n\to\infty} \sup_{m>n} p(Sx_n, Sx_m) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} p(Sx_n, Sz) &\leq \lim_{n\to\infty} \liminf_{m\to\infty} p(Sx_n, SSx_m) \\
&\leq \lim_{n\to\infty} \sup_{m>n} p(Sx_n, SSx_m) \\
&= \lim_{n\to\infty} \sup_{m>n} p(Tx_{n-1}, TSx_{m-1}) \\
&\leq \lim_{n\to\infty} \sup_{m>n} \left(\alpha\, p(Tx_{n-1}, Sx_{n-1}) + \alpha\, p(TSx_{m-1}, SSx_{m-1}) \right) \\
&= 0,
\end{aligned}
$$

we obtain $Sz = z$, by Lemma 2.4. We also have

$$p(Tz, z) \leq \liminf_{n\to\infty} p(Tz, Tx_n)$$
$$\leq \liminf_{n\to\infty} \left(\alpha\, p(Tz, z) + \alpha\, p(Tx_n, Sx_n) \right)$$
$$= \alpha\, p(Tz, z),$$

which implies $p(Tz, z) = 0$. From this and

$$p(Tz, Tz) \leq 2\,\alpha\, p(Tz, Sz) = 2\,\alpha\, p(Tz, z) = 0,$$

we obtain $Tz = z$. Therefore, z is a common fixed point of S and T and $p(z, z) = 0$. Let y be a common fixed point of S and T. Then we can easily prove that $p(y, y) = 0$. We have

$$p(z, y) = p(Tz, Ty) \leq \alpha\, p(Tz, Sz) + \alpha\, p(Ty, Sy) = \alpha\, p(z, z) + \alpha\, p(y, y) = 0,$$

and hence $y = z$, by Lemma 2.2. Therefore the common fixed point z is unique. We next assume (B). We put

$$M = p(Sx_0, Sx_1) + p(Sx_1, Sx_0) + p(SSx_0, SSx_1) + p(SSx_1, SSx_0).$$

Since

$$p(Sx_{n+2}, Sx_{n+1}) = p(Tx_{n+1}, Tx_n)$$
$$\leq \alpha\, p(Tx_{n+1}, Sx_{n+1}) + \alpha\, p(Sx_n, Tx_n)$$
$$= \alpha\, p(Sx_{n+2}, Sx_{n+1}) + \alpha\, p(Sx_n, Sx_{n+1}),$$

and

$$p(Sx_{n+1}, Sx_{n+2}) = p(Tx_n, Tx_{n+1})$$
$$\leq \alpha\, p(Tx_n, Sx_n) + \alpha\, p(Sx_{n+1}, Tx_{n+1})$$
$$= \alpha\, p(Sx_{n+1}, Sx_n) + \alpha\, p(Sx_{n+1}, Sx_{n+2}),$$

we have

$$p(Sx_{n+2}, Sx_{n+1}) \leq r\, p(Sx_n, Sx_{n+1}),$$

and

$$p(Sx_{n+1}, Sx_{n+2}) \leq r\, p(Sx_{n+1}, Sx_n),$$

for $n \in \mathbb{N}$. Hence

$$p(Sx_{n+1}, Sx_n) + p(Sx_n, Sx_{n+1}) \leq r^n\, M,$$

for every $n \in \mathbb{N}$. We can similarly prove that

$$p(SSx_{n+1}, SSx_n) + p(SSx_n, SSx_{n+1}) \leq r^n\, M,$$

for every $n \in \mathbb{N}$. Using these, we can easily prove that

$$\lim_{m,n\to\infty} p(Sx_n, Sx_m) = \lim_{m,n\to\infty} p(SSx_n, SSx_m) = 0.$$

By Lemma 2.4, $\{Sx_n\}$ and $\{SSx_n\}$ are p-Cauchy. Hence by Lemma 2.3, $\{Sx_n\}$ is a Cauchy sequence in the usual sense. Since X is complete, $\{Sx_n\}$ converges to some $z \in X$. We can prove that such z is the unique common fixed point of S and T similar to the case (A). This completes the proof. □

4. Biased mappings

Jungck and Pathak [3] introduced the notion of biased mappings (Definition 4.1). In this section, we prove a fixed point theorem for such maps.

Definition 4.1 ([3])**.** Let (X, d) be a metric space and let S and T be mappings on X. Then the pair (S, T) is said to be S-biased if

$$\limsup_{n\to\infty} d(Sx_n, STx_n) \leq \limsup_{n\to\infty} d(Sx_n, TSx_n),$$

for every sequence $\{x_n\}$ such that $\{Sx_n\}$ and $\{Tx_n\}$ converge to some point $z \in X$.

