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RESTRICTING THE PARAMETER SET OF THE PASCOLETTI-SERAFINI SCALARIZATION

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(Communicated by Nezam Mahdavi-Amiri)

Abstract. A common approach to determine efficient solutions of a multiple objective optimization problem is reformulating it to a parameter dependent scalar optimization problem. This reformulation is called scalarization approach. Here, a well-known scalarization approach named Pascoletti-Serafini scalarization is considered. First, some difficulties of this scalarization are discussed and then removed by restricting the parameter set.
A method is presented to convert a space ordered by a specific ordering cone to an equivalent space ordered by the natural ordering cone. Utilizing the presented conversion, all confirmed results and theorems for multiple objective optimization problems ordered by the natural ordering cone can be extended to multiple objective optimization problems ordered by specific ordering cones.
Keywords: Multiple objective optimization, Pascoletti-Serafini scalarization, ordering cone, parameter set restriction, convexification.
MSC(2010): Primary: 90C29; Secondary: 90C90.

1. **Introduction**

There are various research works on approximating efficient solutions of multiple objective optimization problems (MOPs); e.g., see [2, 4, 6, 7, 9, 12, 14, 20–22, 25]. A common approach to determine efficient solutions of a MOP is reformulating it to a parameter dependent scalar optimization problem. This reformulation is called scalarization. By solving the scalar optimization problem for a variety of parameters, several solutions of the MOP are generated. The most interesting scalarizations are those that by varying their parameters, all efficient solutions of a general MOP can be obtained. For example, we can refer to the well-known Pascoletti-Serafini scalarization [22] method. The Pascoletti-Serafini scalarization [22] is termed variously in the references [9–11, 19, 24, 26], but its application to multiple objective optimization was first given in [22].

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Despite popularity, the Pascoletti-Serafini scalarization has some difficulties as mentioned below:
1. Its parameters are selected from an unbounded set.
2. For some parameters, it is possible that the scalar problem becomes unbounded from below. In practical situations, the user could pick a finite number of points as parameters, corresponding to many (or all) of which the scalar problem becomes unbounded from below.
3. The parameter selection is random and does not appear to have any particular regularity.

Now, the question is whether we can eliminate the mentioned difficulties by restricting the parameter set of this Pascoletti-Serafini scalarization and still obtain all efficient solutions of the MOP? Here, it is shown that the answer to this question is "positive" for MOPs being bounded from below (have utopia points).

The Pascoletti-Serafini scalarization [22] has two parameters which by varying them in $\mathbb{R}^p$, all efficient solutions of a general MOP can be obtained. We show that the parameter set of the scalarization can be restricted to a bounded subset and all efficient solutions of MOPs that have utopia points can still be obtained. This bounded subset is constant for all MOPs and does not depend on the structure of different MOPs. Also, for any parameter of this subset the Pascoletti-Serafini scalarization always is bounded from below.

Some works on restricting the parameter set of the Pascoletti-Serafini scalarization are [9] and [17]. The parameter set restriction method given in [17] is proposed for MOPs with natural ordering cones, and the proposed method in [9] has some difficulties to be addressed in our work. The method proposed here is an extension of the method given in [17] for MOPs with closed pointed ordering cones.

There are several effective methods to approximate efficient solutions of MOPs ordered by the natural ordering cone. We can refer to the ε-constraint method [13], the improved ε-constraint method [8], the weighted Tchebycheff method [15], the free disposal outer approximation method [12], the weighted sum method and the methods reviewed in [4, 6, 16]. Also, in [18] a method is presented to convexify the Pareto front of MOPs ordered by the natural ordering cone with certain properties.

These well-known approaches are not able to determine efficient solutions of MOPs ordered by unnatural ordering cones. Considering that the unnatural ordering cones appear in many practical multiple objective optimization problems [1, 5, 23], generalizations of the mentioned methods to approximate efficient solutions of MOPs ordered by unnatural ordering cones may turn to be worthwhile from both theoretical and practical points of view. To this end, we present a method to convert a space ordered by a specific ordering cone to an equivalent space ordered by the natural ordering cone. The equivalence between the original space and the converted space means that the order between two points in the original space is the same as the order between their corresponding points in the converted space.

The rest of the paper is organized as follows. In Section 2, some preliminaries and basic definitions are provided. In Section 3, the Pascoletti-Serafini scalarization [22] and the parameter set restriction method of [9] are reviewed. In Section 4, a new parameter set restriction method for the Pascoletti-Serafini scalarization is presented. Some numerical results are given in Section 5. In Section 6, a method to convert a space ordered by a specific ordering cone to an equivalent space ordered by the natural ordering cone is presented. Conclusions are given in Section 7.

2. Preliminaries and basic definitions

Consider a multiple objective optimization problem as follows:

$$\boldsymbol{MOP} : \min_{x\in X} f(x) = \Big(f_1(x), f_2(x), ..., f_p(x)\Big), \tag{2.1}$$

where $X \subset \mathbb{R}^n$ is a nonempty set and f is a vector-valued function composed of p ($p \geq 2$) real-valued functions. The image of X under f is denoted by $Y := f(X) \subseteq \mathbb{R}^p$ and referred to as the image space.

Definition 2.1. A nonempty set $\kappa \subseteq \mathbb{R}^p$ is called a cone, if $\lambda\kappa \subseteq \kappa$ for all $\lambda \geq 0$.

Definition 2.2. A cone κ is called pointed if $\kappa \cap (-\kappa) = \{0\}$.

Definition 2.3. Let κ be a convex cone (ordering cone). We say that the space $\mathbb{R}^p$ is ordered by the ordering cone κ, if we use the notation " $\preceq_\kappa$ " to compare the points in $\mathbb{R}^p$. For $x, y \in \mathbb{R}^p$, the notation $x \preceq_\kappa y$ means $y - x \in \kappa$.

Remark 2.4. In the rest of the paper, it is assumed that the ordering cone κ is closed, pointed and $int(\kappa) \neq \emptyset$.

Considering a pointed ordering cone $\kappa \subseteq \mathbb{R}^p$, efficient solutions of MOP (2.1) is defined as follows.

Definition 2.5. A feasible solution $\hat{x} \in X$ is called an efficient solution of (2.1) with respect to the ordering cone κ, if there is no $x \in X$ such that $f(x) \preceq_\kappa f(\hat{x})$ and $f(x) \neq f(\hat{x})$ $\Big($equivalently, $(f(\hat{x}) - \kappa) \cap f(X) = \{f(\hat{x})\}\Big)$.

