ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 747–753

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 747–753
Online ISSN: 1735-8515

FRATTINI SUPPLEMENTS AND FRAT SERIES

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(Communicated by Ali Reza Ashrafi)

ABSTRACT. In this study, Frattini supplement subgroup and Frattini supplemented group are defined by Frattini subgroup. By these definitions, it's shown that finite abelian groups are Frattini supplemented and every conjugate of a Frattini supplement of a subgroup is also a Frattini supplement. A group action of a group is defined over the set of Frattini supplements of a normal subgroup of the group by conjugation and in this study new characterization of primitivity of groups has obtained in terms of Frattini supplemented groups by this action. Moreover, Frat-series of a group is defined based on Frattini supplements of normal subgroups of the group and it is shown that subgroups and factor groups of groups with Frat-series also have Frat-series under some special conditions. Furthermore, we determined a characterization of soluble groups which have Frat-series.
Keywords: Frattini subgroup, primitive group, group actions.
MSC(2010): Primary: 20D25; Secondary: 20B15, 58E40.

1. Introduction

In module theory, *Rad-supplemented* modules were defined as proper generalizations of supplemented modules. Over a ring with identity, a unitial module M is called *Rad-supplemented* if every submodule N of M has Rad-supplement in M, i.e. $N + K = M$ and $N \cap K \leq Rad(K)$ for some submodule K of M, where $Rad(K)$ is the intersection of all maximal submodules of K. Hausen studied supplemented and amply supplemented groups in terms of nilpotency by using Frattini subgroup in [3]. We investigated the properties of these groups in a similar way with [3].

2. Preliminaries

The *Frattini subgroup* of an arbitrary group G is defined to be the intersection of all the maximal subgroups, with the stipulation that it will equal to G if

Article electronically published on 30 June, 2017.
Received: 8 April 2015, Accepted: 11 February 2016.
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G has no maximal subgroup. This subgroup, which is evidently characteristic, is written as $Frat(G)$ [1].

The Frattini subgroup has the remarkable property that it is the set of all nongenerators of the group; here an element g is called a *nongenerator* of G if $G = \langle g, X \rangle$ always implies that $G = \langle X \rangle$ when X is a subset of G [1].

A subgroup H of a group G is *supplemented* in G if there is a subgroup K of G such that $G = HK$. If $H \cap K = \{1\}$ then K is said to be a *complement* of H in G [1].

The derived subgroup of G is defined as $[G, G] = \langle [a, b] | a, b \in G \rangle$ where $[a, b] = a^{-1}b^{-1}ab$, and is written as G' .

This study is based on the following definition.

Definition 2.1. Let G be a group and $N \trianglelefteq G$. The subgroup S of G is called a *Frattini supplement* of N in G if $G = NS$ and $N \cap S \leq Frat(S)$. Clearly G is *Frattini supplement* of $Frat(G)$ in G. If every $N \trianglelefteq G$ has a Frattini supplement in G, then G is said to be *Frattini supplemented*.

Example 2.2. Let G be a group. If $G = Frat(G)$ then G is Frattini supplemented.

Example 2.3. G itself is Frattini supplement of 1_G since $G = 1_G G$ and $1_G \cap G \leq Frat(G)$. So the Frattini supplement of 1_G is G.

Example 2.4. Let G be a group in which every subgroup is normal, and N be a minimal normal subgroup of G. Then G is a Frattini supplement of N.

Example 2.5. For the generalized quaternion group $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \rangle$ $(n \geq 3)$, $\langle y \rangle$ is a Frattini supplement of the normal subgroup $\langle x \rangle$ in Q_{2^n}.

Example 2.6. Let G be a finite abelian group. Then G is Frattini supplemented.

Example 2.7. Let $G = Q_8$ be the group of Hamilton quaternions. It is easy to see that $\langle i \rangle, \langle j \rangle$, and $\langle k \rangle$ are Frattini supplement of each other and G is a Frattini supplement of $\{1, -1\}$ in G, since every subgroup is normal.

