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Author(s):

M. Nasehi

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ON 5-DIMENSIONAL 2-STEP HOMOGENEOUS RANDERS NILMANIFOLDS OF DOUGLAS TYPE

M. NASEHI

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Abstract. In this paper we first obtain the non-Riemannian Randers metrics of Douglas type on two-step homogeneous nilmanifolds of dimension five. Then we explicitly give the flag curvature formulae and the S-curvature formulae for the Randers metrics of Douglas type on these spaces. Moreover, we prove that the only simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type which are Ricci-quadratic have a three-dimensional centre. We also prove that all simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type are never weakly symmetric. The existence of homogeneous Randers spaces of Douglas type with vanishing S-curvature which are never g.o. Finsler spaces is also proved and some examples of locally projectively flat Finsler spaces are also obtained.
Keywords: Two-step homogeneous nilmanifolds, Randers metrics of Douglas type, g.o. Finsler spaces, weakly symmetric spaces.
MSC(2010): Primary: 53C30; Secondary: 53C60.

1. **Introduction**

A two-step nilpotent Lie group which is equipped with a left-invariant Riemannian metric is called a two-step homogeneous nilmanifold [6]. These spaces play an important role in Lie groups, mathematical physics and geometrical analysis. In [11] Lauret classified all homogeneous nilmanifolds (not necessary two-step) of dimension three and four, up to isometry. Then Kowalski and Homolya in [6] gave a classification of two-step homogeneous nilmanifolds of dimension five. Also in [13] we gave a complete classification of totally geodesic and parallel hypersurfaces of these spaces and in [1, 3] we studied some geometrical properties of a class of solvable Lie groups which can be considered as a generalization of nilpotent Lie groups.
On the other hand in the recent years left-invariant Randers metrics, which

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are a generalization of left-invariant Riemannian metrics, on two-step nilpotent Lie groups of dimension five have been investigated. For example, in [14] it is proved that the only two-step nilpotent Lie groups of dimension five which admit a left-invariant Randers metric of Berwald type have a three dimensional centre. In [2] we extended this result for the Randers metrics of Douglas type and proved that each two-step nilpotent Lie group of dimension five admits a left-invariant non-Rimannian Randers metric of Douglas type. Our aim in the present paper is to investigate some curvature properties of these spaces with the Randers metrics of Douglas type.

In Section 2 we report the classification of simply connected five-dimensional two-step homogeneous nilmanifolds and their invariant connections. In Section 3 we explicitly obtain the flag curvature formulae for the Randers metrics of Douglas type on these homogeneous spaces which can be considered as a generalization of the flag curvature formula which is given in [14] for the Randers metrics of Berwald type. Also as an application we obtain some examples of locally projectively flat Finsler spaces. In Section 4 we explicitly give the S-curvature formulae for these spaces with the Randers metrics of Douglas type and apply these formulae to prove the existence of homogeneous Randers spaces of Douglas type with vanishing S-curvature which are never g.o. Finsler spaces. Moreover, we prove that the only simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type which are Ricci-quadratic have a three-dimensional centre. Also, we prove that all simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type are not weakly symmetric spaces.

2. Randers metrics of Douglas type on two-step homogeneous nilmanifolds of dimension five

Let N be a simply connected five-dimensional two-step nilpotent Lie group and g be a left-invariant Riemannian metric on N, which corresponds to an inner product $\langle , \rangle$ on the Lie algebra $\mathcal{N}$ of N. As we already mentioned in the introduction, (N, g) is called a simply connected five-dimensional two-step homogeneous nilmanifold. In order to obtain the Randers metrics of Douglas type on these spaces we recall the classification of these spaces which is given in [6] and their invariant connections which are given in [14].