The set of all efficient solutions of MOP (2.1) with respect to the ordering cone κ is denoted by $X_{E\kappa}$ and $f(X_{E\kappa})$ is called nondominated points with respect to κ and is denoted by $Y_{N\kappa}$.
Considering the natural ordering cone

$$\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p \mid y_i \geq 0,\ i = 1, ..., p\},$$

the set $Y_{N\mathbb{R}^p_{\geqq}}$ is called the Pareto front.

Definition 2.6. The point $a^* \in \mathbb{R}^p$ is called the ideal point of (2.1) with respect to the ordering cone κ, if

$$a^* \preceq_\kappa f(x),\ \forall\ x \in X,$$

and there is no $a \neq a^*$ such that

$$a \preceq_\kappa f(x)\ \forall\ x \in X,\ \text{and}\ a^* \preceq_\kappa a.$$

Remark 2.7. Note that for the ordering cone $\kappa = \mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p \mid y_i \geq 0,\ i = 1, ..., p\}$, we have

$$a^* = (f_1^*, ..., f_p^*)^T,$$

where $f_i^* = \inf_{x \in X} f_i(x)$, for $i = 1, ..., p$.

Definition 2.8. Let $\alpha \in int(\kappa)$ and a^* be the ideal point. The point a^{**} with $a^{**} = a^* - \alpha$ is called the utopia point (with respect to α).

3. A parameter set restriction and its difficulties

The scalar problem of the Pascoletti-Serafini scalarization [22], which from now on is denoted by $P(a, r)$, has two parameters a and r that by varying them in $\mathbb{R}^p$, all efficient solutions of (2.1) with respect to the pointed ordering cone κ can be generated [9]. The formulation of this scalar problem is as follows:

$$\begin{aligned}
&\min t \\
&s.t. \\
&a + tr - f(x) \in \kappa, \ \ \big(P(a, r)\big) \\
&x \in X,\ t \in \mathbb{R}.
\end{aligned}$$

In order to solve $P(a, r)$, the ordering cone $-\kappa$ is moved in the direction r (or $-r$) on the line $a + tr$ starting at the point a till the set $(a + tr - \kappa) \cap f(X)$ is reduced to the empty set. The smallest value of $\bar{t}$ for which $(a+\bar{t}r-\kappa)\cap f(X) \neq \emptyset$ is the optimal value of $P(a, r)$ [9]. Note that, it is possible for some parameters $a, r \in \mathbb{R}^p$, the problem $P(a, r)$ becomes unbounded from below [9].
In the following, Section b of Theorem 2.1 in [9] is given.

Theorem 3.1. *Let $\hat{x}$ be an efficient solution of MOP with respect to the pointed ordering cone κ, then $(0, \hat{x})$ is an optimal solution of $P(a, r)$, for the parameters $a = f(\hat{x})$ and an arbitrary $r \in \kappa \backslash \{0\}$.*

As a result of Theorem 3.1, we can find all efficient solutions of (2.1) for a constant parameter $r \in \kappa\backslash\{0\}$, by varying the parameter $a \in \mathbb{R}^p$ only. In [9], a method is used to limit the choices of a from $\mathbb{R}^p$, and still obtaining all efficient solutions of some MOPs. The procedure of that method is different for two objectives and more than two objectives optimization problems. For the case $p = 2$, it is assumed that the ordering cone κ is closed and pointed.

For example, if $\kappa = \mathbb{R}^2_{\geqq}$, then the proposed method works as follows.
First, the parameter $r \in \mathbb{R}^2\backslash\{0\}$ and the hyperplane $H := \{y \in \mathbb{R}^2 \mid b_1y_1 + b_2y_2 = \beta\}$, with $b^Tr \neq 0$ and $\beta \in \mathbb{R}$, are considered. Then, the points $\bar{x}_1 = \arg\min_{x\in X} f_1(x)$ and $\bar{x}_2 = \arg\min_{x\in X} f_2(x)$ are determined. Afterwards, the points $f(\bar{x}_1)$ and $f(\bar{x}_2)$ are projected in the direction r onto the line H. The projection points $\bar{a}_1 \in H$ and $\bar{a}_2 \in H$ are given by

$$\bar{a}_i := f(\bar{x}_i) - r\frac{b^T f(\bar{x}_i) - \beta}{b^T r},\ i = 1, 2.$$

Theorem 2.17 of [9] shows that it is sufficient to consider the parameters $a \in H^a := \{y \in H \mid y = \lambda\bar{a}_1 + (1-\lambda)\bar{a}_2,\ \lambda \in [0,1]\}$ to approximate the whole efficient set.
For three and more than three objectives optimization problems, the above approach does not work correctly [9]. So, a weaker restriction for the parameter a was proposed in [9] by projecting the image space $f(X)$ in the direction r onto the set H. Thus, the set $\tilde{H} := \{y \in H \mid y + tr = f(x),\ t \in \mathbb{R},\ x \in X\} \subset H$ is determined. The set $\tilde{H} \subset H$ has in general an irregular boundary and is therefore not suitable for a systematic procedure. Hence, the set $\tilde{H}$ is embedded in a $(p-1)$-dimensional cuboid $H^0 \subset \mathbb{R}^p$, which is chosen as optimal as possible. A method for obtaining H^0 is given in [9].

For two objectives optimization problems, the method given in [9] has the following difficulties.
1- If at least one of the problems $\min_{x\in X} f_1(x)$ or $\min_{x\in X} f_2(x)$ has alternative solutions, it is possible that the set H^a becomes unbounded, as illustrated by Example 3.2 below.