3. Frattini supplemented groups

Proposition 3.1. *Let G be a finite group and N be a normal subgroup of G. If S is a Frattini supplement of N in G then S is a minimal supplement of N in G.*

Proof. Since S is a Frattini supplement of N in G, then $G = NS$ and $N \cap S \leq Frat(S)$. Let K be a Frattini supplement of N in G and $K \leq S$. Hence $G = NK$. Since $N \cap S \trianglelefteq S$, if we intersect with S, we get $S = S \cap G = S \cap NK = K(N \cap S) = \langle K, N \cap S \rangle$. Finally, we have $S = K$, since $N \cap S \leq Frat(S)$. $\square$

Proposition 3.2. *Let G be a finite group. If G is a Frattini supplement of G' then G is nilpotent.*

Proof. Since G is a Frattini supplement of G', $G = G'G$ and $G' = G \cap G' \leq Frat(G)$, then G is nilpotent according to [1]. □

Theorem 3.3. *Let G be a group, H be a finite normal subgroup of G and $G = HK$ for some $K \leq G$. If K is minimal, then K is a Frattini supplement of H in G.*

Proof. Assume that K is not a Frattini supplement of H in G. Then $H \cap K \nleq Frat(K)$ and for a maximal subgroup M of K, $H \cap K \nleq M$. So we have $M < (H \cap K)M \leq K$ which implies that $K = (H \cap K)M = HM \cap K$, and so $K \leq HM$. Since $G = HK$, for every $g \in G$, we have $g \in HM$. Hence $G = HM$, which is a contradiction. Therefore, K is a Frattini supplement of H in G. □

Proposition 3.1 and Theorem 3.3 could be merged for finite groups.

Corollary 3.4. *Let G be a finite group, and $N \trianglelefteq G$. Then S is a Frattini supplement of N in G if and only if S is a minimal supplement of N.*

Theorem 3.5. *Let G be a finite group, $N \trianglelefteq G$, H be a Frattini supplement of N in G and $K \leq H$. Then for $K \trianglelefteq G$, H/K is a Frattini supplement of NK/K.*

Proof. Since H is a Frattini supplement of N, $G = NH$ and $H \cap N \leq Frat(H)$. It is easy to see that $G/K = NH/K = (NK/K)(H/K)$ and $(H/K) \cap (NK/K) \leq (H \cap N)K/K \leq (Frat(H))K/K$ by modular law. Then we have $(Frat(H))K/K$
$\leq Frat(H/K)$ by [1], since G is finite. □

Theorem 3.6. *Let G be a finite group, $N \trianglelefteq G$, S be a Frattini supplement of N in G and H be a subgroup of G such that $S \leq H$. Then $N \cap H$ has a Frattini supplement in H.*

Proof. Obviously $N \cap H \trianglelefteq H$. Since S is a Frattini supplement of N in G, $G = NS$ and $S \cap N \leq FratS$. So we have $G = NS$, which implies that $H = G \cap H = (NS) \cap H = (N \cap H)S$ and $(H \cap N) \cap S = H \cap (N \cap S) \leq H \cap Frat(S) \leq Frat(S)$. Hence $N \cap H$ has a Frattini supplement in H. □

Theorem 3.7. *Let G be an abelian group $N, K \leq G$ and let N be a Frattini supplemented group. If a Frattini supplement X of NK in G satisfies $X \cap Y = \{1\}$ for every Frattini supplement Y of the normal subgroups of N then K has a Frattini supplement in G.*

