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STUDY ON MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS VIA OPERATIONAL MATRIX OF HYBRID BASIS FUNCTIONS

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Abstract. In this paper we apply hybrid functions of general block-pulse functions and Legendre polynomials for solving linear and nonlinear multi-order fractional differential equations (FDEs). Our approach is based on incorporating operational matrices of FDEs with hybrid functions that reduces the FDEs problems to the solution of algebraic systems. Error estimate that verifies a convergence of the approximate solutions is considered. The numerical results obtained by this scheme have been compared with the exact solution to show the efficiency of the method.
Keywords: Fractional derivatives and integrals, multi-order fractional differential equations, operational matrix, hybrid functions.
MSC(2010): Primary: 26A33; Secondary: 34A08, 42C05.

1. Introduction

The topic of fractional calculus has been attracted to many scientists, because of providing more accurate models of systems under consideration such as measurement of viscoelastic material properties [8], the fluid dynamic [15], and etc [9, 12]. In this study we consider general form of multi-order FDE

$$D_*^{\alpha} y(t) = f\left(t, y(t), D_*^{\beta_1} y(t), D_*^{\beta_2} y(t), ..., D_*^{\beta_k} y(t)\right), \quad y^{(i)}(0) = y_i,\ i = 0, 1, \ldots, n-1,\ n \in \mathbb{N}, \tag{1.1}$$

where $n - 1 < \alpha \leq n$, $\beta_k < \beta_{k-1} < \ldots < \beta_1 < \alpha$, and for $j = 1, 2, \cdots, k$, $n_j \in \mathbb{N}$, $n_j - 1 < \beta_j \leq n_j$, also D_*^{α} denotes the Caputo fractional derivative of order α. Most FDEs do not have exact analytic solutions and so it motivates us to develop a numerical scheme for their solutions. For this purpose, several methods have been proposed in the literature to solve these problems, such as

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Adomian decomposition method [21], He's variational iteration method [1], operational matrix method [20], homotopy perturbation method [9,11], and other methods [3,13]. In this study we will apply the hybrid of block-pulse function and Legendre polynomials for numerical solution of FDEs. The advantage of hybrid functions is that the orders of block-pulse functions and Legendre polynomials are adjustable to obtain highly accurate numerical solutions, compared to the piecewise constant orthogonal function for solution of the various differential equations. The main characteristic of this technique is that it reduces these problems to nonlinear system of algebraic equations which are suitable for computer programming. In this work we consider multi–order FDEs, and our approach is based upon expanding unknown function as hybrid functions with unknown coefficients. The properties of hybrid functions together with the operational matrix of FDE are used to convert the FDE to an algebraic equation and then, are utilized to evaluate the unknown coefficients. The structure of this paper is as follows:
In Section 2, we introduce the basic definitions of the fractional calculus theory. In Section 3, some relevant properties of the hybrid functions consisting of block-pulse functions and Legendre polynomials, and approximation of function by these basis are presented. Section 4 is devoted to investigate the convergence of the hybrid basis function in order to approximate the solution of FDE, and the accuracy estimation is rendered. At the end of this section, we will apply the hybrid method for solving FDEs, by using the operational matrix of the fractional integration. In Section 5, through the provided examples, our numerical finding reported and the reliability and performance of the proposed scheme is demonstrated. Finally, we conclude the result with some remarks.

2. Preliminaries and notations

First we recall some facts concerning fractional calculus theory which will be used further in this work [6,22].

Definition 2.1. The operator I^{α}, defined on $L_1[0,b]$ by

$$I^{\alpha}y\left(t\right)=\frac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}y\left(s\right)ds,$$

for $0\leq t\leq b$, is called the *Riemann-Liouville fractional integral operator* of order α. Here $\Gamma(.)$ denotes the Gamma function and for $\alpha=0$, we set $I^0=I$, where I is the identity operator.

It may be shown that the fractional integral operator I^{α} transforms the space $L_1[0,b]$ into itself and has some other properties.

Remark 2.2. Let $\alpha,\beta\geq 0$ and $y\in L_1[0,b]$. Then the following hold almost everywhere on $[0,b]$:

1) $I^{\alpha}I^{\beta}y = I^{\beta}I^{\alpha}y$;
2) $I^{\alpha}I^{\beta}y = I^{\alpha+\beta}y$;

Definition 2.3. Let $n = \lceil\alpha\rceil$ ($\lceil.\rceil$ denotes ceiling function, $\lceil x\rceil = min\{z \in \mathbb{Z}: z \geq x\}$). The operator D^{α}, defined by

$$D^{\alpha}y = D^{n}I^{n-\alpha}y,$$

is called the Riemann-Liouville fractional differential operator of order α. For $\alpha = 0$, we set $D^{0} = I$, the identity operator.