Example 3.2. Consider the following bi-objective optimization problem.

$$\min_{x\in\mathbb{R}} \big(f_1(x), f_2(x)\big),$$

where

$$\Big(f_1(x), f_2(x)\Big) = \begin{cases} (1, 4-x), & x \leq 1, \\ (x, 4-x), & 1 \leq x \leq 3, \\ (x, 1), & x \geq 3. \end{cases}$$

The image space of this problem is depicted in Figure 1.
Now, consider the hyperplane $H := \{y \in \mathbb{R}^2 \mid y_1 + y_2 = 0\}$ and $r = (1,1)^T$. As can be seen, the problems $\min_{x\in\mathbb{R}} f_1(x)$ and $\min_{x\in\mathbb{R}} f_2(x)$ have alternative solutions and

$$\arg\min_{x\in\mathbb{R}} f_1(x) = \bar{x}_1 \in (-\infty, 1],$$
$$\arg\min_{x\in\mathbb{R}} f_2(x) = \bar{x}_2 \in [3, +\infty),$$

and if $\bar{x}_1 \longrightarrow -\infty$ and $\bar{x}_2 \longrightarrow +\infty$, then $f_2(\bar{x}_1) \longrightarrow +\infty$ and $f_1(\bar{x}_2) \longrightarrow +\infty$, and this causes that $H^a \longrightarrow H$, and therefore H^a becomes unbounded.
2. For a constant set H, the set H^a is not constant for different MOPs and it

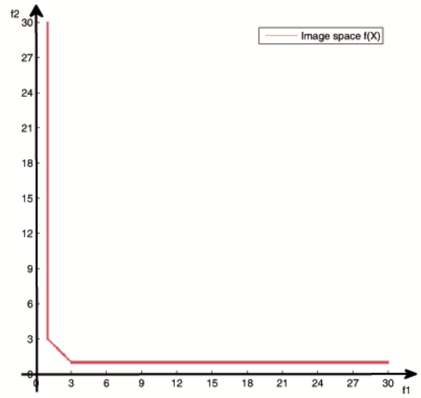

FIGURE 1. The image space of Example 3.2.

is related to the problem under consideration.
3. It is possible that the points $\bar{x}_1$ and $\bar{x}_2$ do not exist, even if the ideal point exists, as illustrated by Example 3.3.

Example 3.3. Consider the following bi-objective optimization problem:

$$\min_{x\in(0,+\infty)} (x, \frac{1}{x}).$$

As shown in Figure 2, the points $\bar{x}_1$ and $\bar{x}_2$ do not exist and the point $(0, 0)$ is the ideal point for the problem.

For three and more than three objectives optimization problems, the method given in [9] has the following difficulties.
1. As discussed in [9], the method might be hard to verify in practice, because obtaining the set H^0 is difficult.
2. For a constant set H, the set H^0 is not constant for different MOPs and is related to the problem under consideration.
3. When a considered MOP is not bounded from above, according to the definition of $\tilde{H}$, it is possible that $\tilde{H}$ and therefore H^0 become unbounded, even if the MOP is bounded from below (the utopia point exists), as illustrated by Example 3.4.

Example 3.4. Consider the following three-objective optimization problem:

$$\min_{x\in[0,+\infty)} (\frac{\sqrt{2}}{2}x, \frac{\sqrt{2}}{2}x, x^2).$$

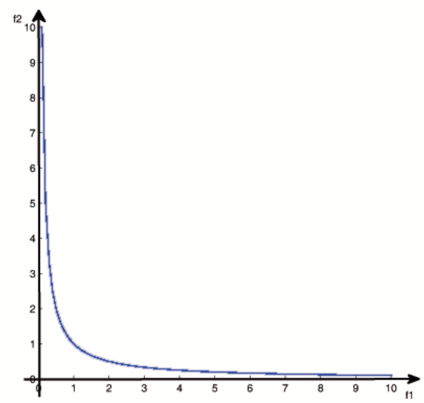

Figure 2. The image space of Example 3.3.

Considering the hyperplane $H := \{y \in \mathbb{R}^3 \mid y_3 = 0\}$ and $r = (0, 0, 1)$, we have

$$\tilde{H} = H^0 := \Big\{(\frac{\sqrt{2}}{2}x, \frac{\sqrt{2}}{2}x, 0) \mid x \in [0, +\infty)\Big\},$$

and so the set H^0 is not bounded.

4. Parameter set restriction: a new method

Here, a new parameter set restriction method for $P(a, r)$ is proposed.
Let a^{**} be the utopia point of (2.1) with respect to the ordering cone κ. If the utopia point is shifted to the origin, then the image space $f(X)$ will be in the interior of the ordering cone κ. In Figure 3, this is shown for a two objective optimization problem with $\kappa = \mathbb{R}^2_{\geqq}$.

In the rest of our work it is assumed that the utopia point exists and it is shifted to the origin. So, the image space $f(X)$ is in the interior of the ordering cone κ.
Now, consider the set R which is defined as follows:

$$R := \{\beta \in \mathbb{R}^p \mid \|\beta\|_2 = 1\} \cap int(\kappa),$$

where $\|.\|_2$ is the Euclidean norm.
A strategy to generate an even spread of combination vectors β has been proposed in [4] that is related to a nonnegative real parameter named δ. This even spread of β generates even spaced points on the set R.

In Theorem 4.1, below it is shown that by considering $a = 0$ and varying $r \in R$ in $P(a, r)$, all efficient solutions of (2.1) with respect to the ordering cone κ can be obtained.

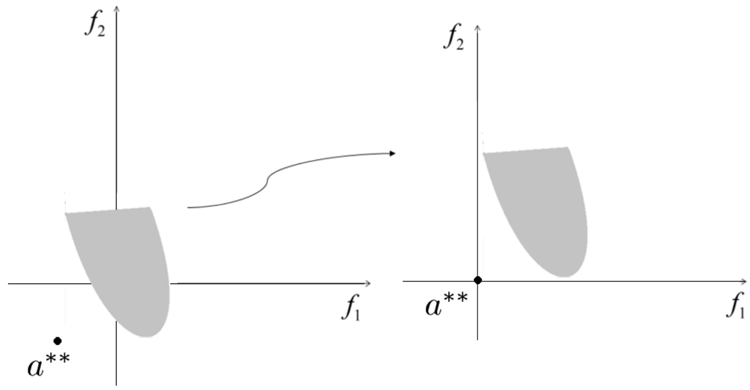

FIGURE 3. The utopia point is shifted to the origin.