Proof. Since X is a Frattini supplement of NK in G, $G = (NK)X$ and $NK \cap X \leq Frat(X)$. Now consider the subgroup $N \cap (KX) \leq N$. For a Frattini supplement Y of $N\cap(KX)$, $N=(N\cap(KX))Y$ and $(N\cap(KX))\cap Y\leq Frat(Y)$. Therefore $Y\cap KX\leq Frat(Y)$. First $G=(NK)X=[(N\cap KX)Y]KX=(N\cap KXY)KX=NKX\cap KXY=G\cap KXY$. Therefore $G=K(XY)$. It is easy to see that $K\cap XY\leq[(NK)\cap X][Y\cap KX]$. Hence $K\cap XY\leq[(NK)\cap X][Y\cap KX]\leq(Frat(X))(Frat(Y))$. Let $xy\in(Frat(X))(Frat(Y))$, for some $x\in Frat(X)$, $y\in Frat(Y)$ and let $XY=\langle xy,A\rangle$ for some $A\subseteq XY$. Since G is an abelian group $XY=\langle xy,A\rangle=\{(xy)^n\prod(x_iy_i)^{\varepsilon_i}\mid x_i\in X,\, y_i\in Y,\, \varepsilon_i=\pm1,\, n\in\mathbb{Z}\}=\{(x^n\prod x_i^{\varepsilon_i})(y^n\prod y_i^{\varepsilon_i})\mid x_i\in X, y_i\in Y,\varepsilon_i=\pm1, n\in\mathbb{Z}\}\leq\langle x,X_i\rangle\langle y,Y_i\rangle$, where $X_i=\bigcup\{x_i\}$, $Y_i=\bigcup\{y_i\}$, x_i and y_i are elements of X and Y respectively. Hence $XY=\langle x,X_i\rangle\langle y,Y_i\rangle$. Since $X\cap Y=\{1\}$, we have $X=\langle x,X_i\rangle$ and $Y=\langle y,Y_i\rangle$. Then we have $X=\langle X_i\rangle$ and $Y=\langle Y_i\rangle$. Finally $XY=\langle X_i\rangle\langle Y_i\rangle\leq\langle A\rangle$ which implies that $XY=\langle A\rangle$. Therefore, $xy\in Frat(XY)$ and XY is a Frattini supplement of K in G. □

Proposition 3.8. *Let G be a group, $N\trianglelefteq G$ and S be the Frattini supplement of N in G and $Frat(G)$ is finite. For $K\trianglelefteq G$, if $K\leq Frat(G)$ then S is a Frattini supplement of NK in G. In particular, if G has no maximal subgroup then S is a Frattini supplement of NK in G for every $K\trianglelefteq G$.*

Proof. $G=NS$ and $N\cap S\leq Frat(S)$ since S is a Frattini supplement of N in G , so for $K\trianglelefteq G$, $G=NKS=(NK)S$. Now we must show that $NK\cap S\leq Frat(S)$. Suppose that $NK\cap S\nleq Frat(S)$. So $NK\cap S\nleq M$, for some maximal subgroup M of S and then $S=\langle NK\cap S,M\rangle$. Hence $S=\langle NK\cap S,M\rangle$ implies $G=NS=N\langle NK\cap S,M\rangle\leq N\langle NK,M\rangle=\langle K,N,M\rangle$. Therefore $G=\langle K,N,M\rangle$ and then $G=NM$ since $K\leq Frat(G)$. But we have $S=M$ by minimality of S from Proposition 3.1, which is a contradiction. Therefore $NK\cap S\leq Frat(S)$ and S is a Frattini supplement of NK in G. In particular, if G has no maximal subgroup then $G=Frat(G)$ and obviously, S is a Frattini supplement of NK in G for every $K\trianglelefteq G$. □

Corollary 3.9. *Let G be a group, $N\trianglelefteq G$ and S be the Frattini supplement of N in G and $Frat(G)$ is finite. If $K\leq Frat(G)$ then $K\cap S\leq Frat(S)$ for $K\trianglelefteq G$.*

Proof. It's obvious since $K\cap S\leq NK\cap S\leq Frat(S)$ by Proposition 3.8. □

4. Primitivity for Frattini supplemented groups

If G is a group and $N\trianglelefteq G$, then the Frattini supplement set of N can be defined. Consider the set $\Sigma_N=\{S\leq G\mid G=NS, N\cap S\leq Frat(S)\}$. One can assume G is a Frattini supplemented to ensure that $\Sigma_N\neq\emptyset$.

