ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 2, pp. 501–514

Title:

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Author(s):

J. Li, D. Yu, G. Chen and W. Shi

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 2, pp. 501–514
Online ISSN: 1735-8515

A CHARACTERIZATION OF SIMPLE K_4-GROUPS OF TYPE $L_2(q)$ AND THEIR AUTOMORPHISM GROUPS

J. LI, D. YU, G. CHEN* AND W. SHI

(Communicated by Ali Reza Ashrafi)

Abstract. In this paper, it is proved that all simple K_4-groups of type $L_2(q)$ can be characterized by their largest element orders together with their orders. Furthermore, the automorphism groups of simple K_4-groups of type $L_2(q)$ are also considered.
Keywords: Simple K_4-groups, the largest element order, characterization.
MSC(2010): Primary: 20D05; Secondary: 20D45, 20D60.

1. Introduction

All groups considered in this paper are finite.

Let G be a group. Let $\pi(G)$ and $\pi_e(G)$ denote the set of primes dividing the order of G and the set of orders of elements of G, respectively. Let $m_1(G)$ and $m_2(G)$ denote the largest element order and the second largest element order of G, respectively. G is said to be a K_n-group if $\pi(G)$ consists of exactly n distinct primes. For a group G, as in [23], we construct its prime graph $\Gamma(G)$ as follows: the vertices are the primes in $\pi(G)$ and two vertices p and r are connected by an edge if and only if G contains an element of order pr. Denote by $T(G) = \{\pi_i(G)|1 \leqslant i \leqslant s(G)\}$ the set of all connected components of the graph $\Gamma(G)$, where $s(G)$ is the number of the connected components of $\Gamma(G)$. If the order of G is even, we always assume that $2 \in \pi_1(G)$. The other notation and terminologies in this paper are standard and the reader is referred to ATLAS [4] and [8] if necessary.

It was conjectured by W.J. Shi in 1980s that every simple group S can be determined by $|S|$ and $\pi_e(S)$ (see [15] for example), which has been proved to be true by Shi and Mazurov et al (see [6, 15, 16, 18–20, 22]). In the proof of this characterization, we see that not all elements of $\pi_e(S)$ should be considered. In

Article electronically published on 30 April, 2017.
Received: 22 March 2015, Accepted: 4 December 2015.
*Corresponding author.

fact, sometimes, we only need some special element orders of S. Additionally, Kantor and Seress proved that the characteristic of a simple group S of Lie type of odd characteristic can be determined by $m_1(S)$ and $m_2(S)$ (see [12]). Hence, it is natural to ask that whether a simple group S can be determined by the order of S and some special element orders of S. Along this direction, many interesting results have been obtained. In [9–11, 27], L.G. He and Q.L. Zhang et al proved that the simple K_3-groups and some $L_2(q)$ were determined by their orders together with their largest element orders or second largest element orders. In [13], Li, Shi and Yu showed that each $L_2(p)$ with $p \neq 7$ a prime can be characterized by its order and largest element order.

Our main result is as follows.

Theorem 1.1. *Let G be a group and S be a simple K_4-group of type $L_2(q)$. Then $G \simeq S$ if and only if $|G| = |S|$ and $m_1(G) = m_1(S)$.*

2. Preliminaries

In this section, we first list some information about simple K_4-groups.

By [2, 17], simple K_4-groups consist of the following five classes of groups.

$\mathcal{C}_1$: $L_2(r)$ with r a prime satisfying

$$r^2 - 1 = 2^a 3^b u^c, \tag{2.1}$$

where $a \geqslant 1, b \geqslant 1, c \geqslant 1, u > 3$ a prime.

$\mathcal{C}_2$: $L_2(2^m)$ with

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t, \end{cases} \tag{2.2}$$

where $m \geqslant 1$, u and t are primes and $t > 3$.

$\mathcal{C}_3$: $L_2(3^m)$ with

$$\begin{cases} 3^m - 1 = 2u \\ 3^m + 1 = 4t, \end{cases} \tag{2.3}$$

where $m \geqslant 1$, u and t are primes.

$\mathcal{C}_4$: $L_2(2^4), L_2(5^2), L_2(7^2), L_2(3^4), L_2(3^5)$.

$\mathcal{C}_5$:

$M_{11}, M_{12}, J_2,$

$A_7, A_8, A_9, A_{10},$

$L_3(4), L_3(5), L_3(7), L_3(8), L_3(17),$

$S_4(4), S_4(5), S_4(7), S_4(9),$

$U_3(4), U_3(5), U_3(7), U_3(8), U_3(9),$

$L_4(3), S_6(2), O_8^+(2), G_2(3), U_4(3), U_5(2), {}^3D_4(2), {}^2F_4(2)', Sz(8), Sz(32).$

In addition, there are 8 simple K_3-groups, which are contained in $\mathcal{C}_6$: $A_5, L_2(7), L_2(8), A_6, L_2(17), L_3(3), U_3(3), U_4(2)$.

The following lemma is a special case of Theorem 1.1, which was proved in [13].