Theorem 4.1. *Let $a = 0$, $R := \{\beta \in \mathbb{R}^p \mid \|\beta\|_2 = 1\} \cap int(\kappa)$ and $\bar{x} \in X$ be an efficient solution of MOP with respect to the ordering cone κ. Then, there are $r \in R$ and $\bar{t} \in \mathbb{R}$ such that $(\bar{t}, \bar{x})$ is an optimal solution of $P(a, r)$.*

Proof. Since $f(X) \subset int(\kappa)$, we have $f(\bar{x}) \neq 0$. Also, it is obvious that $\frac{f(\bar{x})}{\|f(\bar{x})\|_2} \in R$. Considering $r = \frac{f(\bar{x})}{\|f(\bar{x})\|_2} \in R$, $\bar{t} = \|f(\bar{x})\|_2$ and $a = 0$, we have

$$a + \bar{t}r - f(\bar{x}) = 0 + \|f(\bar{x})\|_2 \frac{f(\bar{x})}{\|f(\bar{x})\|_2} - f(\bar{x}) = 0 \in \kappa.$$

So, the point $(\bar{t}, \bar{x})$ is feasible for $P(a, r)$. we claim that the point $(\bar{t}, \bar{x})$ is also an optimal solution of $P(a, r)$, since otherwise, there exist $\hat{t} \in \mathbb{R}$ and $\hat{x} \in X$ such that $(\hat{t}, \hat{x})$ is feasible for $P(a, r)$ and $\hat{t} < \bar{t}$. From the feasibility of $(\hat{t}, \hat{x})$ for $P(a, r)$, we have

$$a + \hat{t}r - f(\hat{x}) = 0 + \hat{t}\frac{f(\bar{x})}{\|f(\bar{x})\|_2} - f(\hat{x}) = 0 + \frac{\hat{t}}{\bar{t}} f(\bar{x}) - f(\hat{x}) \in \kappa. \tag{4.1}$$

Since $\hat{t} < \bar{t}$ and $\bar{t} > 0$, we have $\frac{\hat{t}}{\bar{t}} < 1$. So, from (4.1), we get

$$f(\hat{x}) \preceq_\kappa \frac{\hat{t}}{\bar{t}} f(\bar{x}) \prec_\kappa f(\bar{x}). \tag{4.2}$$

Hence, $f(\hat{x}) \prec_\kappa f(\bar{x})$, and this is a contradiction to $\bar{x}$ being an efficient solution of MOP with respect to the ordering cone κ. □

Next, Theorem 4.2 shows that the problem $P(a, r)$ always is bounded from below for $a = 0$ and $r \in R$.

Theorem 4.2. *For $a = 0$ and $r \in R := \{\beta \in \mathbb{R}^p \mid \|\beta\|_2 = 1\} \cap int(\kappa)$, the problem $P(a, r)$ always is bounded from below. Also, if $f(X)$ is closed, then $P(a, r)$ has an optimal solution.*

Proof. Let $a = 0$ and suppose that $r \in R$ is given. Since $f(X) \subset int(\kappa)$,

$$a + tr - f(x) \in \kappa \Longrightarrow tr \in f(x) + \kappa \Longrightarrow tr \in int(\kappa),$$

and since $R \subset int(\kappa)$, the problem $P(a, r)$ is infeasible for $t \leq 0$. So, $P(a, r)$ is bounded from below.
Now, consider an arbitrary feasible solution $\bar{x} \in X$ and form the set $f(\bar{x}) + \kappa$. Consider $r \in R$ and a line with slope r passing through the origin. Since $f(\bar{x}) + \kappa \subset int(\kappa)$ and $r \in int(\kappa)$, the considered line breaks the set $f(\bar{x}) + \kappa$. Let $\hat{y}$ be the intersection point. So, $\hat{y} \in f(\bar{x}) + \kappa$ or equivalently $f(\bar{x}) \in \hat{y} - \kappa$, and also $\hat{y} = \|\hat{y}\| r$. Thus,

$$f(\bar{x}) \in (0 + \|\hat{y}\| r - \kappa) \cap f(X),$$

and considering $\hat{t} = \|\hat{y}\|$ and $a = 0$, we have

$$f(\bar{x}) \in (a + \hat{t} r - \kappa) \cap f(X).$$

So, $(\hat{t}, \bar{x})$ is feasible for $P(a, r)$ and $\hat{t}$ is an upper bound for the optimal value of $P(a, r)$. Therefore, if $f(X)$ is closed, then $P(a, r)$ has an optimal solution. □

The advantages of the new parameter set restriction are the followings.
1. The parameter a is fixed and is equal to zero. The parameter r is changed in R, which is a constant bounded subset of $\mathbb{R}^p$, and is not related to the problem under consideration.
2. For parameters a and $r \in R$, the scalar problem $P(a, r)$ is always bounded from below.

5. **Numerical results**

Here, the nondominated points of a two bi-objective and a three-objective optimization problems are approximated based on the restriction method given in Section 4.

Example 5.1. Consider the following bi-objective optimization problem with respect to the ordering cone $\mathbb{R}^2_{\geqq}$:

$$\min_{x \in \mathbb{R}} \Big(f_1(x), f_2(x) \Big),$$

where

$$\Big(f_1(x), f_2(x) \Big) = \begin{cases} (1, 4 - x), & x \leq 1, \\ (x, 4 - x), & 1 \leq x \leq 3, \\ (x, 1), & x \geq 3. \end{cases}$$

The image space of the problem is depicted in Figure 4. The set of nondominated points of the problem is $\Big\{ \big(f_1(x), f_2(x) \big) \mid x \in [1, 3] \Big\}$. The utopia point for the problem is $(0.9, 0.9)^T$. After shifting the utopia point to the origin, the set $R := \{\beta \in \mathbb{R}^2 \mid \|\beta\|_2 = 1\} \cap int(\mathbb{R}^2_{\geqq})$ is constructed for $\delta = \frac{1}{20}$, and the

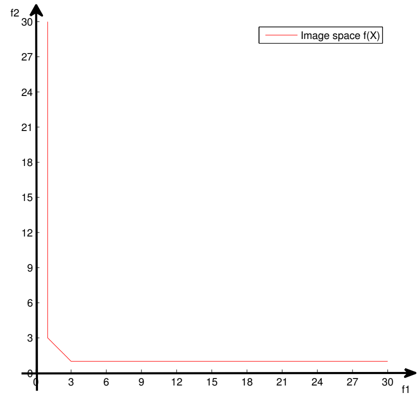

FIGURE 4. The image space of Example 5.1.

problem $P(a, r)$ is solved for $a = 0$ and $r \in R$. The obtained approximation to the set of nondominated points is depicted in Figure 5 using the symbol $*$.

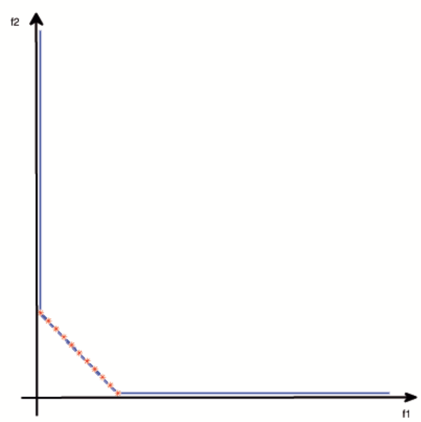

FIGURE 5. The approximated nondominated points for Example 5.1.