Corollary 4.1. *Let G be a Frattini supplemented group and $N\trianglelefteq G$. If $G\in\Sigma_N$ then $\Sigma_N=\{G\}$.*

Proof. Since G is minimal by Proposition 3.1, then $\Sigma_N=\{G\}$. □

Theorem 4.2. *Let G and H be groups, $N \trianglelefteq G$ and S be a Frattini supplement of N in G. If $\varphi : G \to H$ is an isomorphism then $\varphi(S)$ is a Frattini supplement of $\varphi(N)$ in H. In particular, if $\sigma \in Aut(G)$ then $\sigma(S)$ is a Frattini supplement of $\sigma(N)$ in G and if $T \in \Sigma_N$ then for every $g \in G$, $T^g \in \Sigma_N$.*

Proof. Since S is a Frattini supplement of N in G, we have $G = NS$ and $N \cap S \leq Frat(S)$. If $\varphi : G \to H$ is an isomorphism and $N \trianglelefteq G$ then $\varphi(N) \trianglelefteq H$ and obviously $H = \varphi(G) = \varphi(NS) \leq \varphi(N)\varphi(S)$ which implies that $H = \varphi(N)\varphi(S)$. Now we show that $\varphi(N) \cap \varphi(S) \leq Frat(\varphi(S))$. Firstly, if $a \in \varphi(N) \cap \varphi(S)$ then $a = \varphi(n) = \varphi(s)$ for some $n \in N$, and $s \in S$. So $a = \varphi(n) = \varphi(s)$ which implies that $\varphi(n) = \varphi(s)$, and so $\varphi(n)(\varphi(s))^{-1} = 1_H$. It follows that $\varphi(n)\varphi(s^{-1}) = 1_H$, thus $\varphi(ns^{-1}) = 1_H$. Therefore, $ns^{-1} \in Ker(\varphi)$ and $n = s$, since φ is an isomorphism and $Ker(\varphi) = 1_G$. Hence $n = s \in N \cap S$ and we have $a = \varphi(n) = \varphi(s) \in \varphi(N \cap S)$. So $\varphi(N) \cap \varphi(S) \leq \varphi(N \cap S) \leq \varphi(Frat(S))$. Now we will show that $\varphi(Frat(S)) \leq Frat(\varphi(S))$. Let $\varphi(a) \in \varphi(Frat(S))$ for some $a \in Frat(S)$ and let $\varphi(S) = \langle \varphi(a), X \rangle$ for any $X \subseteq \varphi(S)$. If $X \subseteq \varphi(S)$ then $X = \varphi(A)$ for some $A \subseteq S$. Since φ is an isomorphism $\varphi(S) = \langle \varphi(a), \varphi(A) \rangle \leq \varphi(\langle a, A \rangle)$. Therefore, $\varphi(S) = \varphi(\langle a, A \rangle)$ and so $S = \langle a, A \rangle$. Since $a \in Frat(S)$, we have $S = \langle A \rangle$. It follows from $S = \langle A \rangle$ that $\varphi(S) = \varphi(\langle A \rangle) \leq \langle \varphi(A) \rangle = \langle X \rangle$ which implies that $S = \langle X \rangle$, and so $\varphi(a) \in Frat(\varphi(S))$. Hence $\varphi(Frat(S)) \leq Frat(\varphi(S))$. Finally, $\varphi(N) \cap \varphi(S) \leq Frat(\varphi(S))$ and $\varphi(S)$ is a Frattini supplement of $\varphi(N)$ in G. In particular, it is obvious that if $\sigma \in Aut(G)$ then $\sigma(S)$ is a Frattini supplement of $\sigma(N)$ in G, since σ is an isomorphism. If $T \in \Sigma_N$ then for every $\sigma \in Inn(G)$ and for every $g \in G$, $\sigma(T) = T^g$ will be a Frattini supplement of N in G. □

Example 4.3. For the group S_3, $\langle (12) \rangle$ is a Frattini supplement of A_3 in S_3. Anyone can easily see that every conjugate of $\langle (12) \rangle$ in S_3 is also a Frattini supplement of A_3 in S_3.

Let G be a Frattini supplemented group and $N \trianglelefteq G$. Consider the set defined above $\Sigma_N = \{S \leq G \mid G = NS,\ N \cap S \leq Frat(S)\}$. By Theorem 4.2, a group action might be defined as:

The function $G \times \Sigma_N \longrightarrow \Sigma_N$, $(g, S) \to S^g$, then G acts on Σ_N.