($\mathbf{A}_1$) Lie algebras with 1-dimensional centre: In this type there exists an orthonormal basis $\{X_1, \dots, X_5\}$ of $\mathcal{N}$, such that the non-zero Lie brackets are

$$[X_1, X_2] = \lambda X_5, \ [X_3, X_4] = \mu X_5, \tag{2.1}$$

where $\lambda \geq \mu > 0$ and $\{X_5\}$ is a basis for the centre of $\mathcal{N}$. Moreover, the non-zero connection components are given by

$$\nabla_{X_1} X_2 = -\nabla_{X_2} X_1 = \frac{\lambda}{2} X_5, \qquad \nabla_{X_1} X_5 = \nabla_{X_5} X_1 = \frac{-\lambda}{2} X_2,$$

$$\nabla_{X_2}X_5 = \nabla_{X_5}X_2 = \frac{\lambda}{2}X_1, \qquad \nabla_{X_3}X_4 = -\nabla_{X_4}X_3 = \frac{\mu}{2}X_5,$$
$$\nabla_{X_3}X_5 = \nabla_{X_5}X_3 = -\frac{\mu}{2}X_4, \qquad \nabla_{X_4}X_5 = \nabla_{X_5}X_4 = \frac{\mu}{2}X_3. \tag{2.2}$$

(A_2) Lie algebras with 2-dimensional centre: In this type there exists an orthonormal basis $\{X_1, \ldots, X_5\}$ of $\mathcal{N}$, such that the non-zero Lie brackets are

$$[X_1, X_2] = \lambda X_4, \ [X_1, X_3] = \mu X_5, \tag{2.3}$$

where $\lambda \geq \mu > 0$ and $\{X_4, X_5\}$ is a basis for the centre of $\mathcal{N}$. Also the non-zero connection components are given by

$$\nabla_{X_1}X_2 = -\nabla_{X_2}X_1 = \frac{\lambda}{2}X_4, \qquad \nabla_{X_1}X_3 = -\nabla_{X_3}X_1 = \frac{\mu}{2}X_5,$$
$$\nabla_{X_1}X_4 = \nabla_{X_4}X_1 = -\frac{\lambda}{2}X_2, \qquad \nabla_{X_1}X_5 = \nabla_{X_5}X_1 = -\frac{\mu}{2}X_3,$$
$$\nabla_{X_2}X_4 = \nabla_{X_4}X_2 = \frac{\lambda}{2}X_1, \qquad \nabla_{X_3}X_5 = \nabla_{X_5}X_3 = \frac{\mu}{2}X_1.$$

(A_3) Lie algebras with 3-dimensional centre: In this type there exists an orthonormal basis $\{X_1, \ldots, X_5\}$ of $\mathcal{N}$, such that the non-zero Lie bracket is

$$[X_1, X_2] = \lambda X_3, \tag{2.4}$$

where $\lambda > 0$ and $\{X_3, X_4, X_5\}$ is a basis for the centre of $\mathcal{N}$. Also, the non-zero connection components are given by

$$\nabla_{X_1}X_2 = -\nabla_{X_2}X_1 = \frac{\lambda}{2}X_3, \ \nabla_{X_1}X_3 = \nabla_{X_3}X_1 = -\frac{\lambda}{2}X_2,$$
$$\nabla_{X_2}X_3 = \nabla_{X_3}X_2 = \frac{\lambda}{2}X_1. \tag{2.5}$$

Here we recall that a Randers metric F on a smooth manifold M can be written as

$$F(x, y) = \sqrt{a_x(y, y)} + a_x(y, V), \ x \in M, y \in T_xM, \tag{2.6}$$

where $a = a_{ij}dx^i \otimes dx^j$ is the Riemannian metric and V is a vector field with $a_x(V, V) < 1$. Since the Riemannian metric a induces a bijection between 1-forms and vector fields on M, a Randers metric F can be expressed as a general form $F = \alpha + \beta$, where α is the underling Riemannian metric on M and β is a smooth 1-form on M which for all $x \in M$ the length of β with respect to α satisfies $\|\beta\|_x < 1$. If for a Randers metric $F = \alpha + \beta$ on M the 1-form β is closed, then the Randers metric F is said to be of Douglas type. In the case that $M = G$ is a Lie group with a left invariant Randers metric F, where F is defined by an inner product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$ of G and the left-invariant vector field V, then F is of Douglas type if and only if V satisfies in the following condition

$$\langle [m,n], V\rangle = 0, \text{ for all } m,n \in \mathcal{G}. \tag{2.7}$$

For more details see [5, Proposition 7.4]. Here we obtain the exact form of the Randers metrics of Douglas type F on simply connected five-dimensional two-step homogeneous nilmanifolds.