Example 5.2. Consider the following three-objective optimization problem with respect to the ordering cone $\mathbb{R}^3_{\geqq}$:

$$\begin{aligned}&\min\Big(f_1(x), f_2(x), f_3(x)\Big)\\ &s.t.\\ &0 \leq x_1, x_2 \leq 1,\end{aligned}$$

where

$$f_1(x) = cos(\tfrac{x_1\pi}{2})cos(\tfrac{x_2\pi}{2}),$$

$$f_2(x) = cos(\tfrac{x_1\pi}{2})sin(\tfrac{x_2\pi}{2}),$$

$$f_3(x) = sin(\tfrac{x_1\pi}{2}).$$

The image space of the problem is depicted in Figure 6. The utopia point for

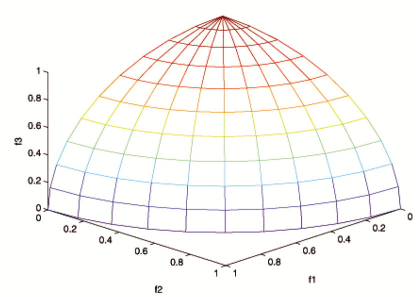

FIGURE 6. The image space of Example 5.2.

the problem is $(-0.1, -0.1, -0.1)^T$. After shifting the utopia point to the origin, the set $R := \{\beta \in \mathbb{R}^3 \mid \|\beta\|_2 = 1\} \cap int(\mathbb{R}^3_{\geqq})$ is constructed for $\delta = \frac{1}{20}$, and the problem $P(a, r)$ is solved for $a = 0$ and $r \in R$. The obtained approximation to the set of nondominated points is depicted in Figure 7 using the symbol $*$.

Example 5.3. Consider the following bi-objective optimization problem:

$$\min_{x\in[-3,3]} \Big(f_1(x), f_2(x)\Big),$$

with respect to the ordering cone $C = \{y \in \mathbb{R}^2 \mid y_2 - \frac{1}{2}y_1 \geq 0,\ 2y_1 - y_2 \geq 0\}$, where

$$\Big(f_1(x), f_2(x)\Big) = \begin{cases} (0, -x), & x \in [-3, 0],\\ (x, 0), & x \in [0, 3].\end{cases}$$

The image space of the problem and the ordering cone C are depicted in Figures 8 and 9, respectively.

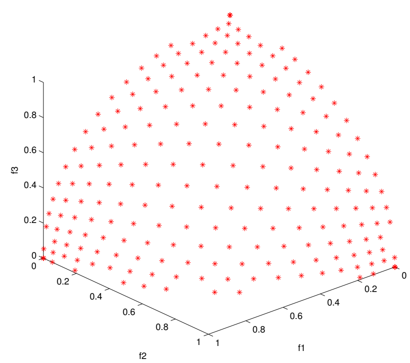

FIGURE 7. The approximated nondominated points for Example 5.2.

The set of nondominated points for the problem is equal to the image space. The utopia point for the problem with respect to the ordering cone C is the point $(-3.1, -3.1)^T$. After shifting the utopia point to the origin, the set $R = \{\beta \in \mathbb{R}^2 \mid \|\beta\|_2 = 1\} \cap int(C)$ is constructed for $\delta = \frac{1}{100}$, and the problem $P(a, r)$ is solved for $a = 0$ and $r \in R$. The obtained approximation to the set of nondominated points is depicted in Figure 10 using the symbol $*$.

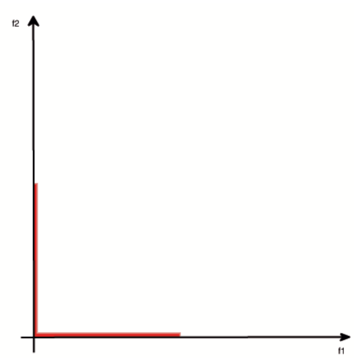

FIGURE 8. The image space of Example 5.3.

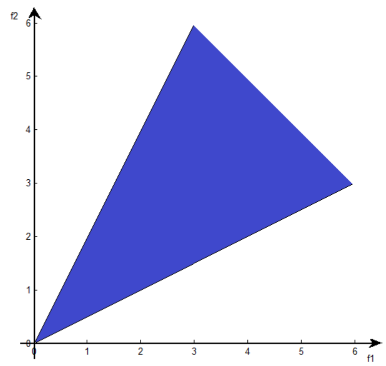

FIGURE 9. The ordering cone C.

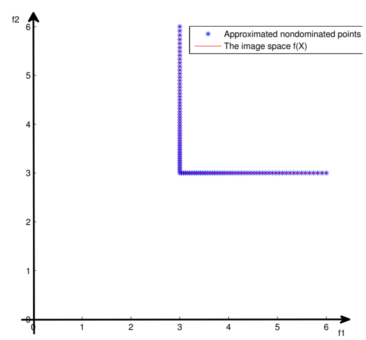

FIGURE 10. The approximated nondominated points for Example 5.3.

6. Converting a MOP to an equivalent MOP with a different ordering cone

Here, it is assumed that the ordering cone κ is a polyhedral cone that is generated by the convex hull of p linear independent vectors $b_i \in \mathbb{R}^p$ $(i = 1, ..., p)$, that is,

$$\kappa = cone\Bigg(conv\Big(\{b_1, b_2, ..., b_p\}\Big)\Bigg).$$

It is not difficult to see that κ is a closed pointed convex cone with a nonempty interior.

Linear independent vectors $b_i \in \mathbb{R}^p$ $(i = 1, ..., p)$ on the boundary of the ordering cone κ generate a basis for $\mathbb{R}^p$. Therefore, the coordinate vector of an arbitrary point $y \in \mathbb{R}^p$ with respect to the basis $\{b_1, b_2, ..., b_p\}$, denoted by $\bar{y}$, is

$$\bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_p \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{p1} \\ b_{12} & b_{22} & \dots & b_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1p} & b_{2p} & \dots & b_{pp} \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}, \tag{6.1}$$

where $b_j = (b_{j1}, ..., b_{jp})^T$.
On the other hand, it is easy to check that

$$\begin{bmatrix} b_{11} & b_{21} & \dots & b_{p1} \\ b_{12} & b_{22} & \dots & b_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1p} & b_{2p} & \dots & b_{pp} \end{bmatrix}^{-1} \kappa = \mathbb{R}^p_{\geqq}. \tag{6.2}$$

Hence, we can redefine Definition 2.5 as the follows.