Using Cayley-like representation by this action we have the function $g : \Sigma_N \to \Sigma_N$, $S \to S^g$ is well-defined so the morphism $\varphi : G \to Sym(\Sigma_N)$, $g \to g^{-1}$ is closed, well defined and a homomorphism. Before Theorem 4.4, consider the transitivity of Σ_N. It may not be found $g \in G$, such that $S^g = T$ for every pair of $S, T \in \Sigma_N$. Hence, let us take the subset A_N of Σ_N such that $A_N = \{S^g | S \in \Sigma_N, g \in G\}$. Obviously G acts transitively on A_N.

Theorem 4.4. *Let G be a Frattini supplemented group, $N \trianglelefteq G$, $G \notin \Sigma_N$ and S be maximal in G for every $S \in \Sigma_N$. Then G acts primitively on the set A_N.*

Proof. Since $S \in \Sigma_N$ is a maximal subgroup of G and $S \leq N_G(S) \leq G$, we have $S = N_G(S)$ or $N_G(S) = G$. First, consider the case $S = N_G(S)$. For some $g \in G \setminus S$, we have $S^g \neq S$, since $S < G$. Then $|A_N| \geq 2$. Now for the subgroup $G_S = \{g \in G \mid S^g = S\}$, it is obvious that $G_S = N_G(S) = S$. Therefore, G_S is a maximal subgroup of G and so G is primitive by [2]. Now consider the second case, when $N_G(S) = G$. Then $S \trianglelefteq G$ and so, for every $g \in G$, we have $S^g = S$. Then $|A_N| = 1$. Hence A_N is a trivial block for G and G is primitive. □

5. **Frat-series**

Definition 5.1. Let G be a group and $1 = G_0 < G_1 < \cdots < G_n = G$ be a normal series of G. If G_i has a Frattini supplement S_i in G_{i+1} for every $1 \leq i \leq n$, then G is said to have a *Frat-series*.

Let G be a group which has a Frat- series, G_{i-1} be a term of the series and S_{i-1} be a Frattini supplement of G_{i-1} in G_i. If G has a subgroup H such that $S_{i-1} \leq H$ for every $1 \leq i \leq n$ then H has a Frat-series. In particular $\langle \{S_i\} \rangle$ has a Frat-series.

Proof. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G. Consider the intersection of H with terms of the series in hypothesis. Obviously $1 = H \cap G_0 < H \cap G_1 < \cdots < H \cap G_n = H$ is a normal series of H. Since S_{i-1} is a Frattini supplement of G_{i-1} in G_i and $S_{i-1} \leq H$, we have $G_i \cap H = (G_{i-1}S_{i-1}) \cap H = (G_{i-1} \cap H)S_{i-1}$. Moreover, $(G_{i-1} \cap H) \cap S_{i-1} = (G_{i-1} \cap S_{i-1}) \cap H \leq FratS_{i-1} \cap H \leq FratS_{i-1}$. So H has a Frat-series. In particular for $H = \langle \{S_i\} \rangle$ we conclude that $\langle \{S_i\} \rangle$ has a Frat-series. □

Theorem 5.2. *Let G be a group that has a Frat-series, G_{i-1} be a term of the series and S_{i-1} be a finite Frattini supplement of G_{i-1} in G_i. If N is a normal subgroup of G such that $N \trianglelefteq S_{i-1}$, then G/N has a Frat-series.*

Proof. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G. One can easily see that the series $1 = N/N = G_0N/N \leq G_1N/N \leq ... \leq G_nN/N = G/N$ which is obtained from the Frat-series of G, is a normal series of G/N. Since S_{i-1} is a finite Frattini supplement of G_{i-1} in G_i for every $1 \leq i \leq n$, G/N has a Frat-series by Theorem 3.5. □

Theorem 5.3. *Let G be a group and $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G and $\sigma \in Aut(G)$. Then $1 = \sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is also a Frat-series of G.*