Remark 2.4. Assume that $c_1, c_2 \in \mathbb{R}$, and let $y, y_1, y_2 \in L_1[0, b]$. Then
1) $D^{\alpha}I^{\alpha}y = y$;
2) $D^{\alpha}(c_1y_1 + c_2y_2) = c_1D^{\alpha}y_1 + c_2D^{\alpha}y_2$.

The other type of fractional derivative that is strongly connected to the Riemann-Liouville fractional derivative is Caputo fractional derivative. Nowadays, this derivative is frequently used in applications. By using the Caputo derivative one can specify the initial conditions of FDEs in classical form. Also, this approach is suitable for real world physical problems, since it requires the initial conditions for definition of fractional order, which have physically meaningful explanation. Accordingly, in this paper we only concentrate on Caputo fractional derivative.

Definition 2.5. The Caputo fractional derivative of $y(t) \in L_1[0, b]$, is defined as

$$D_{*}^{\alpha}y\left(t\right) = \begin{cases} I^{n-\alpha}D^{n}y(t), & n-1 < \alpha < n,\ n \in \mathbb{N}, \\ \frac{d^n}{dt^n}y\left(t\right), & \alpha = n. \end{cases}$$

Note that
$I^{\alpha}D_{*}^{\alpha}y\left(t\right) = y\left(t\right) - \sum_{k=0}^{n-1} y^{(k)}\left(0^{+}\right)\frac{t^k}{k!},\ n-1 < \alpha \leq n,\ n \in \mathbb{N}.$

Lemma 2.6 ([6])**.** *Let $\alpha \geq 0$ and $n = \lceil\alpha\rceil$. Assume that y is such that both $D_{*}^{\alpha}y$ and $D^{\alpha}y$ exist. Then, the relation between Caputo and Riemann-Liouville fractional derivative is as follows:*

$$D_{*}^{\alpha}y\left(t\right) = D^{\alpha}y\left(t\right) - \sum_{k=0}^{n-1}\frac{D^{k}y(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha}.$$

3. Hybrid functions

A set of block-pulse functions $b_n(t), n = 1, 2, \ldots, N$ on the interval $[0, 1)$ is defined as follows [19]:

$$b_n\left(t\right) = \begin{cases} 1, & \frac{n-1}{N} \leq t < \frac{n}{N}, \\ 0, & \text{otherwise}. \end{cases}$$

These functions are disjoint and have the property of orthogonality on $[0,1)$. Hybrid functions $h_{nm}(t), n=1,2,\ldots,N,\ m=0,1,\ldots,M-1$; have three arguments, n and m are the order of block-pulse functions and Legendre polynomials, respectively, and t is the normalized time. Hybrid functions are defined on the interval $[0,1)$ as [18]:

$$h_{nm}(t)=\begin{cases} P_m\left(2Nt-2n+1\right), & t\in\left[\frac{n-1}{N},\frac{n}{N}\right), \\ 0, & \text{otherwise,}\end{cases}$$

where, $P_m(t)$s are the well-known Legendre polynomials of order m which satisfy the following recursive formula,

$$P_0(t)=1, \qquad P_1(t)=t,$$

$$P_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)tP_m(t)-\left(\frac{m}{m+1}\right)P_{m-1}(t), \qquad m=1,2,\ldots.$$

Since $h_{nm}(t)$ consists of block-pulse functions and Legendre polynomials, which are both complete and orthogonal, so a set of hybrid functions based on them is a complete orthogonal set in the Hilbert space.

A function $y(t)$, which is a square integrable function defined over the interval $[0,1)$, may be expanded as [19]

$$y(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}h_{nm}(t),$$

which the hybrid coefficients are

$$c_{nm}=\frac{\langle y(t),h_{nm}(t)\rangle}{\langle h_{nm}(t),h_{nm}(t)\rangle},$$

such that $\langle\cdot,\cdot\rangle$ denotes the inner product. Also, the infinite series may be terminated after $\mu=NM$ terms, that is,

$$y(t)\approx\widetilde{y}_\mu(t)=\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{nm}h_{nm}(t)=C_\mu^T H_\mu(t),$$

where vector forms of $C_\mu, H_\mu(t)$ are

$$C_\mu=\left[c_{10},\ldots,c_{1(M-1)},c_{20},\ldots,c_{2(M-1)},\ldots,c_{N0},\ldots,c_{N(M-1)}\right]^T,$$