Lemma 2.1. *Let G be a group and $p > 3$ a prime. Suppose that $|G| = |L_2(p)| = (p-1)p(p+1)/2$ and $m_1(G)=m_1(L_2(p))=p$. Then G is isomorphic to $L_2(p)$ or to*

$$[[Z_2 \times Z_2 \times Z_2]Z_7]Z_3,$$

a 2-Frobenius group of order 168.

The following lemma is straightforward.

Lemma 2.2. *Suppose that G has a normal series $1 \subseteq H \subseteq K \subseteq G$. Let $\overline{K} = K/H$ be a nonabelian simple group. Then there exists a normal subgroup C of G such that $\overline{K} \lesssim G/C \leqslant Aut(\overline{K})$.*

The next lemma is easy to see and will be used frequently.

Lemma 2.3. *Let G be a group. Let $p, q \in \pi(G)$ such that for every i, j with $1 < q^i \leqslant |G|_q$ and $1 < p^j \leqslant |G|_p$, $p \nmid q^i - 1$ and $q \nmid p^j - 1$. If $pq \notin \pi_e(G)$, then G has a chief factor M/N such that $\{p, q\} \subseteq \pi(M/N)$ and $\{p, q\} \cap \pi(N) = \emptyset$.*

3. Proof of Theorem 1.1

The necessity is obvious and we only need to consider the sufficiency.

By Lemma 2.1 and [11, Theorem 1], we know that the result is true if S is isomorphic to $L_2(2^4)$, $L_2(5^2)$, $L_2(3^4)$ and simple K_4-groups in $\mathcal{C}_1$. Therefore, we discuss the remaining cases in the sequel.

(I) $S = L_2(3^m)$.

By the hypothesis, $|G| = 2^2 3^m tu$, where

$$t = \frac{3^m + 1}{4},\ u = \frac{3^m - 1}{2},\ 3 < t < u.$$

It is easy to see that

$$u = 2t - 1. \tag{3.1}$$

By Table A.1 in [12], it is follows that

$$m_1(G) = m_1(S) = \frac{3^m + 1}{2} = 2t.$$

We claim that G has a chief factor M/N such that $\{t, u\} \subseteq \pi(M/N)$. By Lemma 2.3 and the hypothesis, we only need to show that t does not divide $u - 1$. In fact, if t divides $u - 1$, then for some positive integer k, we have

$$\frac{3^m - 1}{2} - 1 = k \cdot \frac{3^m + 1}{4}.$$

Hence

$$-k-6=(k-2)\cdot 3^m.$$

But, this equation has no solution. Clearly, M/N is a simple non-abelian chief factor, and in particular, t and u are the largest and the second largest primes in $\pi(M/N)$ respectively, which satisfy condition (3.1). Since G is a K_4-group, M/N is a simple K_n-group with $n=3,4$.

$G_2(3)$ and $Sz(8)$ are the only two groups in classes $\mathcal{C}_4$, $\mathcal{C}_5$ and $\mathcal{C}_6$, which satisfy condition (3.1). But the orders of Sylow 2-subgroup of both groups are greater than the order of G, a contradiction. Suppose that M/N is isomorphic to a group $L_2(r)$ in $\mathcal{C}_1$. Then $r=u$. By the order of G and condition (2.1), we have

$$u^2-1=2^2 3^b t, b\leqslant m.$$

It follows from condition (3.1) that

$$(2t-1)^2-1=2^2 3^b t.$$

Thus,

$$t(t-1)=3^b t$$

and

$$t-1=3^b,$$

a contradiction. Assume that M/N is isomorphic to a group in $\mathcal{C}_2$. By condition (2.2), we have

$$u=3t-2. \tag{3.2}$$

Combining (3.1) and (3.2), we have $t=1$, which is impossible.

Therefore, M/N must be isomorphic to a group in $\mathcal{C}_3$ and so one can easily deduce that $G\simeq S$, completing the proof of this case.

(II) $S=L_2(2^m)$.

By the hypothesis, we have $|G|=2^m 3tu$, where

$$t=\frac{2^m+1}{3},\ u=2^m-1,$$

and $u=3t-2$. Note that $m_1(G)=m_1(S)=2^m+1=3t$, by [12] Table A.1. It is clear that t does not divide $u-1$ and so, similar to (I), G has a chief factor M/N such that $\{t,u\}\subseteq\pi(M/N)$. Furthermore, M/N is a simple group. If M/N is isomorphic to a group in classes $\mathcal{C}_4$-$\mathcal{C}_6$, then $M/N\simeq L_4(3)$ or $U_3(8)$, which contradicts the fact that the Sylow 3-subgroups of G have order 3. If M/N is isomorphic to one in $\mathcal{C}_1$, then $t\leqslant 3$. Checking the information shows that $t=1$, a contradiction. Analogous to (I), M/N is not contained in $\mathcal{C}_3$. Thus M/N must be contained in $\mathcal{C}_2$ and therefore $G\simeq S$.

(III) $S=L_2(7^2)$.

It is obvious that $|G|=2^4\cdot 3\cdot 5^2\cdot 7^2$ and $m_1(G)=m_1(S)=\frac{7^2+1}{2}=25$.