Definition 6.1. A feasible solution $\hat{x} \in X$ is called an efficient solution of MOP with respect to the ordering cone κ, if there is no $x \in X$ such that $\bar{f}(x) \preceq_{\mathbb{R}^p_{\geqq}} \bar{f}(\hat{x})$ and $\bar{f}(x) \neq \bar{f}(\hat{x})$ (equivalently $(\bar{f}(\hat{x}) - \mathbb{R}^p_{\geqq}) \cap \bar{f}(X) = \{\bar{f}(\hat{x})\}$), where $\bar{f}(x)$ is defined by (6.1).

Therefore, MOP (2.1) ordered by the ordering cone κ is equal to the following MOP ordered by the natural ordering cone:

$$\min_{x \in X} \bar{f}(x) = \big(\bar{f}_1(x), \bar{f}_2(x), ..., \bar{f}_p(x)\big). \tag{6.3}$$

Thus, instead of solving (2.1) with respect to κ, we solve (6.3) with respect to the natural ordering cone.
Hence, all confirmed results and theorems related to MOPs ordered by the natural ordering cone $\mathbb{R}^p_{\geqq}$ can be generalized to MOPs ordered by the ordering cone κ. Indeed, it is sufficient to convert the set $f(X)$ to $\bar{f}(X)$ and consider $\mathbb{R}^p_{\geqq}$ as the ordering cone instead of the ordering cone κ.
Considering

$$A = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{p1} \\ a_{12} & a_{22} & \dots & a_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1p} & a_{2p} & \dots & a_{pp} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{p1} \\ b_{12} & b_{22} & \dots & b_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1p} & b_{2p} & \dots & b_{pp} \end{bmatrix}^{-1}, \tag{6.4}$$

we have

$$\bar{f}(x) = Af(x),$$

and therefore,

$$\bar{f}_1(x) = \sum_{i=1}^{p} a_{i1} f_i(x), \quad \vdots \quad \bar{f}_p(x) = \sum_{i=1}^{p} a_{ip} f_i(x). \tag{6.5}$$

Based on the results obtained so far, in the following subsections we present extensions of the ε-constraint method [13] and the convexification method given in [18] to (2.1) ordered by the ordering cone κ.

6.1. **An extension of the ε-constraint method.** The ε-constraint method, initially introduced in [13], was discussed in detail in [3]. In this method, one of the objectives is minimized while all the others are bounded from above by means of additional constraints.

The formulation of the ε-constraint method to (2.1), ordered by the natural ordering cone, is as follows:

$$\begin{aligned} &\min f_k(x) \\ &s.t. \\ &f_i(x) \leq \varepsilon_i, \ i \in \{1, ..., p\} \setminus \{k\} \\ &x \in X, \end{aligned} \tag{6.6}$$

where $(\varepsilon_1, ..., \varepsilon_{k-1}, \varepsilon_{k+1}, ..., \varepsilon_p) \in \mathbb{R}^{p-1}$ and $k \in \{1, 2, ..., p\}$.

According to the equivalence of MOP (2.1), ordered by $\mathbb{R}^p_{\geqq}$, and (6.3), ordered by κ, and relation (6.5), we can formulate the ε-constraint method for (2.1), ordered by the ordering cone κ, as follows:

$$\begin{aligned} &\min \sum_{i=1}^{p} a_{ik} f_i(x) \\ &s.t. \\ &\sum_{i=1}^{p} a_{ij} f_i(x) \leq \varepsilon_j, \ j \in \{1, ..., p\} \setminus \{k\} \\ &x \in X. \end{aligned} \tag{6.7}$$

In Example 6.2 below, the set of efficient solutions of a MOP ordered by an unnatural ordering cone is approximated by formulation (6.7).

Example 6.2. Consider the following bi-objective optimization problem with respect to the ordering cone D:

$$\min_{0 \leq x \leq 1} f(x) = (x, 1 - \sqrt{1 - (x-1)^2}). \tag{6.8}$$

The image space of MOP (6.8) and the ordering cone D are depicted in Figures 11 and 12, respectively.

The set of efficient solutions of MOP (6.8) ordered by the ordering cone D is:

$$X_{ED} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid 0.0299 \leq x_1 \leq 0.7575, \ x_2 = 1 - \sqrt{1 - (x_1 - 1)^2} \right\}.$$

The set $f(X_{ED})$ is depicted in Figure 13.

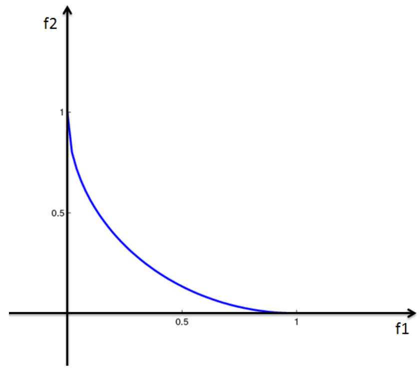

FIGURE 11. The image space of MOP (6.8).

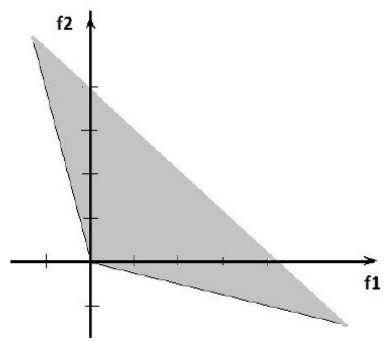

FIGURE 12. The ordering cone D of Example 6.2.

The linear independent vectors $b_1 = (4, -1)^T$ and $b_2 = (-1, 4)^T$, on the boundary of D, form a basis for $\mathbb{R}^2$.
Hence, according to formulation (6.7), we solve the following problem for different choices of ε_1 to approximate the set X_{ED}:

$$\begin{array}{l} \min \frac{1}{15}x + \frac{4}{15} - \frac{4}{15}\sqrt{1-(x-1)^2} \\ s.t. \\ \frac{4}{15}x + \frac{1}{15} - \frac{1}{15}\sqrt{1-(x-1)^2} \leq \varepsilon_1 \\ x \in X. \end{array}$$

It is assumed that $\varepsilon_1 \in [0, 1]$ and the distance between any two choices of the parameter ε_1 is 0.01 unit. The obtained approximation of Y_{ND} is depicted in Figure 14 shown by $*$.