(3.1)

$$H_\mu(t)=\left[h_{10}(t),\ldots,h_{1(M-1)}(t),h_{20}(t),\ldots,h_{2(M-1)}(t),\ldots,h_{N0}(t),\ldots,h_{N(M-1)}(t)\right]^T.$$

4. **Main results**

We wish to show that, approximated solution via hybrid functions converges to the exact solution of FDE. Also, in this section we will show how, for the general form multi-order FDEs, one can carry out our approach by using operational matrix of the fractional integration through a relatively simple formula which will yield improved accuracy.

4.1. **Convergence of the Hybrid Basis Functions.** In fact, [2, 5] have proved that the nonlinear FDE subject to the initial conditions, under some special assumptions, has a unique continuous solution. For more detail about the theorems of the existence and uniqueness of problem one can see above references. Thus, we require that to obtain error estimate that verifies a convergence of approximation. In developing convergence theory, we will require some preliminary results.

Lemma 4.1. *Consider the equation*

$$D_*^{\alpha} y(t) = f\left(t, y(t), D_*^{\beta} y(t)\right), \quad y^{(i)}(0) = y_i, \;\; 0 \le i \le \lceil \alpha \rceil - 1. \tag{4.1}$$

Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, $n = \lceil \alpha \rceil, k = \lceil \beta \rceil$ and $\beta < \alpha$. Then the initial value problem (4.1) is equivalent to the following integral equation

$$y(t) = \sum_{i=0}^{n-1} y_i \frac{t^i}{i!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \frac{1}{\Gamma(k-\beta)} \int_0^s (s-\tau)^{k-\beta-1} y^{(k)}(\tau) d\tau\right) ds. \tag{4.2}$$

Proof. By using definition of the Caputo fractional derivative and applying operator I^{α} to both side of Eq. (4.1) it is obvious. □

Lemma 4.2. ([4]) Assume that $y(t) \in H^{\kappa}(\Omega)$ (Sobolev space), and $\widetilde{y}_{\mu}(t), \widetilde{y}'_{\mu}(t)$ be the best approximations of $y(t), y'(t)$, respectively, in L^2-norm. Then for $\Omega = (0,1)$ and $\kappa \ge 2$, the following estimates are hold

$$\|y - \widetilde{y}_{\mu}\|_{L^2(\Omega)} \le C_0 \mu^{-\kappa} \|y\|_{H^{\kappa}(\Omega)},$$
$$\|y' - \widetilde{y}'_{\mu}\|_{L^2(\Omega)} \le C_0 \mu^{1-\kappa} \|y\|_{H^{\kappa}(\Omega)},$$

where C_0 is a positive constant, which depends on the selected norm and is independent of $y(t)$ and $\widetilde{y}_{\mu}(t)$, where

$$\|y\|_{L^2(\Omega)} = \left(\int_0^1 y^2(t) dt\right)^{\frac{1}{2}},$$
$$\|y\|_{H^{\kappa}(\Omega)} = \left(\sum_{i=0}^{\kappa} \int_0^1 |y^{(i)}(t)|^2 dt\right)^{\frac{1}{2}}.$$

Theorem 4.3. *Let $f(t,\cdot,\cdot)$ be satisfied in the Lipschitz condition respect to the last two components, with Lipschitz constants l_1 and l_2, respectively. Assume that for $0<\beta<\frac{1}{2},\alpha>\frac{1}{2}$, $\widetilde{y}_\mu(t)$ is the numerical solution of Eq. (4.1). Then, in the view of the previous lemma assumptions, we have*

$$\sup_{t\in\Omega}|y(t)-\widetilde{y}_\mu(t)|\leq C_0\left(\frac{l_1}{\sqrt{2\alpha-1}\;\Gamma(\alpha)}+\frac{l_2\mu}{\sqrt{1-2\beta}}\right)\mu^{-\kappa}\|y\|_{H^\kappa(\Omega)}.$$

Proof. Consider the Eq. (4.1) and its transformed fractional integral Eq. (4.2). Assume that the exact solution of Eq. (4.2) is

$$y(t)=\sum_{i=0}^{n-1}y_i\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,y(s),D_*^\beta y(s)\right)ds.$$