Theorem 2.1. *Let (N, g) be a simply connected five-dimensional two-step homogeneous nilmanifold and $\langle,\rangle$ be an inner product on the the Lie algebra $\mathcal{N}$ of N corresponding to the left-invariant metric g. Then the non-Riemannian Randers metrics of Douglas type on (N, g) are given as follows*

(a) ***For the type* ($\mathbf{A}_1$)*:*** *F is defined by an inner product $\langle,\rangle$ on $\mathcal{N}$ and $V = a_1X_1 + \ldots + a_4X_4$, more precisely,*

$$F(s) = \sqrt{\sum_{i=1}^{5} K_i^2} + \sum_{i=1}^{4} K_i a_i, \text{ with } 0 < \sqrt{\sum_{i=1}^{4} a_i^2} < 1, \tag{2.8}$$

(b) ***For the type* ($\mathbf{A}_2$)*:*** *F is defined by an inner product $\langle,\rangle$ on $\mathcal{N}$ and $V = a_1X_1 + a_2X_2 + a_3X_3$, more precisely,*

$$F(s) = \sqrt{\sum_{i=1}^{5} K_i^2} + \sum_{i=1}^{3} K_i a_i, \text{ with } 0 < \sqrt{\sum_{i=1}^{3} a_i^2} < 1, \tag{2.9}$$

(c) ***For the type* ($\mathbf{A}_3$)*:*** *F is defined by an inner product $\langle,\rangle$ on $\mathcal{N}$ and $V = a_1X_1 + a_2X_2 + a_4X_4 + a_5X_5$, more precisely,*

(2.10)

$$F(s) = \sqrt{\sum_{i=1}^{5} K_i^2} + \sum_{i=1}^{2} K_i a_i + \sum_{i=4}^{5} K_i a_i, \text{ with } 0 < \sqrt{\sum_{i=1}^{2} a_i^2 + \sum_{i=4}^{5} a_i^2} < 1,$$

where $s = K_1X_1 + \ldots + K_5X_5$ is a vector in the Lie algebra $\mathcal{N}$ of N.

Proof. Type (A_1): In this type we assume that $V = a_1X_1 + \ldots + a_5X_5$. Then the formula (2.7) and the Lie bracket operations given in (2.1) give us $V = a_1X_1 + \ldots + a_4X_4$. Then by replacing V and s in the formula (2.6) we get the result. Types (A_2) and (A_3) can be proved in a similar way. □

Convention 2.2. A simply connected five-dimensional two-step homogeneous nilmanifold (N, g) which is equipped with the Randers metrics of Douglas type given in Theorem 2.1 is said to be a simply connected five-dimensional two-step homogeneous Randers nilmanifold of Douglas type.

We note that if a simply connected five-dimensional two-step homogeneous Randers nilmanifold of Douglas type (N, F) is Riemannian i.e., $V = 0$, then the above definition is just the definition of a simply connected five-dimensional two-step homogeneous nilmanifold.

3. Flag curvatures on 5-dimensional two-step homogeneous Randers nilmanifolds of Douglas type

The flag curvature which generalizes the sectional curvature is one of the quantities that helps us to enrich our knowledge about the curvature properties of spaces. Here as a generalization of [14] we obtain these formulaes for simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type.

Theorem 3.1. *Let (N, g) be a simply connected five-dimensional two-step homogeneous nilmanifold. Also, let (P, s) be a flag in the Lie algebra $\mathcal{N}$ of N such that $\{s = K_1X_1 + \ldots + K_5K_5, t = K_1'X_1 + \ldots + K_5'X_5\}$ is an orthonormal basis of P with respect to the inner product $\langle , \rangle$ on $\mathcal{N}$.*
***Case 1**. If $\mathcal{N}$ has type (A_1) with the non-Riemannian Douglas metric (2.8), then the flag curvature of the flag (P, s) in T_eN is given by*