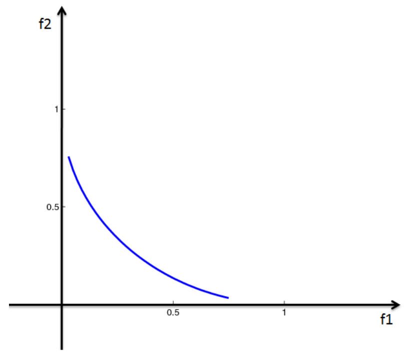

FIGURE 13. The set $f(X_{ED})$.

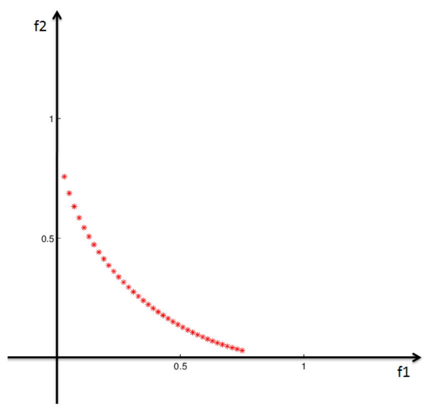

FIGURE 14. The obtained approximation of Y_{ND} for Example 6.2.

6.2. **An extension of the convexification method given in [18].** In [18], a method is proposed to convexify the Pareto front of a MOP with certain properties, in which all objective functions are nonnegative. By replacing all the objective functions of a MOP by their rth power, a converted MOP is formed. It is shown that the set of efficient solutions of the original MOP is equal to the set of efficient solutions of the converted MOP. If the value of r is chosen to be large enough, the Pareto front of the converted MOP is convex. Therefore, applying techniques capable of approximating all efficient solutions of a MOP with convex Pareto front, we can approximate all efficient solutions

of a MOP with a nonconvex front. We can refer to the weighted sum method as the well-known approach to solve a MOP with a convex Pareto front.
In Example 6.3, it is shown that if the ordering cone is unnatural, then it is possible that the set of efficient solutions of the original MOP and the converted MOP be different. Thus, the convexification method given in [18] is not correct for MOPs ordered by unnatural ordering cones. Afterwards, we present an extension of the method given in [18] corresponding to MOPs ordered by unnatural ordering cones.

Example 6.3. Consider the following bi-objective optimization problem with respect to the ordering cone C:

$$\min_{0\le x\le 1} f(x) = \big(f_1(x), f_2(x)\big) = (x, \sqrt{1-x^2}). \tag{6.9}$$

The ordering cone C is defined as

$$C = cone\Big(conv\big(\{b_1, b_2\}\big)\Big),$$

where $b_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$.
On the other hand, we have

$$C = \left\{ \alpha \begin{bmatrix} 6 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 6 \end{bmatrix} \,\Big|\, \alpha, \beta \ge 0 \right\}.$$

The image space of MOP (6.9) and the ordering cone C are depicted in Figures 15 and 16, respectively.

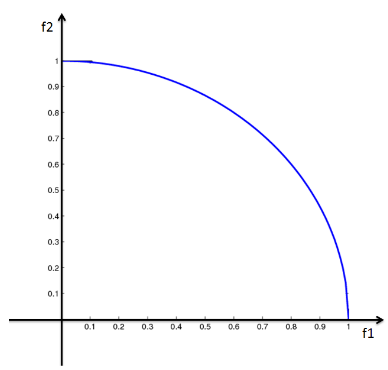

FIGURE 15. The image space of MOP (6.9).

Considering $r = 4$, the MOP (6.10) is:

$$\min_{0\le x\le 1} \big(f_1^4(x), f_2^4(x)\big) = \big(x^4, (1-x^2)^2\big). \tag{6.10}$$

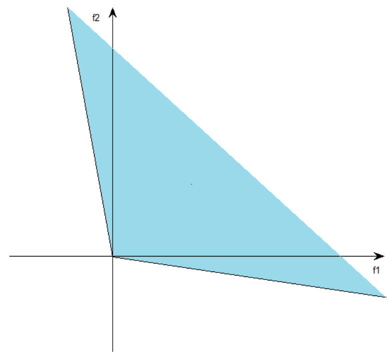

FIGURE 16. The ordering cone C.

The image space of MOP (6.10) is depicted in Figure 17.

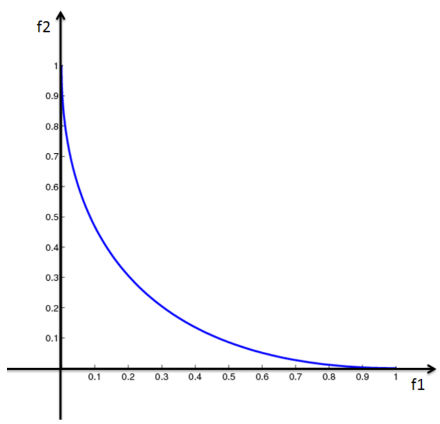

FIGURE 17. The image space of MOP (6.10).

In the following, it is shown that the set of efficient solutions of MOP (6.9) with respect to the ordering cone C is different from the set of efficient solutions of MOP (6.10) with respect to the ordering cone C. To this end, it is shown that the feasible solution $\hat{x} = 1$ is efficient for the MOP (6.9), but it is not efficient for the MOP (6.10).
According to Definition 2.5 we show that $(f(1) - C) \cap f(X) = \{f(1)\}$. Assume to the contrary that $(f(1) - C) \cap f(X) \neq \{f(1)\}$. Thus, there exists an $x = 1 - \varepsilon$ with $0 < \varepsilon \leq 1$ such that

$$f(x) \in (f(1) - C \backslash \{0\}).$$

Hence,

$$f(1) - f(x) = \begin{bmatrix} 1 - (1-\varepsilon) \\ 0 - \sqrt{1-(1-\varepsilon)^2} \end{bmatrix} \in C\backslash\{0\}.$$

Since

$$f(1) - f(x) = \begin{bmatrix} 1 - (1-\varepsilon) \\ 0 - \sqrt{1-(1-\varepsilon)^2} \end{bmatrix} = \alpha \begin{bmatrix} 6 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 6 \end{bmatrix},$$

with

$$\beta = \frac{\varepsilon - 6\sqrt{2\varepsilon - \varepsilon^2}}{35} < 0,$$

thus, according to the definition of the ordering cone C, we get

$$\begin{bmatrix} 1 - (1-\varepsilon) \\ 0 - \sqrt{1-(1-\varepsilon)^2} \end{bmatrix} \notin C\backslash\{0\},$$

So, $\hat{x} = 1$ is efficient for the MOP (6.9) with respect to the ordering cone C.