We first show that G is insoluble. Note that $35\notin\pi_e(G)$. Replacing $\{p,q\}$ with $\{5,7\}$ in Lemma 2.3 guarantees the existence of a chief factor M/N of

G such that $\{5,7\} \subseteq \pi(M/N)$ and $\{5,7\} \cap \pi(N) = \emptyset$. This implies that G is insoluble. Let K be the largest soluble normal subgroup of G. Write $\overline{G} = G/K$. Again, Lemma 2.3 shows that K is a $\{5,7\}'$-group.

Denote $\overline{L} = Soc(\overline{G})$, where $Soc(\overline{G})$ denotes the socle of $\overline{G}$. Then $\overline{L} = L_1 \times L_2 \times \cdots \times L_r$, where L_i's are nonabelian simple groups. By the order of G and the orders of simple K_n-groups with $n = 3,4$, we have $r = 1$. In fact, $\overline{G}$ is an almost simple group such that $\overline{L} \leqslant \overline{G} \leqslant Aut(\overline{L})$. By [4], $\overline{L}$ is isomorphic to one of

$$L_2(5), L_2(7), L_2(7^2).$$

Suppose that $\overline{L} \simeq L_2(5)$ or $L_2(7)$. Since

$$|Out(L_2(5))| = 2, \quad |Out(L_2(7))| = 2,$$

both 5 and 7 divide the order of K, which contradicts that K is a $\{5,7\}'$-group. Hence $\overline{L}$ must be isomorphic to $L_2(7^2)$ and therefore $G \simeq L_2(7^2)$, as desired.

(IV) $S = L_2(3^5)$.

First, we have $|G| = 2^2 \cdot 3^5 \cdot 11^2 \cdot 61$ and $m_1(G) = m_1(S) = \frac{3^5+1}{2} = 2 \cdot 61$, by Table A.1 in [12].

Since $11 \cdot 61 \notin \pi_e(G)$, Lemma 2.3 implies G has a chief factor M/N such that $\{11, 61\} \subseteq \pi(M/N)$. By the order of G, we see that M/N is simple. By [4], M/N is isomorphic to $L_2(3^5)$ and so is G, as wanted.

Thus, the proof is complete. □

4. Automorphism groups of simple K_4-groups

In the ensuing analysis, we concern the automorphism groups of simple K_4-groups of type $L_2(q)$. It is well known that if $q = p^d$, then $Aut(L_2(q)) \simeq PGL(2,q).Z_d$, where Z_d denotes a cyclic group of order d. In particular, $Aut(L_2(p)) \simeq PGL(2,p)$, where p is a prime. By [7, Table 3], we know that

$$m_1(Aut(L_2(q))) = q + 1.$$

In [13], we show that the automorphism group of $L_2(p)$ with p a prime can be determined by their orders and largest element orders.

Theorem 4.1. *Let G be a group. Then $G \simeq PGL(2,p)$ if and only if $|G| = |PGL(2,p)|$ and $m_1(G){=}m_1(PGL(2,p)){=}p+1$, where p is a prime.*

Lemma 4.2. *Let G be a Frobenius group of even order with H and K its kernel and complement, respectively. Then $s(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$.*

Proof. See [3, Theorem 1]. □

The following lemma is well known.

Lemma 4.3. *Let G be a Frobenius group with kernel F and complement C. Then the following assertions hold.*

(1) *F is a nilpotent group.*

(2) $|F| \equiv 1$ *(mod $|C|$).*

(3) *Every subgroup of C of order $p \cdot q$, with p, q primes (not necessarily distinct), is cyclic. In particular, every Sylow subgroup of C of odd order is cyclic and every Sylow 2-subgroup of C is either cyclic or a generalized quaternion group. If C is a insoluble group, then C has a subgroup of index at most 2 isomorphic to $SL(2,5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|, 30) = 1$.*

Lemma 4.4. *Let G be a finite group with $s(G) \geq 2$ with $2 \in \pi_1(G) := \pi_1$. Then G is one of the following groups:*

(1) *G is a Frobenius or 2-Frobenius group. In particular, a 2-Frobenius group is soluble.*

(2) *G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that H is a nilpotent π_1-group, G/K is a π_1-group and K/H is a finite non-abelian simple group such that $|G/H|$ divides $|Aut(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.*

Proof. See [23]. □

From now on, we discuss the automorphism groups of simple K_4-groups in classes $\mathcal{C}_2 - \mathcal{C}_4$ case by case.

The following lemma will be useful in our discussing.

Lemma 4.5 ([26, Theorem 1.1])**.** *Let G be a simple group of order $2^n \cdot 3 \cdot p_1 \cdot p_2 \cdots p_m$, where $p_1, p_2, \ldots, p_m$ are distinct primes greater than 3. Then G is isomorphic to J_1 or some $L_2(q)$.*

Proposition 4.6. *Let G be a group and S a simple K_4-group of type $L_2(2^m)$. If $m_1(G) = m_1(Aut(S))$ and $|G| = |Aut(S)|$, then $G \simeq Aut(S)$.*

Proof. Since S is a simple K_4-group, by Table 3 in [7], we have

$$m_1(Aut(S)) = 2^m + 1 = 3t.$$

Since $2^m - 1 = u$ is a prime, m is also a prime. By Fermat's little theorem, we have $m|u-1$ and $m|t-1$ and so it is obvious that $m \neq u, t$ and $u \nmid m-1$.