$$K(P, s) = \frac{3(a_1\lambda K_2K_5 - a_2\lambda K_1K_5 + a_3\mu K_4K_5 - a_4\mu K_5K_3)^2 + \sum_{i=1}^5 K_i^2}{4(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^4 K_ia_i)^4}$$
$$\frac{\times(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^4 K_ia_i)^2[6\lambda\mu(K_1K_2' - K_2'K_1)(K_3K_4' - K_4K_3')}{4(\sqrt{\sum K_i^2} + \sum_{i=1}^4 K_ia_i)^4}$$
$$\frac{-3(\lambda^2(K_1K_2' - K_2K_1')^2 + \mu^2(K_3K_4' - K_3'K_4)^2) + \lambda^2((K_5K_1' - K_1K_5')^2}{4(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^4 K_ia_i)^4}$$
$$\frac{+(K_5K_2' - K_5'K_2)^2) + \mu^2((K_5K_3' - K_3K_5')^2 + (K_5K_4' - K_4K_5')^2)]-}{4(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^4 K_ia_i)^4}$$
(3.1)
$$\frac{2(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^4 K_ia_i)(K_5^2)(-\lambda^2(K_1a_1 + K_2a_2) - \mu^2(K_3a_3 + K_4a_4))}{4(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^4 K_ia_i)^4}.$$

***Case 2.** If $\mathcal{N}$ has type (A_2) with the non-Riemannian Douglas metric (2.9), then the flag curvature of the flag (P, s) in T_eN is determined by*

$$K(P,s) = \frac{3(a_1(\lambda K_2K_4 + \mu K_3K_5) - a_2\lambda K_1K_4 - a_3\mu K_1K_5)^2 - 2(\sqrt{\sum_{i=1}^5 K_i^2}}{4(\sqrt{\sum_{i=1}^5 K_i^2} + \sum_{i=1}^3 K_ia_i)^4}$$

$$\frac{+\sum_{i=1}^{3}K_ia_i)[(-\lambda^2K_4^2-\mu^2K_5^2)K_1a_1-(\lambda^2K_2K_4^2+\lambda\mu K_3K_5K_4)a_2}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{3}K_ia_i)^4}$$

$$\frac{-(\mu\lambda K_2K_4K_5+\mu^2K_3K_5^2)a_3]+(\sum_{i=1}^{5}K_i^2)(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{3}K_ia_i)^2\times}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{3}K_ia_i)^4}$$

$$\frac{[(2\lambda\mu(K_5K_2'-K_2K_5')(K_4K_3'-K_3K_4')+(K_3K_2'-K_2K_3')(K_4K_5'-K_5K_4'))}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{3}K_ia_i)^4}$$

$$\frac{-3(\lambda^2(K_2K_1'-K_1K_2')^2+\mu^2(K_3K_1'-K_1K_3')^2)+\lambda^2((K_4K_1'-K_1K_4')^2}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{3}K_ia_i)^4}$$

$$\frac{+(K_4K_2'-K_2K_4')^2)+\mu^2((K_5K_1'-K_1K_5')^2+(K_5K_3'-K_3K_5)^2)]}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{3}K_ia_i)^4}. \tag{3.2}$$

Case 3. *If $\mathcal{N}$ has type* (A_3) *with the non-Riemannian Douglas metric* (2.10), *then the flag curvature of the flag (P,s) in T_eN is given by*

$$K(P,s)=\frac{2\lambda^2K_3^2(K_1a_1+K_2a_2)(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{2}K_ia_i+\sum_{i=4}^{5}K_ia_i)+}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{2}K_ia_i+\sum_{i=4}^{5}K_ia_i)^4}$$

$$\frac{(\sum_{i=1}^{5}K_i^2)(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{2}K_ia_i+\sum_{i=4}^{5}K_ia_i)^2\lambda^2[(K_3K_1'-K_1K_3')^2}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{2}K_ia_i+\sum_{i=4}^{5}K_ia_i)^4}$$

$$\frac{+(K_3K_2'-K_2K_3')^2-3(K_2K_1'-K_1K_3')^2]+3(\lambda K_2K_3a_1-\lambda K_1K_3a_2)^2}{4(\sqrt{\sum_{i=1}^{5}K_i^2}+\sum_{i=1}^{2}K_ia_i+\sum_{i=4}^{5}K_ia_i)^4}.$$