If we define the numerical solution of Eq. (4.1) by $\widetilde{y}_\mu(t)$, then

$$\widetilde{y}_\mu(t)=\sum_{i=0}^{n-1}y_i\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,\widetilde{y}_\mu(s),D_*^\beta\widetilde{y}_\mu(s)\right)ds,$$

and we have

$$|y(t)-\widetilde{y}_\mu(t)|=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f\left(s,y(s),D_*^\beta y(s)\right)-f\left(s,\widetilde{y}_\mu(s),D_*^\beta\widetilde{y}_\mu(s)\right)\right|ds.$$

Since f satisfies in the Lipschitz condition with respect to the second and third components, by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sup_{t\in\Omega}|y(t)-\widetilde{y}_\mu(t)|\leq&\frac{l_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|y(s)-\widetilde{y}_\mu(s)|\,ds+\\&\frac{l_2}{\Gamma(\alpha)\Gamma(1-\beta)}\int_0^t(t-s)^{\alpha-1}\left(\int_0^s(s-\tau)^{-\beta}\left|y'(\tau)-\widetilde{y}'_\mu(\tau)\right|d\tau\right)ds\\\leq&\frac{l_1\|y-\widetilde{y}_\mu\|_{L^2(\Omega)}}{\Gamma(\alpha)}\left(\int_0^t(t-s)^{2\alpha-2}ds\right)^{\frac{1}{2}}+\\&\frac{l_2\left\|y'-\widetilde{y}'_\mu\right\|_{L^2(\Omega)}}{\Gamma(\alpha)\Gamma(1-\beta)}\int_0^t(t-s)^{\alpha-1}\left(\int_0^s(s-\tau)^{-2\beta}d\tau\right)^{\frac{1}{2}}ds\\=&\frac{l_1\|y-\widetilde{y}_\mu\|_{L^2(\Omega)}}{\Gamma(\alpha)}\left(\frac{t^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}+\frac{l_2t^{\frac{1}{2}+\alpha-\beta}\left\|y'-\widetilde{y}'_\mu\right\|_{L^2(\Omega)}}{\sqrt{1-2\beta}\;\Gamma(\alpha)\Gamma(1-\beta)}B(\alpha,\frac{3}{2}-\beta)\\<&\frac{l_1\|y-\widetilde{y}_\mu\|_{L^2(\Omega)}}{\sqrt{2\alpha-1}\;\Gamma(\alpha)}+\frac{l_2\left\|y'-\widetilde{y}'_\mu\right\|_{L^2(\Omega)}}{\sqrt{1-2\beta}},\end{aligned}$$

where $B(.,.)$ denotes Beta function. By using Lemma 4.2, we have

$$\sup_{t\in\Omega}|y(t)-\widetilde{y}_\mu(t)|\leq\left(\frac{l_1}{\sqrt{2\alpha-1}\;\Gamma(\alpha)}+\frac{l_2\mu}{\sqrt{1-2\beta}}\right)C_0\mu^{-\kappa}\|y\|_{H^\kappa(\Omega)},$$

and this completes the proof. On the other word, since $\kappa\geq2$, we can see for $y(t)\in H^\kappa(\Omega)$, as μ trends to infinity implies $\widetilde{y}_\mu(t)\to y(t)$. For generic

FDE 1.1, by applying the appropriate Riemann-Liouville fractional integration operator on it, we can often use the result of this theorem. □

4.2. **Operational matrix and implementation method.** In order to obtain the hybrid function operational matrix of the fractional integration, let

$$I^{\alpha}H_{\mu}(t) \approx P^{\alpha}_{\mu\times\mu}H_{\mu}(t). \tag{4.3}$$

$P^{\alpha}_{\mu\times\mu}$ is called the operational matrix of the fractional integration for hybrid function and is obtained by the following formula:

$$P^{\alpha}_{\mu\times\mu} = \Phi_{\mu\times\mu}F^{\alpha}_{\mu\times\mu}\Phi^{-1}_{\mu\times\mu}, \tag{4.4}$$

where matrix $\Phi_{\mu\times\mu}$ is an invertible matrix and define by using vector $H_{\mu}(t)$ in collocation points $t_i = \frac{2i-1}{2\mu},\ i = 1, 2, \ldots, \mu$, as following

$$\Phi_{\mu\times\mu} = \left[H_{\mu}\left(\frac{1}{2\mu}\right) \quad H_{\mu}\left(\frac{3}{2\mu}\right) \ \ldots \ H_{\mu}\left(\frac{2\mu-1}{2\mu}\right)\right],$$