Note that $|G| = |Aut(S)| = 2^m \cdot 3 \cdot t \cdot u \cdot m$ with $m > 3$ and $u = 3t+2$. It is to check that $t \nmid u-1$ by the hypothesis and so, in view of Lemma 2.3, there exists a chief factor M/N of G such that $\{t, u\} \subseteq \pi(M/N)$ and $\pi(N) \cap \{t, u\} = \emptyset$. Obviously, M/N is a simple non-abelian group with order $2^n \cdot 3^i \cdot t^j \cdot u^k \cdot m^l$, where $n \leqslant m$ and $i, j, k, l \in \{0, 1\}$. If $i = 0$, then $3 \nmid |M/N|$ and M/N is a simple K_3-group or K_4-group. Then one can easily check that M/N is isomorphic to $Sz(8)$ with $t = 7, u = 13$ or $Sz(32)$ with $t = 31, u = 41$. In both cases, however, t, u do not satisfy $u = 3t + 2$. Hence, $3 \in \pi(M/N)$. Now, by Lemma

4.5, M/N is isomorphic to J_1 or some $L_2(q)$. Since J_1 is a K_6-group, that is $|\pi(J_1)| = 6$, M/N is isomorphic to some $L_2(q)$. Thus, by direct calculation, we have $M/N \simeq L_2(2^m)$. By Lemma 2.2, there exists a normal subgroup C of G such that $S \leqslant G/C \leqslant Aut(S)$. If $m \in \pi(C)$, then G has an element of order um since $m-1$ is not divisible by u, a contradiction. Hence $C = 1$ and so $G \simeq Aut(S)$, as wanted. □

Proposition 4.7. *Let G be a group and S a simple K_4-group of type $L_2(3^m)$. If $m_1(G) = m_1(Aut(S))$ and $|G| = |Aut(S)|$, then*

$$G \simeq Aut(S) \text{ or } Z_2.L_2(3^m).Z_m.$$

Proof. First, by the hypothesis and Table 3 in [7], we have

$$|G| = |Aut(S)| = 2^3 \cdot 3^m \cdot t \cdot u \cdot m,\ m_1(G) = m_1(Aut(S)) = 3^m + 1 = 4t,$$

where $3^m - 1 = 2u$ with $u > 3$ a prime. It is easy to see that m is a prime and $u = 2t - 1$. By Fermat's little theorem, $m|u-1$ and $m|t-1$ and consequently $m \neq t, u$. By the hypothesis and Lemma 2.3, G/N has a chief factor M/N such that $\pi(N) \cap \{t, u\} = \emptyset$ and $\{t, u\} \subseteq \pi(M/N)$. Clearly, M/N is simple. Since $m_1(G) = 4t$, a Sylow 2-subgroup G_2 of G has a cyclic maximal subgroup of order 4. G_2 can not be cyclic because G is insoluble by above argument. Therefore, G_2 is one of the following types:

(1) the direct product of a cyclic group of order 4 and one of order 2,
(2) the dihedral group D_8,
(3) the quaternion group Q_8.

According to Theorem 8.6, Theorem 8.7 and Theorem 11.1 of Chapter 6 in [21], M/N is isomorphic to A_7 or $L_2(q)$ for an odd prime power $q > 3$. If $M/N \simeq A_7$, then $t = 5, u = 7$, which do not satisfy $u = 2t - 1$. Hence, $M/N \simeq L_2(q)$ for some odd prime power q. Then we can easily obtain $q = 3^m$ and so $M/N \simeq L_2(3^m) = S$.

By Lemma 2.2, there exists a normal subgroup C of G such that $M/N \leqslant G/C \leqslant Aut(M/N)$. Thus, $|C| \leqslant 2m$. If $m \in \pi(C)$, then C_m is normal in G, where C_m is a Sylow m-subgroup of C. Let g be an element of G of order t. Since by Fermat's little theorem $t \nmid m-1$, g acts trivially on C_m and therefore $mt \in \pi_e(G)$, which violates the largest element order in G is $4t$ when $m > 4$. If $m \leqslant 4$, then $m = 3$ since S is a simple K_4-group isomorphic to $L_2(3^3)$. But now, $3 \cdot 13 \in \pi_e(G)$ since an element in G with order 13 acts trivially on C_m. This contradiction shows that 3 does not divide $|C|$. Hence, $m \nmid |C|$.