Proof. **Type ($\mathbf{A}_1$)**. In this type by equation (2.1) and the following equation

$$2\langle U(X,Y),Z\rangle=\langle[Z,X],Y\rangle+\langle[Z,Y],X\rangle, \tag{3.3}$$

where $X,Y,Z\in\mathcal{N}$ we obtain

$$U(s,t)=\frac{1}{2}\{\lambda(K_2'K_5+K_5'K_2)X_1-\lambda(K_1'K_5+K_5'K_1)X_2 \\ +\mu(K_4'K_5+K_4K_5')X_3-\mu(K_5K_3'+K_5'K_3)X_4\},$$

which implies that

$$\langle U(s,s),V\rangle=a_1\lambda K_2K_5-a_2\lambda K_1K_5+a_3\mu K_4K_5-a_4\mu K_5K_3, \tag{3.4}$$

where V is given in (2.8). Also, we find that

$$\langle U(s,U(s,s),V\rangle=\frac{1}{2}K_5^2\{-\lambda^2(K_1a_1+K_2a_2)-\mu^2(K_3a_3+K_4a_4)\}. \tag{3.5}$$

Then we can use the following formula which is given in [4, Theorem 2.1]

$$K(P,s)=\frac{\langle s,s\rangle^2}{F^2(s)}\bar{K}+\frac{1}{4F(s)^4}(3\langle U(s,s),V\rangle^2-4F(s)\langle U(s,U(s,s)),V\rangle), \tag{3.6}$$

where $\bar{K}$ is the sectional curvature of the left-invariant Riemannian metric g. If we replace (2.8), (3.4) and (3.5) in (3.6) and use the sectional curvature given in [14], then we obtain the formula (3.1).

Type (A$_2$). In this type by equations (2.3) and (3.3) we obtain

$$\begin{aligned} U(s,t)=&\frac{1}{2}\{(\lambda(K_2'K_4+K_2K_4')+\mu(K_3'K_5+K_5'K_3))X_1\\ &-\lambda(K_1'K_4+K_4'K_1)X_2-\mu(K_1'K_5+K_1K_5')X_3\}, \end{aligned}$$

which implies that

$$\langle U(s,s),V\rangle=a_1(\lambda K_2K_4+\mu K_3K_5)-a_2\lambda K_1K_4-a_3\mu K_1K_5, \tag{3.7}$$

where V is given in (2.9). Also, we find that

$$\begin{aligned} \langle U(s,U(s,s)),V\rangle=&\frac{1}{2}\{(-\lambda^2K_4^2-\mu^2K_5^2)K_1a_1-(\lambda^2K_2K_4^2+\lambda\mu K_3K_5K_4)a_2\\ &-(\mu\lambda K_2K_4K_5+\mu^2K_3K_5^2)a_3\}. \end{aligned} \tag{3.8}$$

Then if we replace the equations (2.9), (3.7) and (3.8) in the equation (3.6) and use the sectional curvature which is given in [14] we obtain the equation (3.2).

Type (A$_3$). In this type by equations (2.4) and (3.3) we obtain that

$$U(s,t)=\frac{1}{2}\{(\lambda(K_2K_3'+K_2'K_3)X_1-\lambda(K_1K_3'+K_1'K_3)X_2\},$$

which implies that

$$\langle U(s,s),V\rangle=\lambda K_2K_3a_1-\lambda K_1K_3a_2, \tag{3.9}$$

where V is given in (2.10). Also, we obtain that

$$\langle U(s,U(s,s)),V\rangle=-\frac{\lambda^2}{2}K_3^2(K_1a_1+K_2a_2). \tag{3.10}$$

By replacing the equations (2.10), (3.9) and (3.10) in the equation (3.6) and using the sectional curvature which is given in [14] we have the result. □

In order to obtain some examples of locally projective flat Finsler spaces we recall the following theorem from [12].

Theorem 3.2. *A Finsler metric F on a manifold M ($\dim M\geq 3$) is locally projectively flat if and only if F is a Douglas metric with scalar flag curvature.*

So as a consequence of Theorems 3.1 and 3.2 we get the following result.