and

$$F^{\alpha}_{\mu\times\mu} = \frac{1}{\mu^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_{\mu-1} \\ 0 & 1 & \varepsilon_1 & \ldots & \varepsilon_{\mu-2} \\ 0 & 0 & 1 & \ldots & \varepsilon_{\mu-3} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with $\varepsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, for $k = 1, 2, \cdots, \mu-1$.
On the other hand by taking $B_{\mu}(t) = [b_1(t), b_2(t), ..., b_{\mu}(t)]^T$, that b_is are block-pulse functions, hybrid functions may be expanded into μ-term block-pulse functions, as

$$H_{\mu}(t) = \Phi_{\mu\times\mu}B_{\mu}(t), \tag{4.5}$$

and since $F^{\alpha}_{\mu\times\mu}$ is the block-pulse operational matrix of the fractional integration we get [14],

$$I^{\alpha}B_{\mu}(t) \approx F^{\alpha}_{\mu\times\mu}B_{\mu}(t). \tag{4.6}$$

Finally from Eqs. (4.4)-(4.6), one can conclude that

$$I^{\alpha}H_{\mu}(t) = I^{\alpha}\Phi_{\mu\times\mu}B_{\mu}\,(t) = \Phi_{\mu\times\mu}I^{\alpha}B_{\mu}(t) \approx \Phi_{\mu\times\mu}F^{\alpha}_{\mu\times\mu}B_{\mu}(t) = P^{\alpha}_{\mu\times\mu}\Phi_{\mu\times\mu}B_{\mu}(t). \tag{4.7}$$

At the end of this section, we show that how one can implement this approach for solving linear and nonlinear multi-order FDEs in general case. To solve FDE 1.1, we approximate $D^{\alpha}_{*}y(t)$ by the hybrid functions as

$$D^{\alpha}_{*}y(t) = C^T_{\mu}H_{\mu}(t), \tag{4.8}$$

where C_μ is an unknown vector. Accordingly, by applying the fractional integral operator to both sides of the above equation, we get

$$I^\alpha D_*^\alpha y(t) = C_\mu^T I^\alpha H_\mu(t) \Longrightarrow y(t) = C_\mu^T I^\alpha H_\mu(t) + \sum_{i=0}^{n-1} \frac{y_i}{i!} t^i,$$

and by relatin (4.7)

$$y(t) = C_\mu^T P_{\mu\times\mu}^\alpha \Phi_{\mu\times\mu} B_\mu(t) + \sum_{i=0}^{n-1} \frac{y_i}{i!} t^i \tag{4.9}$$

and

$$D_*^{\beta_j} y(t) = C_\mu^T P_{\mu\times\mu}^{\alpha-\beta_j} \Phi_{\mu\times\mu} B_\mu(t) + \sum_{i=n_j}^{n-1} \frac{y_i}{(i-n_j)!} t^{i-n_j}, \tag{4.10}$$

where $n_j - 1 < \beta_j \le n_j$. By substituting Eqs. (4.8)-(4.10) in Eq. (1.1), we obtain a system of algebraic equations, which can be solved to find unknown function. Implementation of this approach is given in the next section via numerical experiments.

5. Applications and results

In this section we illustrate the applicability of the presented numerical scheme in previous section to solve multi-order FDEs. For this purpose, we consider some linear and nonlinear multi-order FDEs and compare the results obtained by using our approach with the analytical solution or the estimated solutions by using the other methods.

Example 5.1. Let us consider the Bagley-Torvik equation that governs the motion of a rigid plate immersed in a Newtonian fluid,

$$aD_*^2 y(t) + bD_*^{1.5} y(t) + cy(t) = h(t), \quad t \in [0, T]. \tag{5.1}$$

Following [7, 10, 17], we consider the case $h(t) = c(1 + t)$ and $a = b = c = 1$, subject to the initial states: $y(0) = y'(0) = 1$.
The exact solution of this problem is $y(t) = t + 1$. Here we use the hybrid operational matrices of the fractional integration to solve it. Let $D_*^2 y(t) = C_\mu^T H_\mu(t)$, together with the initial conditions and using Eq. (4.9), then we have

$$y(t) = C_\mu^T P_{\mu\times\mu}^2 H_\mu(t) + t + 1. \tag{5.2}$$

By substituting the above equation into Eq. (5.1), we get

$$C_\mu^T \left[I_\mu + P_{\mu\times\mu}^{0.5} + P_{\mu\times\mu}^2\right] H_\mu(t) = 0. \tag{5.3}$$

As respects, $I_\mu + P_{\mu\times\mu}^{0.5} + P_{\mu\times\mu}^2$ and $H_\mu(t)$ are the nonzero matrices, thus the solution of Eq. (5.3) is $C_\mu = 0$, which by using Eq. (5.2) we achieve the exact solution.