Assume $|C| = 2$. Then $C \leqslant Z(G)$. It follows that G is isomorphic to

$$Z_2.L_2(3^m).Z_m \text{ or } L_2(3^m).Z_m \times Z_2.$$

Clearly, $m_1(L_2(3^m)) \leqslant m_1(L_2(3^m).Z_m)$ since $L_2(3^m)$ is normal in $L_2(3^m).Z_m$. In fact,

$$m_1(L_2(3^m)) = m_1(L_2(3^m).Z_m).$$

In order to prove the validity of above equality, we investigate the structure of maximal subgroups of $L_2(3^m).Z_m$. Take $q = 3^m$ and $T = L_2(3^m).Z_m$. By [5, Theorem 1.3], the maximal subgroups of $L_2(3^m).Z_m$ are:

(1) $L_2(3^m)$,
(2) $Z_3^m \rtimes Z_{(q-1)/2}$,
(3) $N_T(D_{q-1})$,
(4) $N_T(D_{q+1})$.

The largest element orders of groups in (1) and (2) are less than $2t$. Since

$$N_T(D_{q-1}) \simeq (Z_u \rtimes Z_2) \rtimes Z_m,$$

by [14, Table I], $N_T(D_{q-1})$ has no element of order mu or $2m$ and so $m_1(N_T(D_{q-1})) \leqslant 2t$. For $N_T(D_{q+1}) \simeq (Z_{2t} \rtimes Z_2) \rtimes Z_m$, by [14, Table I], $N_T(D_{q+1})$ has no element of mt and so $m_1(N_T(D_{q+1})) \leqslant 2t$. Hence, by the above argument, $L_2(3^m).Z_m$ and $L_2(3^m)$ have the same largest element order. This means that

$$m_1(L_2(3^m).Z_m \times Z_2) = 2t.$$

Since $Z_2.L_2(3^m).Z_m$ is a non-split central extension and $m_1(L_2(3^m).Z_m) = 2t$, we have

$$m_1(Z_2.L_2(3^m).Z_m) = 4t,$$

which satisfies our assumption on G. Therefore, in this case,

$$G \simeq Z_2.L_2(3^m).Z_m.$$

If $C = 1$, then $G \cong Aut(S)$. Thus, the proof is complete. □

Proposition 4.8. *Let G be a group and S a simple K_4-group in $\mathcal{C}_4$. Suppose that $m_1(G) = m_1(Aut(S))$ and $|G| = |Aut(S)|$. Then G has a unique chief factor M/N isomorphic to S. Furthermore, the following cases hold.*

(1) *If $S \simeq L_2(2^4)$, then $G \simeq Aut(S)$.*
(2) *If $S \simeq L_2(5^2)$, then $G \simeq Aut(S)$ or one of the following:*

$$L_2(5^2).Z_2 \times Z_2,\ Z_2.L_2(5^2) \times Z_2,\ L_2(5^2) \times Z_2 \times Z_2.$$

(3) *If $S \simeq L_2(3^4)$, then $G \simeq Aut(S)$ or one of the following:*

$$Z_2.L_2(3^4).Z_4,\ L_2(3^4).Z_4 \times Z_2,$$
$$L_2(3^4).(Z_2 \times Z_2) \times Z_2,\ L_2(3^4).Z_2 \times Z_2 \times Z_2,$$
$$L_2(3^4) \times Z_2 \times Z_2 \times Z_2,\ Z_2.L_2(3^4) \times Z_2 \times Z_2,\ Z_2.L_2(3^4).Z_2 \times Z_2.$$

(4) *If $S \simeq L_2(7^2)$, then $G \simeq Aut(S)$ or one of the following:*

$$Z_2.L_2(7^2).Z_2,\ L_2(7^2).Z_2 \times Z_2,\ Z_2.L_2(7^2) \times Z_2,\ L_2(7^2) \times Z_2 \times Z_2.$$

(5) *If $S \simeq L_2(3^5)$, then $G \simeq Aut(S)$ or $Z_2.L_2(3^5).Z_5$.*

Proof. We discuss the automorphism groups of simple K_4-groups of type $L_2(q)$ in $\mathcal{C}_4$ case by case.

(I) $S = L_2(2^4)$.

By the hypothesis and Table 3 in [7], we have

$$|G| = |Aut(S)| = 2^6 \cdot 3 \cdot 5 \cdot 17,\ m_1(G) = m_1(Aut(S)) = 2^4 + 1 = 17.$$

Obviously, Since $m_1(G) = 17$, it follows that 17 is an isolated point of $\Gamma(G)$ and $s(G) \geq 2$.

First, note that G is insoluble. This follows directly from Lemma 2.3.

Second, G is not a Frobenius group. If G is a Frobenius group, then by Lemma 4.2 it follows that $s(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$ with H, K its kernel and complement, respectively. Obviously, $\pi(H) = \{2, 3, 5\}$ and $\pi(K) = \{17\}$. By Lemma 4.3, we have $|K| \mid (|H_p| - 1)$, where H_p is a Sylow subgroup of H and $p \in \{2, 3, 5\}$. It follows that 17 divides $|H_5| - 1 = 4$, a contradiction.

Third, as G is insoluble, by Lemma 4.4, we get that G is not a 2$-$Frobenius group.

Hence, it follows from Lemma 4.4 and Table 1 in [25] that G has a chief factor K/H isomorphic to $S = L_2(2^4)$. In addition, $|H| \leqslant 4$.

If $|H| = 1$, then $G \cong Aut(S)$.