Proof. First, we will show that $1 = \sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is a normal series of G. It is obvious that $\sigma(G_i) < \sigma(G_{i+1})$ for every i. Also, it is easy to see that $\sigma(G_i) \trianglelefteq G$. Since $1 = G_0 < G_1 < \cdots < G_n = G$ is a Frat-series of G, then there exists $S_i \leq G_{i+1}$ such that $G_{i+1} = G_iS_i$ and $G_i \cap S_i \leq Frat(S_i)$ for every i. Furthermore, the restriction of σ to G_{i+1} is an isomorphism from G_{i+1} to $\sigma(G_{i+1})$ and $\sigma(S_i)$ is a Frattini supplement of $\sigma(G_i)$

in $\sigma(G_{i+1})$ by Theorem 4.2. Finally, $1 = \sigma(G_0) < \sigma(G_1) < \cdots < \sigma(G_n) = G$ is a Frat-series of G. □

Theorem 5.4. *Let G be a group, $1 = G_0 < G_1 < \cdots < G_n = G$ be a Frat-series of G, G_{i-1} be a term of the series and $S_{i-1} \trianglelefteq G$ be a Frattini supplement of G_{i-1} in G_i for every $1 \leq i \leq n-1$ and $S_{n-1} \trianglelefteq G$ be a complement of G_{n-1} in G. If S_{i-1} is a complement of G_{i-1} in G_i then G_{i-1} has a Frattini supplement in G_{i+1}.*

Proof. Since G_{i+1} and G_i are terms of the Frat-series of G, we have $G_{i+1} = G_{i-1}(S_{i-1}S_i)$, $1 = G_{i-1} \cap S_{i-1} \leq Frat(S_{i-1})$, and $1 = G_i \cap S_i \leq Frat(S_i)$ for some $S_{i-1}, S_i \trianglelefteq G$. Let a be an element of $G_{i-1} \cap (S_{i-1}S_i)$. Then $a = xy$ for some $x \in S_{i-1}$ and $y \in S_i$. Therefore $a = xy$ implies that $y = x^{-1}a \in S_{i-1}G_{i-1} = G_{i-1}S_{i-1} = G_i$ and then $y \in G_i \cap S_i = 1$. So we have $a = x \in G_{i-1} \cap S_{i-1} = 1$ and $a = 1$. Hence $1 = G_{i-1} \cap (S_{i-1}S_i) \leq Frat(S_{i-1}S_i)$. Therefore $S_{i-1}S_i$ is a Frattini supplement of G_{i-1} in G_{i+1}. □

Theorem 5.5. *Let $1 = G_0 < G_1 < \cdots < G_n = G$ be Frat-series of G. If $S_i^{'} \leq G_i$ for every $0 \leq i < n$ where S_i is a Frattini supplement of G_i in G_{i+1} then G is soluble.*

Proof. Since $1 = G_0 < G_1 < \cdots < G_n = G$ is a Frat-series of G, then $G_{i+1} = G_iS_i$ and $G_i \cap S_i \leq Frat(S_i)$ for every i. Now, we will show that the Frat-series of G is also a derived series. Consider the element $[x, y]$ of $S_i^{'}$. So $[x, y] \in G_i$ and then $S_i^{'} \leq G_i \cap S_i$ since $S_i^{'} \leq S_i$. Hence, for every $[x, y] \in S_i^{'}$, $[x, y](G_i \cap S_i) = G_i \cap S_i$. Therefore $x^{-1}y^{-1}xy(G_i \cap S_i) = G_i \cap S_i$ and so $xy(G_i \cap S_i) = yx(G_i \cap S_i)$ and we obtain $x(G_i \cap S_i)y(G_i \cap S_i) = y(G_i \cap S_i)x(G_i \cap S_i)$ for every $x, y \in G_i$. Therefore the factor $S_i/G_i \cap S_i$ is abelian. Since $S_i/G_i \cap S_i \simeq G_iS_i/G_i = G_{i+1}/G_i$, we have G_{i+1}/G_i is abelian. So, $1 = G_0 < G_1 < \cdots < G_n = G$ is a derived series of G and G is soluble. □

Acknowledgements

The authors appreciate the comments and valuable suggestions of the editor and the reviewer. Their advice helped to improve the clarity and presentation of this paper.

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