Corollary 3.3. *All simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type are locally projectively flat Finsler spaces.*

4. S-curvatures and their applications on 5-dimensional 2-step homogeneous Randers nilmanifolds of Douglas type

In this section to investigate the S-curvature formulae on simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type we recall some facts from [5]. Suppose that G is a Lie group with a left-invariant Randers metric F which is defined by an inner product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$ of G and the left-invariant vector field V. Then the S-curvature is given by

$$S(e, y) = \frac{n+1}{2}\left\{\frac{\langle [V, y], \langle y, V\rangle V - y\rangle}{F(y)} - \langle [V, y], V\rangle\right\}, \tag{4.1}$$

and the Randers metric F has vanishing S-curvature if and only if the linear endomorphism $ad(V)$ of $\mathcal{G}$ is skew symmetric with respect to the inner product $\langle , \rangle$. (See [5, Proposition 7.5].)

Theorem 4.1. *Let (N, g) be a simply connected two-step homogeneous nilmanifold of dimension five. Then*

(a) *The S-curvature on the type (A_1) with the non-Riemannian Douglas metric (2.8) is given by*

$$S(e, y) = -3\left\{\frac{K_5(\lambda a_1 K_2 - a_2\lambda K_1 + \mu a_3 K_4 - \mu a_4 K_3)}{\sqrt{\sum_{i=1}^{5} K_i^2} + \sum_{i=1}^{4} K_i a_i}\right\}, \tag{4.2}$$

(b) *The S-curvature on the type (A_2) with the non-Riemannian Douglas metric (2.9) is determined by*

$$S(e, y) = 3\left\{\frac{-\lambda a_1 K_4 K_2 + a_2\lambda K_1 K_4 - \mu a_1 K_3 K_5 + \mu a_3 K_1 K_5}{\sqrt{\sum_{i=1}^{5} K_i^2} + \sum_{i=1}^{3} K_i a_i}\right\}, \tag{4.3}$$

(c) *The S-curvature on the type (A_3) with the non-Riemannian Douglas metric (2.10) is given by*

$$S(e, y) = 3\left\{\frac{-K_3\lambda(a_1 K_2 - a_2 K_1)}{\sqrt{\sum_{i=1}^{5} K_i^2} + \sum_{i=1}^{2} K_i a_i + \sum_{i=4}^{5} K_i a_i}\right\}, \tag{4.4}$$

where $y = K_1X_1 + \ldots + K_5X_5$ is a vector in the Lie algebra $\mathcal{N}$ of N.

Proof. **Type $(\mathbf{A}_1)$:** In this type by the representation for V which is given in (2.8) we obtain that

$$[V, y] = (\lambda a_1 K_2 - a_2\lambda K_1 + \mu a_3 K_4 - \mu a_4 K_3)X_5, \ [[V, y], V] = 0$$

$$\text{and } \langle y, V\rangle V - y = \sum_{i=1}^{4}((\sum_{i=1}^{4} a_i K_i)a_i - K_i)X_i - K_5X_5, \tag{4.5}$$

which implies that

$$\langle [V, y], \langle y, V\rangle V - y\rangle = -K_5(\lambda a_1 K_2 - a_2\lambda K_1 + \mu a_3 K_4 - \mu a_4 K_3). \tag{4.6}$$

If we replace the equations (2.8), (4.5) and (4.6) in the equation (4.1), then we obtain the equation (4.2).

Type (A$_2$): In this type by the representation for V which is given in (2.9) we find that

$$[V,y]=\lambda(a_1K_2-a_2K_1)X_4+\mu(a_1K_3-a_3K_1)X_5,\ [[V,y],V]=0,$$

$$\text{and } \langle y,V\rangle V-y=\sum_{i=1}^{3}((\sum_{i=1}^{3}a_iK_i)a_i-K_i)X_i-K_4X_4-K_5X_5. \tag{4.7}$$

These equations yield that

$$\langle[V,y],\langle y,V\rangle V-y\rangle=-\lambda a_1K_4K_2+a_2\lambda K_1K_4-\mu a_1K_3K_5+\mu a_3K_1K_5. \tag{4.8}$$

Replacing the equations (2.9), (4.7) and (4.8) in the equation (4.1) gives us the equation (4.3).