If $|H| = 2$, then $H \leqslant Z(G)$. It follows that $2 \cdot 17 \in \pi(G)$, which violates the largest element order in G is 17.

If $|H| = 4$, it follows that H is non-cyclic. Otherwise, $4 \cdot 17 \in \pi(G)$, which violates the largest element order in G is 17. Then H is an elementary abelian and $G/H \simeq S$. Since $H < C_G(H)$, $G = C_G(H)$ and so $H \leqslant Z(G)$. It follows that G has an element of order $2 \cdot 17$, a contradiction.

(II) $S = L_2(5^2)$.

By the hypothesis and Table 3 in [7], we have

$$|G| = |Aut(S)| = 2^5 \cdot 3 \cdot 5^2 \cdot 13,\ m_1(G) = m_1(Aut(S)) = 5^2 + 1 = 26.$$

Clearly, $5 \cdot 13 \notin \pi_e(G)$. By Lemma 2.3 and the hypothesis, G has a chief factor M/N such that $\{5, 13\} \subseteq \pi(M/N)$. Thus, it follows that G has a chief factor M/N isomorphic to $S = L_2(5^2)$. By Lemma 2.2, there exists a normal subgroup C of G such that $L_2(5^2) \leqslant G/C \leqslant Aut(L_2(5^2))$. Hence $|C| \leqslant 4$.

If $|C| = 1$, then $G \simeq Aut(S)$.

If $|C| = 2$, then $C \leqslant Z(G)$ and $G/C \simeq L_2(5^2).Z_2$. Then one can easily obtain that G is isomorphic to $Z_2.L_2(5^2).Z_2$ or $L_2(5^2).Z_2 \times Z_2$. By [4], we see that G can not be isomorphic to the former case since

$$m_1(Z_2.L_2(5^2).Z_2) \neq 26.$$

Since by [4], $m_1(L_2(5^2).Z_2) = 13$ or 26, we have

$$m_1(L_2(5^2).Z_2 \times Z_2) = 26.$$

Hence, we have $G \simeq L_2(5^2).Z_2 \times Z_2$, which satisfies the hypothesis.

Let $|C| = 4$ and suppose that H is cyclic. It follows that that $4 \cdot 13 \in \pi(G)$, which violates the largest element order in G is $2 \cdot 13$. Therefore, H is an elementary abelian group. Hence we have G is isomorphic to

$$L_2(5^2) \times Z_2 \times Z_2 \text{ or } Z_2.L_2(5^2) \times Z_2,$$

both of which have the largest element order 26, because $m_1(L_2(5^2)) = 13$ and $m_1(Z_2.L_2(5^2)) = 26$ (see [4]).

(III) $S = L_2(3^4)$.

By the hypothesis and Table 3 in [7], we have

$$|G| = |Aut(S)| = 2^7 \cdot 3^4 \cdot 5 \cdot 41, \ m_1(G) = m_1(Aut(S)) = 3^4 + 1 = 2 \cdot 41.$$

We first prove that G is insoluble. Since $3 \cdot 41 \notin \pi_e(G)$, by Lemma 2.3, G has a chief factor M/N such that $\{3, 41\} \cap \pi(N) = \emptyset$ and $\{3, 41\} \subseteq \pi(M/N)$. Hence G is insoluble.

Let K be the largest soluble normal subgroup of G. Then it is easy to see that G/K is an almost simple group such that $L/K \leqslant G/K \leqslant Aut(L/K)$, where L/K is simple. Similar to above, we see that L/K can not be a simple K_3-group. Therefore, L/K is a simple K_4-group and moreover, $L/K \simeq L_2(3^4)$. By the order of the outer automorphism of $L_2(3^4)$, we know that $|K| \leqslant 8$. Obviously, $41 \notin \pi(K)$. Now, we distinguish the following cases.

If $|K| = 1$, then $G \cong Aut(S)$.

If $|K| = 2$, then $K \leqslant Z(G)$. It follows that

$$G/K \simeq L_2(3^4).Z_4, \text{ or } G/K \simeq L_2(3^4).(Z_2 \times Z_2).$$

Since $L \simeq Z_2.L_2(3^4)$ or $L_2(3^4) \times Z_2$, we conclude that G is isomorphic to one of the following:

$$Z_2.L_2(3^4).Z_4, \ Z_2.L_2(3^4).(Z_2 \times Z_2), \ (L_2(3^4).Z_4) \times Z_2, \ (L_2(3^4).(Z_2 \times Z_2)) \times Z_2.$$

With the help of Magma [1], we know that

$$m_1(L_2(3^4).Z_4) = 41 \text{ and } m_1(L_2(3^4).(Z_2 \times Z_2)) = 82.$$

Hence, $m_1(Z_2.L_2(3^4).(Z_2 \times Z_2)) = 164$ as $Z_2.L_2(3^4).(Z_2 \times Z_2)$ is a non-split central extension of $L_2(3^4).(Z_2 \times Z_2)$. On the other hand, it is easy to see that

$$Z_2.L_2(3^4).Z_4, \ (L_2(3^4).Z_4) \times Z_2, \ (L_2(3^4).(Z_2 \times Z_2)) \times Z_2$$

have the same largest element order 82, which fulfill our hypothesis on G.