Motivated by the above definition, we introduce the τ-distance version of biased mappings.

Definition 4.2. Let (X, d) be a metric space with a τ-distance p, and let S and T be mappings on X. Then the pair (S, T) is said to be (p, S)-biased if

$$\limsup_{n\to\infty} p(Sx_n, STx_n) \leq \limsup_{n\to\infty} p(Sx_n, TSx_n),$$

for every sequence $\{x_n\}$ such that $\{Sx_n\}$ and $\{Tx_n\}$ are p-Cauchy and converge to some point $z \in X$.

We note that a convergent sequence is not necessarily p-Cauchy. The following lemma is obvious.

Lemma 4.3. *If $ST = TS$, then (S, T) is (p, S)-biased.*

Lemma 4.4. *For every metric space X and mappings S, T on X, (S, T) is (d, S)-biased if and only if (S, T) is S-biased.*

Proof. We note that a sequence $\{x_n\}$ in X is d-Cauchy if and only if $\{x_n\}$ is a Cauchy sequence in the usual sense. □

Now we prove the following theorem.

Theorem 4.5. *Let (X, d) be a complete metric space with a τ-distance p. Let S and T be mappings on X satisfying the following conditions:*

(a) *$T(X) \subset S(X)$;*

(b) *if $\{x_n\}$ is p-Cauchy and converges to $z \in X$, then $\{Sx_n\}$ is also p-Cauchy and converges to Sz;*

(c) *(S, T) is (p, S)-biased;*

(d) *there exists a continuous function φ from $[0, \infty)$ into itself satisfying $\varphi(t) < t$ for all $t \in (0, \infty)$ and $p(Tx, Ty) \leq \varphi(p(Sx, Sy))$ for all $x, y \in X$.*

Then S and T have a unique common fixed point $z \in X$ such that $p(z, z) = 0$.

Remark 4.6. While Conditions (b) and (d) in this theorem are stronger than the corresponding conditions in of Theorem 3.1, Condition (c) in this theorem is weaker than the corresponding one in Theorem 3.1.

Proof. Since $\varphi(t) < t$ for all $t \in (0, \infty)$ and φ is continuous, we have $\varphi(0) = 0$. Thus $\varphi(t) \leq t$ for all $t \in [0, \infty)$. Considering a function $t \mapsto \max\{\varphi(s) : s \in [0, t]\}$, without loss of generality, we may assume that φ is nondecreasing. By (a), we can define a mapping I on X satisfying $SIx = Tx$ for all $x \in X$. We first show that

$$\lim_{n \to \infty} p(SI^n x, SI^n y) = 0,$$

for all $x, y \in X$. Since

$$p(SI^{n+1}x, SI^{n+1}y) = p(TI^n x, TI^n y)$$

$$\leq \varphi(p(SI^n x, SI^n y)) \leq p(SI^n x, SI^n y),$$

for all $n \in \mathbb{N}$, $\{p(SI^n x, SI^n y)\}$ converges to some nonnegative number t_1. If $t_1 > 0$, then as n tends to ∞, we have

$$t_1 = \lim_{n \to \infty} p(SI^{n+1}x, SI^{n+1}y) \leq \lim_{n \to \infty} \varphi(p(SI^n x, SI^n y)) = \varphi(t_1),$$

which leads to a contradiction. Therefore $t_1 = 0$. We next fix $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = I^n x_0$ for $n \in \mathbb{N}$. We note that

$Sx_{n+1} = SIx_n = Tx_n$ for all $n \in \mathbb{N}$. We will show that

$$\lim_{m,n\to\infty} p(Sx_n, Sx_m) = 0.$$

Fix $\varepsilon > 0$. There exists $\nu_1 \in \mathbb{N}$ such that

$$\max \{p(Sx_n, Sx_n), \, p(Sx_n, Sx_{n+1}), \, p(Sx_{n+1}, Sx_n)\} < \varepsilon - \varphi(\varepsilon),$$

for all $n \geq \nu_1$. We assume that $\limsup_{m,n} p(Sx_n, Sx_m) > \varepsilon$. Then there exist $k, \ell \in \mathbb{N}$ such that $k \geq \nu_1$, $\ell \geq \nu_1$ and either of the following holds

(i) $p(Sx_k, Sx_\ell) \leq \varepsilon < p(Sx_k, Sx_{\ell+1})$;
(ii) $p(Sx_k, Sx_\ell) \leq \varepsilon < p(Sx_{k+1}, Sx_\ell)$.