Example 5.2. Conforming [10, 17], we consider the fractional order variable coefficient linear differential equation
(5.4)
$$\begin{cases} aD_*^2 y(t) + b(t)D_*^{\beta_1} y(t) + c(t)D_* y(t) + e(t)D_*^{\beta_2} y(t) + k(t)y(t) = h(t), \\ 0 \le t \le 1, \;\; 0 < \beta_2 < 1, \;\; 1 < \beta_1 < 2, \;\; y(0) = 2, \;\; y'(0) = 0, \end{cases}$$
where
$$h(t) = -a - \frac{b(t)}{\Gamma(3-\beta_1)} t^{2-\beta_1} - c(t)t - \frac{e(t)}{\Gamma(3-\beta_2)} t^{2-\beta_2} + k(t)(2 - \frac{t^2}{2}),$$
and the exact solution is $y(t) = 2 - \frac{t^2}{2}$.
Let $D_*^2 y(t) = C_\mu^T H_\mu(t)$ and using initial states, we have

$$D_*^{\beta} y(t) = C_\mu^T P_{\mu\times\mu}^{2-\beta} H_\mu(t), \quad \text{for} \quad \beta = \beta_2, 1, \beta_1, \tag{5.5}$$

also, $y(t) = C_\mu^T P_{\mu\times\mu}^2 H_\mu(t) + 2$.
Therefore, the corresponding algebraic system for representation FDE (5.4), is
$$C_\mu^T \left[aI_\mu + b(t)P_{\mu\times\mu}^{2-\beta_1} + c(t)P_{\mu\times\mu}^1 + e(t)P_{\mu\times\mu}^{2-\beta_2} + k(t)P_{\mu\times\mu}^2 \right] H_\mu(t) = h(t) - 2k(t).$$
We apply the collocation method with taking the collocation points $t_i = \frac{2i-1}{2\mu}, i = 1, 2, \ldots, \mu$, for processing the above system. By solving this system, we can find the numerical results for different values of coefficients, μ and $\beta_i, i = 1, 2$. Likely [10, 17], we present numerical solution of Eq. (5.4) by our method for $a = 1, b(t) = \sqrt{t}, c(t) = t^{\frac{1}{3}}, e(t) = t^{\frac{1}{4}}, k(t) = t^{\frac{1}{5}}, \beta_2 = 0.333$ and $\beta_1 = 1.234$. Table 1 shows the absolute errors of the numerical solutions for $\mu = 8, 24, 72$. It is evident from the Table 1 that, the numerical solution by presented hybrid method in Section 4, converges to the exact solution. Moreover, the absolute errors decrease as the step size is decreased. The same trend is observed for other values of variable coefficients, μ and $\beta_i, i = 1, 2$. Clearly, the approximations obtained by the hybrid method are in agreement with those obtained with other mentioned numerical methods in [10, 17].

TABLE 1. Absolute errors with $M = 4, N = 2, 6, 18$ in different values of t for Example 5.2.

t Values	$N = 2$	$N = 6$	$N = 18$
0.1	6.52451×10^{-4}	7.29339×10^{-5}	8.13767×10^{-6}
0.2	6.52338×10^{-4}	7.33444×10^{-5}	8.20893×10^{-6}
0.3	6.47673×10^{-4}	7.30876×10^{-5}	8.20212×10^{-6}
0.4	6.37365×10^{-4}	7.21244×10^{-5}	8.11286×10^{-6}
0.5	6.21304×10^{-4}	7.05097×10^{-5}	7.94738×10^{-6}
0.6	6.00712×10^{-4}	6.83381×10^{-5}	7.71643×10^{-6}
0.7	5.76486×10^{-4}	6.57189×10^{-5}	7.43254×10^{-6}
0.8	5.49523×10^{-4}	6.27639×10^{-5}	7.10847×10^{-6}
0.9	5.20722×10^{-4}	5.95782×10^{-5}	6.75633×10^{-6}