Now suppose that $|K| = 4$. If K is cyclic, it follows that $4 \cdot 41 \in \pi(G)$, which violates our hypothesis for G. Hence K is an elementary abelian group with order 4. Since $G/C_G(K) \lesssim Aut(K)$, $G/C_G(K) = 1$ or $G/C_G(K) \simeq Z_2$. If $G/C_G(K) = 1$, then $K \leqslant Z(G)$. Since $L/K \simeq L_2(3^4)$, we have that $L \simeq L_2(3^4) \times Z_2 \times Z_2$ or $L \simeq Z_2.L_2(3^4) \times Z_2$. Since $G/K \simeq L_2(3^4).Z_2$, it follows that

$$G \simeq L_2(3^4).Z_2 \times Z_2 \times Z_2 \text{ or } G \simeq Z_2.L_2(3^4).Z_2 \times Z_2.$$

With the help of Magma [1], we have that $m_1(L_2(3^4).Z_2) = 41$ or 82. Hence

$$m_1(L_2(3^4).Z_2 \times Z_2 \times Z_2) = 82.$$

However, since $Z_2.L_2(3^4).Z_2$ is a non-split central extension of $L_2(3^4).Z_2$, we have

$$m_1(Z_2.L_2(3^4).Z_2 \times Z_2) = 82,$$

if $m_1(L_2(3^4).Z_2) = 41$, and

$$m_1(Z_2.L_2(3^4).Z_2 \times Z_2) = 164$$

if $m_1(L_2(3^4).Z_2) = 82$.

If $G/C_G(K) \simeq Z_2$, then $C_G(K)/K \simeq L/K$ and so $K \leqslant Z(L)$. Similarly, we get the same conclusion as above.

At last, we assume that $|K| = 8$. Then $G/K \simeq L_2(3^4)$. If $K \simeq Z_8$ or $Z_4 \times Z_2$, then $4 \times 41 \in \pi_e(G)$, which contradicts that $m_1(G) = 2 \times 41$. Suppose that $K \simeq Z_2 \times Z_2 \times Z_2$. Since $G/C_G(K) \lesssim Aut(K)$ and $K \leqslant C_G(K)$, we obtain that $G = C_G(K)$. It follows that

$$G \simeq L_2(3^4) \times Z_2 \times Z_2 \times Z_2 \text{ or } G \simeq Z_2.L_2(3^4) \times Z_2 \times Z_2.$$

It is obvious that $m_1(L_2(3^4) \times Z_2 \times Z_2 \times Z_2) = 82$, since $m_1(L_2(3^4)) = 41$. Because $Z_2.L_2(3^4)$ is a non-split central extension, we have

$$m_1(Z_2.L_2(3^4)) = 82,$$

and so

$$m_1(Z_2.L_2(3^4) \times Z_2 \times Z_2) = 82.$$

If K is a non-abelian group, then $K \simeq D_8$ or Q_8. Since $G/C_G(K) \lesssim Aut(K)$ and $Aut(K) \simeq D_8$ or S_4, 41 divides the order of $C_G(K)$. But K has an element of order 4. Hence $4 \times 41 \in \pi_e(G)$, a contradiction.

Thus, the proof is complete.

(IV) $S = L_2(7^2)$.

By the hypothesis and Table 3 in [7], we have

$$|G| = |Aut(S)| = 2^6 \cdot 3 \cdot 5^2 \cdot 7^2,\ m_1(G) = m_1(Aut(S)) = 7^2 + 1 = 2 \cdot 5^2.$$

We assert that G is insoluble. Otherwise, G contains a Hall $\{5, 7\}-$subgroup H of order $5^2 \cdot 7^2$. Since $m_1(G) = 2 \cdot 5^2$, H_5 is a cyclic subgroup of order 25, where H_5 is a Sylow 5-subgroup of H. Let H_7 be a Sylow 7-subgroup of H. Then H_7 is normal in H as H_5 is cyclc. Since $(5, |Aut(H_7)|) = 1$, G has an element of order $5^2 \cdot 7$, which contradicts that $m_1(G) = 2 \cdot 5^2$.

Let K be the largest soluble normal subgroup of G and $\overline{G} = G/K$. Then, by the hypothesis and the orders of the simple K_3-groups and simple K_4-groups, we know that G/K has a unique minimal non-abelian simple subgroup L/K such that $L/K \leqslant G/K \leqslant Aut(L/K)$. By [4], L/K is isomorphic to one of $L_2(2^2),\ L_2(7),\ L_2(7^2)$. If $L/K \simeq L_2(4)$, then $|K| = 2^i \cdot 5 \cdot 7^2$, where $i = 3, 4$. Let H/K be a Sylow 5-subgroup of G/K. Then H is a soluble group and, as above, $5^2 \cdot 7 \in \pi_e(H)$, a contradiction by the hypothesis. If $L/K \simeq L_2(7)$, then $|K| = 2^i \cdot 5^2 \cdot 7$, where $i = 2, 3$. Similarly, we can deduce a contradiction. Hence, L/K must be isomorphic to $L_2(7^2)$. Note that $|K| \leqslant 4$.