Type (A$_3$): In this type by the representation for V which is given in (2.10) we obtain that

$$[V,y]=\lambda(a_1K_2-a_2K_1)X_3,\ [[V,y],V]=0,$$

$$\text{and } \langle y,V\rangle V-y=\sum_{i=1}^{2}\left(\left(\sum_{i=1}^{2}a_iK_i+\sum_{i=4}^{5}a_iK_i\right)a_i-K_i\right)X_i-K_3X_3$$

$$+\sum_{i=4}^{5}\left(\left(\sum_{i=1}^{2}a_iK_i+\sum_{i=4}^{5}a_iK_i\right)a_i-K_i\right)X_i, \tag{4.9}$$

These equations give us

$$\langle[V,y],\langle y,V\rangle V-y\rangle=-K_3\lambda(a_1K_2-a_2K_1). \tag{4.10}$$

Then by replacing the equations (2.10), (4.9) and (4.10) in the equation (4.1) we get the result. □

As it can be seen the S-curvature formula (4.1) for the case $a_1=a_2=0$ is equal to $S=0$ which helps us to obtain some curvature properties of these spaces. So here we give a proof for this fact.

Theorem 4.2. *The only simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type (N,F) with vanishing S-curvature have a three-dimensional centre.*

Proof. **Type (A$_1$):** In this type we assume that F has the vanishing S-curvature. Then $ad(V)$, where $V=a_1X_1+\ldots+a_4X_4$ is skew-symmetric with respect to the inner product $\langle,\rangle$. This implies that we have

$$\langle[V,X],Y\rangle+\langle[V,Y],X\rangle=0, \tag{4.11}$$

where X, and Y are in the Lie algebra $\mathcal{N}$ of N. If we replace (X,Y) in (4.11) by $(X_2,X_5),(X_1,X_5)$, and (X_3,X_5), (X_4,X_5), where $\{X_1,\ldots,X_5\}$ is

an orthonormal basis of $\mathcal{N}$ and use (2.1), then we have $a_1 = a_2 = a_3 = a_4 = 0$ which give us the contradiction $V = 0$.
Type (A$_2$): In this type by an argument similar to the type (A$_1$) we have the result.
Type (A$_3$): In this type by considering $V = a_1X_1 + a_2X_2 + a_4X_4 + a_5X_5$ which is given in (2.10), it can be seen that the equation (4.11) for all $X, Y \in \mathcal{N}$ is satisfied if and only if $a_1 = a_2 = 0$. Then the Randers metric F which is defined by an inner product $\langle , \rangle$ and the left-invariant vector field $V = a_4X_4 + a_5X_5$ has the vanishing S-curvature which gives us the result. □

Recall that a Finsler metric is said to be Ricci-quadratic if its Ricci curvature $Ricc(x, y)$ is quadratic in y. By [5, Theorem 7.9] a homogeneous Randers space is Ricci quadratic if and only if it is of Berwald type which yields the following result.

Theorem 4.3. *The only simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type (N, F) which are Ricci-quadratic have a three-dimensional centre.*

Proof. **Type (A$_1$):** In this type we assume that (N, F) is Ricci-quadratic. Then by [5, Theorem 7.9] F is of Berwald type. This implies that $V = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$, which is given in (2.8), is parallel with respect to $\langle , \rangle$, i.e., for all $A \in \mathcal{N}$ we have $\nabla_A V = 0$. But by the equation (2.2) this gives us the contradiction $V = 0$.
Type (A$_2$): In this type by a similar argument given for the type (A_1) we have the result.
Type (A$_3$): In this type we assume that (N, F) is not Ricci-quadratic. Then by [5, Theorem 7.9] F is not of Berwald type, i.e., $V = a_1X_1 + a_2X_2 + a_4X_4 + a_5X_5$ given in (2.10) is not parallel. But by the equation (2.5) in the case that $a_1 = a_2 = 0$, V is parallel which gives us a contradiction. □

Suppose that the Randers metric F on the Lie group G is defined by an inner product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$ of G and a left invariant vector field V, if for all $X, Y, Z \in \mathcal{G}$ we have $\langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle = 0$ and $\langle [X, Y], V\rangle = 0$, then (G, F) is naturally reductive. (For more details see [2, Theorem 3.3].) Naturally reductive Randers metrics have nice simple geometrical properties. For example they are of Berwald type [8]. Recently it is proved that these metrics are of Douglas type. (See [2, Theorem 3.3].) Here to show that the converse of this fact is not true, we recall the following theorem from [10].