Now, it is shown that the feasible solution $\hat{x} = 1$ is not efficient for the MOP (6.10). To this end, it is shown that

$$f^4(0.9) \in f^4(1) - C\backslash\{0\}.$$

Since

$$f^4(1) - f^4(0.9) = \alpha \begin{bmatrix} 6 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

with $\alpha = 0.0579 > 0$ and $\beta = 0.0036 > 0$, we have

$$f^4(0.9) \in f^4(1) - C\backslash\{0\}.$$

Therefore, the set of efficient solutions of the MOP (6.9) and the MOP (6.10), ordered by the ordering cone C, are different. Thus, the convexification method given in [18] is not correct for MOPs with unnatural ordering cones.

In order to apply the convexification method given in [18] to a MOP ordered by unnatural ordering cone, the MOP is first converted to a MOP ordered by the natural ordering cone, using our proposed approach here. If the new MOP satisfies the assumptions of [18], then the convexification approach is applied. Therefore, we can use the weighted sum method to find an approximation of the set of efficient solutions for the new MOP with convex Pareto front. Since efficient solutions of the new MOP and the original MOP are the same, an approximation of the set of efficient solutions of the original MOP is at hand. This is illustrated by Example 6.4.

Example 6.4. Consider the following bi-objective optimization problem ordered by the ordering cone C, which was defined in Example 6.3:

$$\text{(6.11)} \quad \min_{0 \le x \le 1} f(x) = \big(f_1(x), f_2(x)\big) = (1 + 6x - \sqrt{1-x^2}, 1 - x + 6\sqrt{1-x^2}).$$

The image space of MOP (6.11) is depicted in Figures 18. According to our

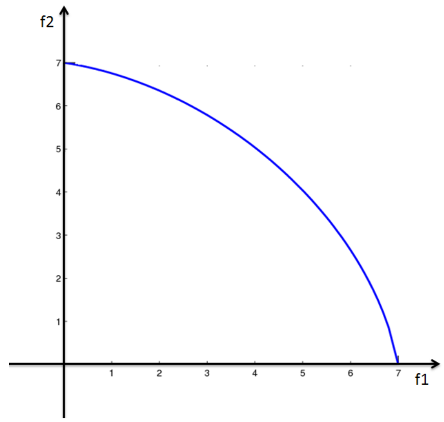

FIGURE 18. The image space of MOP (6.11).

proposed method, the MOP (6.11) ordered by the ordering cone C is equal to the MOP (6.12) below ordered by the natural ordering cone:

$$\min_{0\leq x\leq 1} \bar{f}(x) = \big(\bar{f}_1(x), \bar{f}_2(x)\big) = (\frac{7}{35} + x, \frac{7}{35} + \sqrt{1-x^2}). \tag{6.12}$$

The image space of the MOP (6.12) is depicted in Figure 19. As seen, the

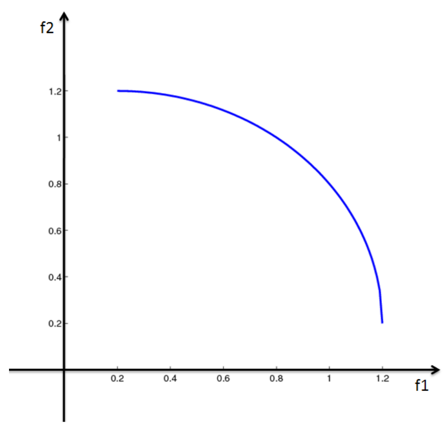

FIGURE 19. The image space of MOP (6.12).

Pareto front of MOP (6.12) is not convex. In order to convexify the Pareto front, we consider $r = 4$ and construct the following MOP:

$$\min_{0\leq x\leq 1} \bar{f}^4(x) = \big(\bar{f}_1^4(x), \bar{f}_2^4(x)\big) = \big((\frac{7}{35} + x)^4, (\frac{7}{35} + \sqrt{1-x^2})^4\big). \tag{6.13}$$

The image space of the MOP (6.13) is depicted in Figure 20. We solved the

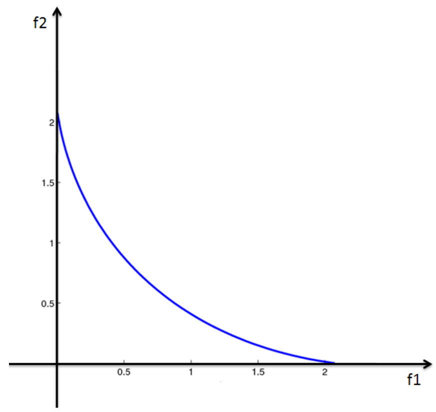

FIGURE 20. The image space of MOP (6.13).

MOP (6.13) using the weighted sum method. The image of the obtained efficient solutions corresponding to the MOP (6.11) is depicted in Figure 21.

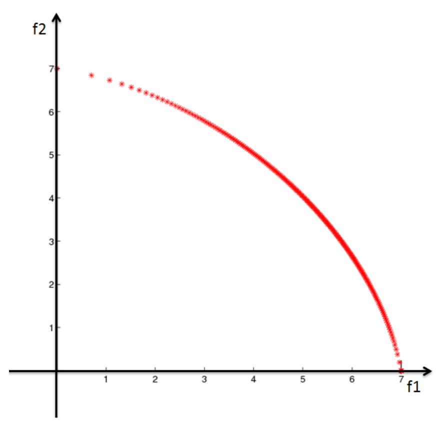

FIGURE 21. Approximate Pareto front for MOP (6.11).

7. Conclusions

We considered the well-known Pascoletti-Serafini scalarization and MOPs being bounded from below. It was first shown that the parameters of this scalarization could be restricted to a bounded subset of $\mathbb{R}^p$, and still all efficient solutions could be obtained. The obtained subset is constant and is not dependent on the structure of the problem under consideration. Also, it was shown that for parameters of the obtained subset, the scalar problem is always bounded from below. Then, a method to convert a MOP with respect to a specific ordering cone to an equivalent MOP ordered by the natural ordering cone was introduced. Based on the proposed conversion, it was concluded that the available methods for solving MOPs ordered by the natural ordering cone could be extended to solve MOPs ordered by specific unnatural ordering cones. Extensions of the ε-constraint method [13] and the convexification method of [18] were presented.

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