If $|K| = 1$, then $G \simeq Aut(L_2(7^2))$.

If $|K| = 2$, then $K \leqslant Z(G)$. Then $L \simeq Z_2.L_2(7^2)$ or $L_2(7^2) \times Z_2$. It follows that G is isomorphic to

$$Z_2.L_2(7^2).Z_2 \text{ or } L_2(7^2).Z_2 \times Z_2.$$

By Magma [1], $m_1(L_2(7^2).Z_2) = 25$ or 50. Hence, if

$$m_1(L_2(7^2).Z_2) = 25,$$

then

$$m_1(Z_2.L_2(7^2).Z_2) = 50,$$

since $Z_2.L_2(7^2).Z_2$ is a non-split central extension of $L_2(7^2).Z_2$. However, in both cases, we always have

$$m_1(L_2(7^2).Z_2 \times Z_2) = 50.$$

Assume that $|K| = 4$. Then, as above, K must be an elementary abelian group. Notice that $G/K \simeq L_2(7^2)$. It is easy to see that $K \leqslant Z(G)$. Hence, $G \simeq Z_2.L_2(7^2) \times Z_2$ or $L_2(7^2) \times Z_2 \times Z_2$. Since $m_1(L_2(7^2)) = 25$ and $Z_2.L_2(7^2)$ is a non-split central extension, $m_1(Z_2.L_2(7^2)) = 50$, therefore

$$m_1(Z_2.L_2(7^2) \times Z_2) = 50.$$

In addition, it is obvious that

$$m_1(L_2(7^2) \times Z_2 \times Z_2) = 50.$$

(V) $S = L_2(3^5)$.

By the hypothesis and Table 3 in [7], we have

$$|G| = |Aut(S)| = 2^3 \cdot 3^5 \cdot 5 \cdot 11^2 \cdot 61, \ m_1(G) = m_1(Aut(S)) = 3^5 + 1 = 2^2 \cdot 61.$$

By Lemma 2.3 and the hypothesis, we know that G has a chief factor M/N isomorphic to $S = L_2(3^5)$ and $\{11, 61\} \subseteq \pi(M/N)$. Also, by Lemma 2.2, there exists a normal subgroup C of G such that $M/N \leqslant G/C \leqslant Aut(M/N)$. Hence $|C| \leqslant 10$. If $5 \in \pi(C)$, then G has an element of order $5 \cdot 61$, which violates the largest element order in G is $4 \cdot 61$. Hence $|C| \leqslant 2$. If $|C| = 1$, then $G \simeq Aut(S)$. If $|C| = 2$, then $C \leqslant Z(G)$ and $G/C \simeq L_2(3^5).Z_5$. It follows that G is isomorphic to

$$Z_2.L_2(3^5).Z_5 \text{ or } L_2(3^5).Z_5 \times Z_2.$$

Since $m_1(L_2(3^5).Z_5) = 2 \cdot 61$ with the help of Magma [1], we know that

$$m_1(L_2(3^5).Z_5 \times Z_2) = 2 \cdot 61,$$

while

$$m_1(Z_2.L_2(3^5).Z_5) = 2^2 \cdot 61$$

for that $Z_2.L_2(3^5).Z_5$ is a non-split central extension. Hence G must be isomorphic to $Z_2.L_2(3^5).Z_5$.

Thus, the proof of this result is complete. □

Remark 4.9. In Proposition 4.8, the largest element orders of the groups of types $Z_2.L_2(3^4).Z_2$ and $Z_2.L_2(7^2).Z_2$ are not unique, which have been mentioned in the foregoing argument. In fact, by Magma, we have

$$m_1(Z_2.L_2(3^4).Z_2) = 82 \text{ or } 164,$$
$$m_1(Z_2.L_2(7^2).Z_2) = 50 \text{ or } 100.$$

Acknowledgements

We cordially thank an anonymous referee for suggesting to us the Lemma 2.3 and for other helpful comments which greatly improved the exposition. This work was partially supported by the NNSF of China (11501071, 11671063, 11671324) and the Natural Science Foundation Project of CSTC (cstc2014jcyjA 00054, cstc2015jcyjA00052).

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(Jinbao Li) Department of Mathematics and Chongqing Key Laboratory of GGTA, Chongqing University of Arts and Sciences, Chongqing 402160, P.R. China.
E-mail address: leejinbao25@163.com

(Dapeng Yu) Department of Mathematics and Chongqing Key Laboratory of GGTA, Chongqing University of Arts and Sciences, Chongqing 402160, P.R. China.
E-mail address: yudapeng0@sina.com

(Guiyun Chen) School of Mathematics and Statistics, Southwest University, Chongqing 400715, P.R. China.
E-mail address: gychen1963@163.com

(Wujie Shi) Department of Mathematics and Chongqing Key Laboratory of GGTA, Chongqing University of Arts and Sciences, Chongqing 402160, P.R. China.
E-mail address: wjshi@suda.edu.cn