Theorem 4.4. *Let $(M = \frac{G}{H}, F)$ be a homogeneous Randers space with F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field V. If (M, F) is naturally reductive, then the underling Riemannian metric (M, a) is naturally reductive.*

Also we recall that a Riemannian homogeneous space $(\frac{G}{H}, g)$ is said to be naturally reductive if there exists a reductive decomposition $\mathcal{G} = \mathcal{M} + \mathcal{H}$ of $\mathcal{G}$ satisfying the condition

$$\langle [X, Y]_{\mathcal{M}}, Z\rangle + \langle [X, Z]_{\mathcal{M}}, Y\rangle = 0, \tag{4.12}$$

for all $X, Y, Z \in \mathcal{M}$. Moreover, when $H = \{e\}$, then $\mathcal{M} = \mathcal{G}$ and the condition (4.12) is just the condition

$$\langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle = 0, \tag{4.13}$$

where X, Y, Z are in the Lie algebra $\mathcal{G}$ of G. (See [8]).
Now we can prove the following result.

Theorem 4.5. *All simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type (N, F) are never naturally reductive.*

Proof. **Type ($\mathbf{A}_1$):** In this type we should prove that the Douglas metric F which is given in (2.8) is not naturally reductive. Suppose, conversely, that the Douglas metric F is naturally reductive. Then by Theorem 4.4 the underling Riemannian metric (N, g) is naturally reductive and by (4.13) for all X,Y, and Z in the Lie algebra $\mathcal{N}$ of N we have $\langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle = 0$. If we replace (X, Y, Z) by (X_1, X_2, X_5), where $\{X_1, \ldots, X_5\}$ is an orthonormal basis of $\mathcal{N}$ and use (2.1), then we have the contradiction $\lambda = 0$. For the Types (A_2) and (A_3) we have a similar proof. □

Suppose that (M, F) is a homogeneous Finsler space. Then (M, F) is said to be a geodesic orbit (*g.o.*) space if every geodesic in M is an orbit of a one-parameter group of isometries. (For more details on geodesic vectors see [9] and [7].) In [7] the relationship between the S-curvature and homogeneous geodesics is given and it is proved that if (M, F) is a g.o. Finsler space, then the S-curvature vanishes. (See [7, Corollary 5.3].) The following lemma helps us to prove the existence of homogeneous Randers spaces with vanishing S-curvature which are not g.o. Finsler spaces.

Lemma 4.6. *All simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type (N, F) are not g.o Finsler spaces.*

Proof. **Types ($\mathbf{A}_1$) and ($\mathbf{A}_2$):** In these types if (N, F) is a g.o. Finsler space, then by [7, Corollary 5.3] the S-curvature vanishes which contradicts Theorem 4.2.
Type ($\mathbf{A}_3$): In this type if (N, F) where F is given in (2.10) is a g.o Finsler space, then by [15, Proposition 6.6] (N, g) is a g.o. Riemannian space but it is known that up to dimension 5, every g.o. Riemannian manifold is naturally reductive. This implies that (N, g) is naturally reductive which yields that for all X,Y, and Z in the Lie algebra $\mathcal{N}$ we have $\langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle = 0$. But if we replace (X, Y, Z) by (X_1, X_2, X_3), where $\{X_1, \ldots, X_5\}$ is an orthonormal basis of $\mathcal{N}$, then by (2.4) we have the contradiction $\lambda = 0$. □

Theorem (4.5) and Lemma (4.6) give us the following result.

Theorem 4.7. *Simply connected five-dimensional two-step homogeneous Randes nilmanifolds of Douglas type with a three-dimensional center have vanishing S-curvature while they are never g.o. Finsler spaces.*

Recall that a connected Finsler space (M, F) is called a weakly symmetric space if for every two points p and q in M there exists an isometry σ in the full group of isometries $I(M, F)$ such that $\sigma(p) = q$ and $\sigma(q) = p$. As it is proved in [5, Theorem 6.3] a weakly symmetric Finsler space must be a Finsler g.o. space. Thus Lemma 4.6 gives us the following result.

Theorem 4.8. *All simply connected five-dimensional two-step homogeneous Randers nilmanifolds of Douglas type are never weakly symmetric spaces.*

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(Mehri Nasehi) Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran.

E-mail address: m.nasehi@math.iut.ac.ir