Example 5.3. Lastly, in order to assess the advantages and the accuracy of the hybrid method presented in this article for solving nonlinear multi-order FDEs, we consider the following initial values problem

$$\begin{cases} aD_*^{\alpha}y(t) + bD_*^{\beta_1}y(t) + c\left(D_*^{\beta_2}y(t)\right)^i + e\,(y(t))^3 = h(t), \quad 0 \le t \le 1, \\ 0 < \beta_2 < 1, \quad 1 \le \beta_1 < 2, \quad 2 \le \alpha < 3, \quad y(0) = y'(0) = y''(0) = 0, \end{cases} \tag{5.6}$$

with

$$h(t) = \frac{2a}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{2b}{\Gamma(4-\beta_1)}t^{3-\beta_1} + c\left(\frac{2}{\Gamma(4-\beta_2)}t^{3-\beta_2}\right)^i + e\left(\frac{t^3}{3}\right)^3,$$

and the exact solution $y(t) = \frac{t^3}{3}$ in three cases:
Case I ([16]): $a = b = c = e = 1, i = 1, \beta_2 = 0.333, \beta_1 = 1.234, \alpha = 2$.
Case II([17]): $a = b = c = e = 1, i = 1, \beta_2 = 0.75, \beta_1 = 1.25, \alpha = 2.2$.
Case III([10]): $a = 1, b = c = e = 0.5, i = 2, \beta_2 = 0.276, \beta_1 = 1.999, \alpha = 2$.
Implementation of our method is utterly similar in above cases. Let $D_*^{\alpha}y(t) = C_\mu^T H_\mu(t)$, using Eq. (4.9) we have

$$y(t) = C_\mu^T P_{\mu\times\mu}^{\alpha} \Phi_{\mu\times\mu} B_\mu(t).$$

By letting

$$C_\mu^T P_{\mu\times\mu}^{\alpha} \Phi_{\mu\times\mu} = [d_1, d_2, \ldots, d_\mu],$$

and use the properties of block-pulse function, we have

$$y^3(t) = \left[d_1^3, d_2^3, \ldots, d_\mu^3\right] B_\mu(t) = D^T B_\mu(t).$$

Substituting these equations into FDE (5.6), we attain the following system of nonlinear algebraic equations

$$aC_\mu^T H_\mu(t) + bC_\mu^T P_{\mu\times\mu}^{\alpha-\beta_1} H_\mu(t) + c\left(C_\mu^T P_{\mu\times\mu}^{\alpha-\beta_2} H_\mu(t)\right)^i + eD^T B_\mu(t) = h(t).$$

By solving the above system, we can find the vector C_μ and subsequently, solution of FDE (5.6) is obtained. In Table 2, the absolute errors for Example 5.3 with $\mu = 64, 96, 100$, obtained by the hybrid method in some points $t \in (0, 1)$, are given. Also, this outcomes comparison with Refs. [16, 17]. About Case III, in [10] the absolute errors in collocation points is not available, and we only known that, the maximal error is 5.76168×10^{-4} for $m = 1000$.

6. **Conclusions**

Hybrid methods have different resolution capability for expanding of different functions. In this paper, the hybrid method of block-pulse function and Legendre polynomials has been adopted for numerical solution of the initial value problems for linear and nonlinear multi-order FDEs. Our approach in this study is more generalized and unproblematic to implement. Also, the presented method yields to very accurate results. Another important advantage

TABLE 2. Absolute errors of Example 5.3, in comparison with the Refs. [17] and [16]

	CaseI		*CaseII*		*CaseIII*
t	[16] ($m = 96$)	Our approach ($\mu = 64$)	[17] ($m = 128$)	Our approach ($\mu = 96$)	Our approach ($\mu = 100$)
0.1	5.25376×10^{-6}	5.49976×10^{-6}	4.79×10^{-6}	3.291×10^{-6}	2.49667×10^{-6}
0.2	1.26534×10^{-5}	1.01598×10^{-5}	8.11×10^{-6}	6.614×10^{-6}	4.99209×10^{-6}
0.3	1.85804×10^{-5}	1.41604×10^{-5}	1.189×10^{-5}	9.672×10^{-6}	7.48669×10^{-6}
0.4	1.89244×10^{-5}	1.75918×10^{-5}	1.835×10^{-5}	1.241×10^{-5}	9.97956×10^{-6}
0.5	3.17167×10^{-5}	2.05188×10^{-5}	2.357×10^{-5}	1.481×10^{-5}	1.24684×10^{-5}
0.6	2.69467×10^{-5}	2.29932×10^{-5}	2.608×10^{-5}	1.688×10^{-5}	1.49488×10^{-5}
0.7	3.97022×10^{-5}	2.50581×10^{-5}	2.442×10^{-5}	1.865×10^{-5}	1.74134×10^{-5}
0.8	4.45965×10^{-5}	2.67483×10^{-5}	2.689×10^{-5}	2.014×10^{-5}	1.98503×10^{-5}
0.9	3.74835×10^{-5}	2.80885×10^{-5}	3.577×10^{-5}	2.136×10^{-5}	2.22416×10^{-5}