In the first case, we have

$$\begin{aligned}
p(Sx_k, Sx_{\ell+1}) &\leq p(Sx_k, Sx_{k+1}) + p(Sx_{k+1}, Sx_{\ell+1}) \\
&\leq p(Sx_k, Sx_{k+1}) + \varphi(p(Sx_k, Sx_\ell)) \\
&< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon,
\end{aligned}$$

which is a contradiction. In the second case, we have

$$\begin{aligned}
p(Sx_{k+1}, Sx_\ell) &\leq p(Sx_{k+1}, Sx_{\ell+1}) + p(Sx_{\ell+1}, Sx_\ell) \\
&\leq \varphi(p(Sx_k, Sx_\ell)) + p(Sx_{\ell+1}, Sx_\ell) \\
&< \varphi(\varepsilon) + \varepsilon - \varphi(\varepsilon) = \varepsilon,
\end{aligned}$$

which is again a contradiction. Hence $\limsup_{m,n} p(Sx_n, Sx_m) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{m,n} p(Sx_n, Sx_m) = 0$. By Lemma 2.4, $\{Sx_n\}$ is p-Cauchy. Hence, by Lemma 2.3, $\{Sx_n\}$ is a Cauchy sequence in the usual sense. Since X is complete, $\{Sx_n\}$ converges to some $z \in X$. Hence $\{Tx_n\}$ also converges to z because $Sx_n = Tx_{n-1}$. By the assumptions, $\{STx_n\}$ is p-Cauchy and $Sz = \lim_n STx_n$. Since $\{STx_n\}$ is p-Cauchy, there exist $\nu_2 \in \mathbb{N}$ and $v \in X$ such that

$$\sup \{p(v, STx_n) : \, n \geq \nu_2\} \leq 1.$$

Since $\lim_{m,n} p(Sx_n, Sx_m) = 0$, there exists $\nu_3 \geq \nu_2$ such that $p(Sx_n, Sx_{\nu_3}) \leq 1$, for all $n \in \mathbb{N}$ with $n \geq \nu_3$. For each $n \geq \nu_3$ we have

$$\begin{aligned}
&\sup \{p(Sx_n, STx_n) : \, n \geq \nu_3\} \\
&\leq \sup \{p(Sx_n, Sx_{\nu_3}) + p(Sx_{\nu_3}, v) + p(v, STx_n) : \, n \geq \nu_3\} \\
&\leq 2 + p(Sx_{\nu_3}, v) < \infty.
\end{aligned}$$

Hence $\{p(Sx_n, STx_n)\}$ is bounded. We assume $t_2 := \limsup_n p(Sx_n, STx_n) > 0$. Then we have

$$
\begin{aligned}
t_2 &= \limsup_{n\to\infty} p(Sx_n, STx_n) \\
&\leq \limsup_{n\to\infty} p(Sx_n, TSx_n) = \limsup_{n\to\infty} p(Tx_{n-1}, TSx_n) \\
&\leq \limsup_{n\to\infty} \varphi\big(p(Sx_{n-1}, SSx_n)\big) = \limsup_{n\to\infty} \varphi\big(p(Sx_{n-1}, STx_{n-1})\big) \\
&= \varphi(t_2) < t_2,
\end{aligned}
$$

which is a contradiction. Hence $\lim_n p(Sx_n, STx_n) = 0$. Again by Lemma 2.4, $\lim_n d(Sx_n, STx_n) = 0$. This implies $Sz = z$. Using this, we have

$$
\begin{aligned}
\limsup_{n\to\infty} p(Sx_n, Tz) &= \limsup_{n\to\infty} p(Tx_{n-1}, Tz) \\
&\leq \limsup_{n\to\infty} p(Sx_{n-1}, Sz) = \limsup_{n\to\infty} p(Sx_{n-1}, z) \\
&\leq \limsup_{n\to\infty} \liminf_{m\to\infty} p(Sx_{n-1}, Sx_m) \\
&\leq \lim_{n\to\infty} \sup_{m\geq n} p(Sx_{n-1}, Sx_m) = 0,
\end{aligned}
$$

which implies $\lim_n d(Sx_n, Tz) = 0$ and hence $Tz = z$. Therefore z is a common fixed point of S and T. We can prove uniqueness of z similar to the proof of this assertion in Theorem 3.1. $\qquad\square$

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