is that, hybrid method is capable of greatly reducing the size of computational work while accurately providing the series solution with fast convergence rate. To highlight the convergence, the numerical results are presented for different values of $\mu = MN$. The solutions obtained by using hybrid method are in perfect agreement with the exact solutions and as observed, we get good approximation with low terms of basis. Another direction for further research would be to extend the presented method to the systems of FDEs. This work is currently in progress.

References

[1] S. Abbasbandy, An approximation solution of a nonlinear equation with Riemann-Liouville's fractional derivatives by He's variational iteration method, *J. Comput. Appl. Math.* **207** (2007), no. 1, 53–58.

[2] A. Ahmadkhanlu and M. Jahanshahi, On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales, *Bull. Iranian Math. Soc.* **38** (2012), no. 1, 241–252.

[3] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, Fractional Calculus Models and Numerical Methods, World Scientific, Singapore, 2012.

[4] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods on Fluid Dynamics, Springer-Verlag, Berlin, 1988.

[5] J. Deng and L. Ma, Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, *Appl. Math. Lett.* **23** (2010), no. 6, 676–680.

[6] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010.

[7] K. Diethelm and N.J. Ford, Numerical solution of the Bagley-Torvik equation, *BIT* **42** (2002), no. 3, 490–507.

[8] M. Di Paola, G. Failla and A. Pirrotta, Stationary and non-stationary stochastic response of linear fractional viscoelastic systems, *Probabilist. Eng. Mech.* **28** (2012) 85–90.

[9] A.A. Elbeleze, A. Kilicman and B.M. Taib, Homotopy perturbation method for fractional Black-Scholes European option pricing equations using Sumudu transform, *Math. Probl. Eng.* **2013** (2013), Article ID 524852, 7 pages.

[10] A. El-Mesiry, A. El-Sayed and H. El-Saka, Numerical methods for multi-term fractional (arbitrary) orders differential equations, *Appl. Math. Comput.* **160** (2005), no. 3, 683–699.

[11] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, *Int. J. Nonlinear Sci. Numer. Simul.* **6** (2005), no. 2, 207–208.

[12] J.H. He, Some applications of nonlinear fractional differential equations and their approximations, *Bull. Sci. Technol.* **15** (1999) 86–90.

[13] A. Kazemi Nasab, A. Kilicman, Z. Pashazadeh Atabakan and S. Abbasbandy, Chebyshev wavelet finite difference method: A new approach for solving initial and boundary value problems of fractional order, *Abstr. Appl. Anal.* **2013** (2013), Article ID 916456, 15 pages.

[14] A. Kilicman and Z.A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, *Appl. Math. Comput.* **187** (2007), no. 1, 250–265.

[15] V.V. Kulish and J.L. Lage, Application of fractional calculus to fluid mechanics, *J. Fluid. Eng.* **124** (2002) 803–806.

[16] Y. Li, Solving a nonlinear fractional differential equation using Chebyshev wavelets, *Commun. Nonlinear Sci. Numer. Simulat.* **15** (2010), no. 9, 2284–2292.

[17] Y. Li and W. Zhao, Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, *Appl. Math. Comput.* **216** (2010), no. 8, 2276–2285.

[18] K. Maleknejad and E. Hashemizadeh, A numerical approach for Hammerstein integral equation of mixed type using operational matrices of hybrid functions, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **73** (2011), no. 3, 95–104.

[19] K. Maleknejad, M. Khodabin and M. Rostami, Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block-pulse functions, *Math. Comput. Model. Dyn. Syst.* **55** (2012), no. 3-4, 791–800.

[20] K. Maleknejad, K. Nouri and L. Torkzadeh, Operational matrix of fractional integration based on the shifted second kind Chebyshev polynomials for solving fractional differential equations, *Mediterr. J. Math.* **13** (2016), no. 3, 1377–1390.

[21] Z. Odibat and S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, *Appl. Math. Model.* **32** (2008) 28–39.

[22] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, New York, 1999.

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